THE GLOBAL CONNECTED DOMINATION IN GRAPHS

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ABSTRACT. A subset S of vertices of a graph G is called a *global connected dominating set* if S is both a global dominating set and a connected dominating set. The *global connected domination number* is the minimum cardinality of a global connected dominating set of G and is denoted by $\gamma_{gc}(G)$. In this paper, sharp bounds for γ_{gc} are supplied, and all graphs attaining those bounds are characterized. We also characterize all graphs of order n with $\gamma_{gc} = k$ where $3 \leq k \leq n-1$. Exact values of this number for trees and cycles are presented as well.

1. INTRODUCTION

Domination is an active subject in graph theory, and has numerous applications to distributed computing [1, 5], the web graph [3], and ad hoc networks [4, 7]. For a comprehensive introduction to theoretical and applied facets of domination in graphs the reader is directed to the books [8, 9].

A set S of vertices is called a *dominating set* of a graph G if each vertex not in S is joined to some vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

Many variants of the domination number have been studied. For instance, a dominating set S of a connected graph G is called a *connected dominating set* (CD-set) if the induced subgraph $\langle S \rangle$ is connected. Similarly, a dominating set S of a connected graph G is called a *nonsplit dominating set* (NSD-set) if the induced subgraph $\langle V(G) - S \rangle$ is connected. A set S is called a *global dominating set* (GD-set) of G if S is a dominating set of both G and its complement $\overline{G}$. The *connected domination number* of G, written $\gamma_c(G)$, is the minimum cardinality of a CD-set of G. The *nonsplit domination number* $\gamma_{ns}(G)$ and *global domination number* $\gamma_g(G)$ are defined analogously.

Recently Kulli and Janakiram [6] introduced the concept of global nonsplit domination. Let G and $\overline{G}$ be connected graphs. A set $S \subseteq V(G)$ is called a *global nonsplit dominating set* (GNSD-set) if S is an NSD-set of both G and $\overline{G}$. The *global nonsplit domination number* $\gamma_{gns}(G)$ of G is the minimum cardinality of a GNSD-set of G.

Clearly, the global nonsplit domination number γ_{gns} is not defined for many connected graphs. For instance, for any connected graph having a universal vertex (that is, a vertex joined to all other vertices in the graph) such as K_n and $K_{1,n-1}$.

In the present paper, we introduce a new graph parameter, the global connected domination number, for a connected graph G. We call $S \subseteq V(G)$ a *global connected*

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dominating set (GCD-set) of G if S is both a global dominating and a connected dominating set of G. The minimum cardinality of a GCD-set of G is the *global connected domination number* $\gamma_{gc}(G)$. Note that any GCD-set of a graph G (e.g., $V(G)$ is a GCD-set of G) has to be connected in G (but not necessarily in $\overline{G}$). Hence, the global connected domination number γ_{gc} is well-defined for any connected graph.

The parameter γ_{gc} differs significantly from γ_g. For instance, for a cycle C_n of order $n \geq 6$ (see [10]), $\gamma_g(C_n) = \lceil n/3 \rceil$, while $\gamma_{gc}(C_n) = n - 2$ (see Corollary 5). The parameter γ_{gc} also differs from γ_c significantly. For example, for all integers $n \geq 2$, $\gamma_c(K_n) = 1$, while $\gamma_{gc}(K_n) = n$ (see Theorem 1).

All graphs considered in this paper are simple, finite, undirected and connected. For all graph-theoretic terminology not defined here, the reader is referred to [2]. The union of two vertex-disjoint graphs G and H is denoted by $G \cup H$. We use $G \cong H$ to denote that G and H are isomorphic.

In Section 2, sharp bounds for γ_{gc} are supplied, and the graphs attaining those bounds are characterized. In Section 3, we characterize the graphs with $\gamma_{gc} = k$ where $3 \leq k \leq n-1$. Exact values of this number for trees are presented in Section 4. We witness all possible values of γ_{gc}, and also prove that there exists a connected graph G of order $n \geq 5$ such that $\overline{G}$ is connected, and $\gamma_{gc}(G) \neq \gamma_{gc}(\overline{G})$. We conclude in the last section with a few directions for future work.

2. Bounds

In this section, we present lower and upper bounds on $\gamma_{gc}(G)$ for a general graph in terms of its order, and we characterize the graphs attaining these bounds.

Fix $n \geq 2$ an integer. We define a family $\mathcal{F}$ of graphs of order n as follows. Fix A and B two disjoint subsets of vertices such that $|A \cup B| = n - 2$ ($A = B = \emptyset$ when $n = 2$), and let $a, b \notin A \cup B$. Let $V = \{a, b\} \cup A \cup B$. Denote by $F(A, B)$ the set of graphs with vertex set V which satisfy the following properties:

(1) The vertex a is joined to b.
(2) The vertex a is joined to each vertex in A whenever $A \neq \emptyset$.
(3) The vertex b is joined to each vertex in B whenever $B \neq \emptyset$.
(4) None of $V(G) - \{a, b\}$ is joined to both a and b.

Let $\mathcal{F} = \bigcup_{A \cap B = \emptyset, |A \cup B| = n-2} F(A, B)$.

Theorem 1. *Let G be a graph of order $n \geq 2$. Then*

(1) $2 \leq \gamma_{gc}(G) \leq n$.
(2) $\gamma_{gc}(G) = n$ *if and only if* $G \cong K_n$.
(3) $\gamma_{gc}(G) = 2$ *if and only if* $G \in \mathcal{F}$.

Proof. For item (1), note that $1 \leq \gamma_{gc}(G) \leq n$, it suffices to show that $\gamma_{gc}(G) \neq 1$. To the contrary, we may assume that there exists a GCD-set $S = \{v\}$. Hence, v is joined to all vertices in $V(G) - \{v\}$, and so v is isolated in $\overline{G}$. This contradicts that $\{v\}$ is a GCD-set of G.

For item (2), note that the graph $\overline{K_n}$ consists of n isolated vertices, so the proof of sufficiency is trivial. To show necessity, we take a spanning tree of G, say T. Let v be a leaf of T, and let $S = V(G) - \{v\}$. Note that v is joined to some vertex in S, and the subgraph in G induced by S is connected. Hence, S is a CD-set of G with size $n-1$. As $\gamma_{gc}(G) = n$, S is not a GCD-set of G. Therefore, v is isolated in $\overline{G}$ implying that v is joined to all vertices of S in G. Now we take another spanning tree T' consisting of all edges incident with v. Note that every vertex $u \in V(G) - \{v\}$ is a leaf of T', and $S' = V(G) - \{u\}$ is a CD-set of G with size $n-1$. By a similar argument, one can show that u is joined to all vertices in S'. Hence, $G \cong K_n$.

For item (3), we prove sufficiency first. Let $G = F(A, B) \in \mathcal{F}$. By the construction, we know that $\{a, b\}$ is a GCD-set of G. Hence, $\gamma_{gc}(G) \leq 2$. By item (1), $\gamma_{gc}(G) = 2$. Now we prove necessity. Let G be an arbitrary graph with $\gamma_{gc}(G) = 2$. Suppose that $S = \{a, b\}$ is a GCD-set of G. Then $ab \in E(G)$. Since S is a GCD-set of G, there exists no vertex joined to both a and b. As S is a dominating set of G, each vertex $u \in V(G) - S$ must be joined to either a or b. Hence, for each vertex $u \in V(G) - S$, u is joined to either a or b but not both. Denote by A and B the sets of vertices which are joined to a and b, respectively. Clearly, $G \in F(A, B)$, and so $G \in \mathcal{F}$. □

Corollary 2. *For all positive integers p and q,*

$$\gamma_{gc}(K_{p,q}) = 2.$$

3. Characterizing graphs with $\gamma_{gc} = k$ $(3 \leq k \leq n-1)$.

In this section, we first characterize the graphs with $\gamma_{gc} = n - 1$.

Theorem 3. *For any graph G of order $n \geq 3$, $\gamma_{gc}(G) = n - 1$ if and only if $G \cong K_n - e$, where e is an edge of K_n.*

Proof. Sufficiency. If $G \cong K_n - e$, where $e = uv \in E(K_n)$, then $\overline{G}$ is a graph consisting of an edge uv and $n-2$ isolated vertices. Hence, every GCD-set of G must contain all vertices of $V(G) - \{u, v\}$ and at least one of u and v. Hence, $\gamma_{gc}(G) \geq n-1$. The inequality $\gamma_{gc}(G) \leq n-1$ follows from the fact that $V(G) - \{u\}$ is a GCD-set of G. Hence, $\gamma_{gc}(G) = n - 1$.

Necessity. We may assume that $n \geq 4$, as the assertion holds by Theorem 1 (3) when $n = 3$. Suppose that S is a GCD-set of G with size $n-1$. Let v be the unique vertex outside of S. Since S is a GCD-set of G, v must be joined to some vertex in S but not all. Therefore, there exist vertices $x, y \in S$ such that $xv \in E(G)$ and $yv \notin E(G)$. As S is a GCD-set of G, there exists a spanning tree T of the induced subgraph $\langle S \rangle$. Let $w \in S - \{x\}$ be a leaf of T. Since S is a GCD-set of G with smallest size $n-1$, $S - \{w\}$ is not a GCD-set of G. Note that $S - \{w\}$ is a CD-set of G. Hence, $S - \{w\}$ cannot be a GD-set of G, which implies that w is not joined to any vertex of $S - \{w\}$ in $\overline{G}$. It turns out that w is joined to all vertices of $S - \{w\}$ in G. Let T' be the subgraph induced by all edges incident with w in $\langle S \rangle$. It is clear that T' is a spanning tree of $\langle S \rangle$ with $n-2$ leaves. Take any leaf $z \neq x$ on T'. By a similar argument to the above, we can derive that z is joined to all vertices of $S - \{z\}$. Thus, $\langle S \rangle \cong K_{n-1}$. To complete our proof, we only need to show that v is joined to all vertices of $S - \{y\}$. To the contrary, suppose that there is one vertex $u \in S - \{y\}$ such that $uv \notin E(G)$. Then $S' = V(G) - \{u, y\}$ is a GCD-set of G with size $n-2$, which contradicts the assumption $\gamma_{gc}(G) = n - 1$. □

Next we characterize the graphs with $\gamma_{gc} = k$, where $3 \leq k \leq n-2$. We define a family $\mathcal{H}_k$ of graphs as follows. For each graph $G \in \mathcal{H}_k$, there exists a subset S of vertices such that $|S| = k$ and $\langle S \rangle$ is connected, and for every $v \in V(G) \backslash S$, v is joined to at least one vertex in S but not all of them.

Theorem 4. *Let G be a graph of order $n \geq 5$. For any integer $3 \leq k \leq n-2$, $\gamma_{gc}(G) = k$ if and only if $G \in \mathcal{H}_k - \mathcal{H}_{k-1}$.*

Proof. Observe that $\gamma_{gc}(G) \leq k$ for every $G \in \mathcal{H}_k$. Thus the proof of sufficiency is trivial. Now we prove necessity. Let S be a GCD-set of G with size k. By definition, the induced subgraph $\langle S \rangle$ is connected, and for every $v \in V(G) \backslash S$, v is joined to at least one vertex in S but not all of them. So, $G \in \mathcal{H}_k$. It is clear that $G \notin \mathcal{H}_{k-1}$, as $\gamma_{gc}(G) = k$. Thus, $G \in \mathcal{H}_k - \mathcal{H}_{k-1}$. □

We remark that the proof of Theorem 4 also works for $k = n - 1$.

Corollary 5. *For all $n \geq 4$,*

$$\gamma_{gc}(C_n) = n - 2.$$

Proof. The assertion holds for $n = 4$ obviously. Hence, we may assume that $n \geq 5$. Clearly, $C_n \in \mathcal{H}_{n-2} - \mathcal{H}_{n-3}$. Thus, it follows from Theorem 4 that $\gamma_{gc}(C_n) = n - 2$. □

4. More on the parameter γ_{gc}

In this section, we first study the global connected domination number γ_{gc} for trees in terms of their orders. Our next theorem, whose proof will be given later, gives the exact values of γ_{gc} for trees.

Theorem 6. *Let T be a tree of order $n \geq 3$. If $T \ncong K_{1,k}$ where $k \geq 2$ is an integer, then $\gamma_{gc}(T) = n - \mu(T)$, where $\mu(T)$ is the number of leaves of T.*

Corollary 7. *Let T be a tree of order $n \geq 4$. Then $\gamma_{gc}(T) = n - 2$ if and only if $T \cong P_n$.*

Proof. We only prove necessity, as sufficiency follows from Theorem 6 (1). Suppose that T is a tree of order $n \geq 4$ with $\gamma_{gc}(T) = n - 2$. By Theorem 6, we know that there are exactly 2 leaves in the tree T. It is straightforward to show that any tree with exactly 2 leaves must be isomorphic to a path on n vertices. □

The following lemma is useful in proving Theorem 6. A *cut-vertex* of a graph is a vertex whose deletion increases the number of components of the graph.

Lemma 8. *Let G be a graph and let S be a CD-set of G. Then S contains all cut vertices (if exist) of G.*

Proof. Suppose that there exists a cut vertex $v \in V(G)$ such that $v \notin S$. Note that $S \subset V(G) - \{v\}$ and $G - v$ is disconnected. This is a contradiction, as S forms a CD-set of G. □

Proof of Theorem 6. Note that T has at least two leaves. Let U be the set of all leaves in T. Since $T \ncong K_{1,k}$, $T - U$ is nontrivial and connected. Note that the subgraph induced by $V(T) - U$ is a tree, so it is connected. As every leaf of T is joined to some vertex in $V(T) - U$ but not all, $V(T) - U$ is a GCD-set of T. Thus, $\gamma_{gc}(T) \leq n - \mu(T)$. Note that every vertex in $V(T) - U$ is a cut vertex in T. By Lemma 8, $\gamma_{gc}(T) \geq n - \mu(T)$. □

Theorem 9. *There exists a connected graph G of order $n \geq 5$ such that $\overline{G}$ is connected, and $\gamma_{gc}(G) \neq \gamma_{gc}(\overline{G})$.*

Proof. Let G be a path $P_n = v_1, v_2, \ldots, v_n$ on $n \geq 5$ vertices. By Corollary 7, we have that $\gamma_{gc}(G) = n - 2$. In the following we will show that $\gamma_{gc}(\overline{G}) = \lceil n/3 \rceil$, so it follows that $\gamma_{gc}(G) \neq \gamma_{gc}(\overline{G})$.

By the definition, any GCD-set of $\overline{G}$ is a dominating set of $\overline{\overline{G}}$. As $\overline{\overline{G}} \cong P_n$, every GCD-set S of $\overline{G}$ is a dominating set of P_n. Hence, $\gamma_{gc}(\overline{G}) \geq \gamma(P_n) = \lceil n/3 \rceil$. It is not hard to construct a GCD-set S of $\overline{G}$ with order $\lceil n/3 \rceil$. For instance,

$$S = \begin{cases} \bigcup_{k=1}^{\lceil n/3 \rceil - 1} \{v_{3k-1}\} \cup \{v_n\} & n = 1 \pmod 3; \\ \bigcup_{k=1}^{\lceil n/3 \rceil} \{v_{3k-1}\} & \text{otherwise.} \end{cases}$$

From the construction of S, we know that S is a GCD of $\overline{G}$ with size $\lceil n/3 \rceil$. Thus, $\gamma_{gc}(\overline{G}) \leq \lceil n/3 \rceil$. □

By Theorem 1 (1) we have that

$$2 \leq \gamma_{gc}(G) \leq n$$

for any connected graph G of order $n \geq 2$. Are all these possible values of γ_{gc} witnessed? The last theorem answers this question in the affirmative.

Theorem 10. *Let $n \geq 2$ be an integer. For each k satisfying $2 \leq k \leq n$, there is a connected graph G such that*

$$\gamma_{gc}(G) = k.$$

Proof. By Theorems 1, 3, 4 and Corollary 7, we only need to show the cases when $4 \leq k \leq n - 3$. We construct a tree G as follows. Add $n - k - 2$ vertices to a path P on $k + 2$ vertices so that the resulting graph G is still a tree with $n - k$ leaves. By Theorem 6, we know that $\gamma_{gc}(G) = n - (n - k) = k$. □

5. Open problem and future work

Many graph products exist, such as the cartesian, categorical, strong, and lexicographic products. We will investigate how the parameter γ_{gc} acts with respect to these products in future work.

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