On the Laplacian spectral radii of tricyclic graphs *

Liu Mu-huo[1,2] , Wei Fu-yi[1], Bolian Liu[2]
[1] Department of Applied Mathematics, South China Agricultural University, Guangzhou, P. R. China, 510642
[2] College of Mathematic Science, South China Normal University, Guangzhou, P. R. China, 510631

December 11, 2008

Abstract: A graph G of order n is called a tricyclic graph if G is connected and the number of edges of G is $n+2$. Let $\mathcal{T}_n$ denote the set of all tricyclic graphs on n vertices. In this paper, we determine the first to nineteenth largest Laplacian spectral radii among the all graphs in the class $\mathcal{T}_n$ $(n \geq 11)$ together with the corresponding graphs.
Keywords: Laplacian matrix, Laplacian spectral radius, Tricyclic graph.
2000 Mathematics Subject Classification: 05C50; 05C75

1 Introduction

Throughout the paper, $G = (V, E)$ is a connected undirected simple graph with $V = \{v_1, v_2, ..., v_n\}$ and $E = \{e_1, e_2, ..., e_m\}$, i.e., $|V| = n$ and $|E| = m$. If $m = n+c-1$, then G is called a $c-$cyclic graph. Especially, if $c = 1, 2, 3$, then G is called a unicyclic, bicyclic, or tricyclic graph, respectively. The notation $\mathcal{T}_n$ is used to denote the class of tricyclic graphs of order n. The neighbors of a vertex v is denoted by $N(v)$. Write $d(v)$ for the degree of vertex v. Specially, Δ denotes the maximum degree of G.

The adjacency matrix $A(G) = [a_{ij}]$ of G is an $n \times n$ symmetric matrix of 0's and 1's with $a_{ij} = 1$ if and only if v_i and v_j is joined by an edge. Suppose the degree of vertex v_i equals $d(v_i)$ for $i = 1, 2, ..., n$, and let $D(G)$ be the diagonal matrix whose (i,i)-entry is $d(v_i)$. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, and the *Lapalacian characteristic polynomial*

*The first author is supported by the fund of South China Agricultural University (No. 2008K012); The third author is the corresponding author who is supported by NNSF of China. E-mail address: liubl@scnu.edu.cn

of G is denoted by $\Phi(G,\lambda)$, i.e., $\Phi(G,\lambda)$=det($\lambda I - L(G)$). The maximum eigenvalue of $L(G)$, denoted by $\mu(G)$, is called the *Laplacian spectral radius* of $L(G)$. Our terminology and notation are standard except as indicated. For terminology and notation not defined here, we refer the readers to [1-10] and the references therein.

The investigation on the spectral radii of $L(G)$ is an important topic in the theory of graph spectral. Up to now, the first to fourteenth largest spectral radii of $L(G)$ were determined in the class of trees in [1-3]. The first to thirteenth largest spectral radii of $L(G)$ were given in the class of unicyclic graphs in [4-6]. Recently, Jia-Yu Shao et al. [7] obtained the first four largest spectral radii of $L(G)$ in the class of bicyclic graphs. Actually, we have determined the first eight largest spectral radii of $L(G)$ by using the different method from [1-7] in [8]. In this paper, we obtain the first to nineteenth largest Laplacian spectral radii among the all graphs in the class $\mathcal{T}_n$ ($n \geq 11$) together with the corresponding graphs.

2 Main results

A vertex of degree 1 is called a *pendant vertex* of G. Suppose B is a square matrix, let $a_{ii}(B)$ denote the entry appearing in the i-th row and the i-th column of B. In [9], we have obtained a new method to calculate the Lapalacian characteristic polynomial of an n-vertex graph as follows

Lemma 2.1 [9] *Let G be a graph on $n-k$ ($1 \leq k \leq n-2$) vertices with $V(G) = \{v_n, v_{n-1}, ..., v_{k+1}\}$. If G' is obtained from G by attaching k new pendant vertices, say v_1, ..., v_k, to v_{k+1}, then*

$$\Phi(L(G'),\lambda) = (\lambda-1)^k \cdot det(\lambda I_{n-k} - L(G) - B_{n-k}),$$

where $a_{11}(L(G))$ is corresponding to the vertex v_{k+1}, and $B_{n-k} = diag\{k+\frac{k}{\lambda-1}, 0, ..., 0\}$.

Example 2.1 *Let G', G be the graphs as shown in Fig 1. It is easy to see that G' is obtained from G by attaching $n-4$ new pendant vertices to the vertex v_{n-3} of G. By Lemma 2.1, we have*

$\Phi(L(G'),\lambda) = (\lambda-1)^{n-4} det(M)$,

$$\text{where } M = \begin{pmatrix} \lambda-(n-2)-\frac{n-4}{\lambda-1} & 1 & 1 & 0 \\ 1 & \lambda-3 & 1 & 1 \\ 1 & 1 & \lambda-2 & 0 \\ 0 & 1 & 0 & \lambda-1 \end{pmatrix}.$$

By using "Matlab", it is easy to obtain that

$\Phi(L(G'),\lambda) = \lambda(\lambda-1)^{n-5}(\lambda^4-(n+5)\lambda^3+(6n+3)\lambda^2-(9n-5)\lambda+3n)$.

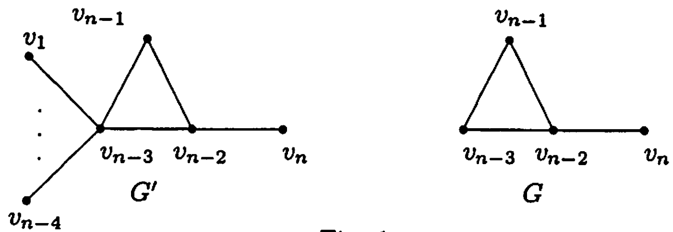


Fig. 1

Lemma 2.2 [10] *$\mu(G) \le max\{d(v) + m(v) : v \in V\}$, where $m(v) = \sum_{u\in N(v)} d(u)/d(v)$.*

Lemma 2.3 [11] *$\Delta + 1 \le \mu(G) \le n$, the left equality holds if and only if $\Delta = n - 1$.*

Proposition 2.1 *Suppose $c \ge 0$ and G is a $c-$cyclic graph with $\Delta \le n-3$. If $n \ge max\{c+7, 2c+5\}$, then $\mu(G) \le n-1$.*

Proof. By Lemma 2.2, we only need to prove that $max\{d(v) + m(v) : v \in V\} \le n-1$.

Suppose $max\{d(v)+m(v) : v \in V\}$ occurs at the vertex u. Three cases arise $d(u) = 1$, $d(u) = 2$, or $3 \le d(u) \le n-3$.

Case 1. $d(u) = 1$. Suppose $v \in N(u)$. Since $d(v) \le \Delta \le n-3$, thus $d(u) + m(u) = d(u) + d(v) \le n-2 < n-1$.

Case 2. $d(u) = 2$. Suppose that $v,w \in N(u)$. Note that G is a $c-$cyclic graph, then $|N(v) \cap N(w)| \le c+1$ and $|N(v) \cup N(w)| \le n$. Therefore, $d(u) + m(u) = 2 + \frac{d(v)+d(w)}{2} \le 2 + \frac{n+c+1}{2} \le n-1$.

Case 3. $3 \le d(u) \le n-3$. Note that $3 \le d(u) \le n-3$, then $d(u) + m(u) \le d(u) + \frac{2m-d(u)-2}{d(u)} = d(u) - 1 + \frac{2m-2}{d(u)}$. Next we shall prove that $d(u) - 1 + \frac{2m-2}{d(u)} \le n-1$, equivalently, $d(u)(n - d(u)) \ge 2m-2$. Once this is proved, we are done. Let $f(x) = (n-x)x$.

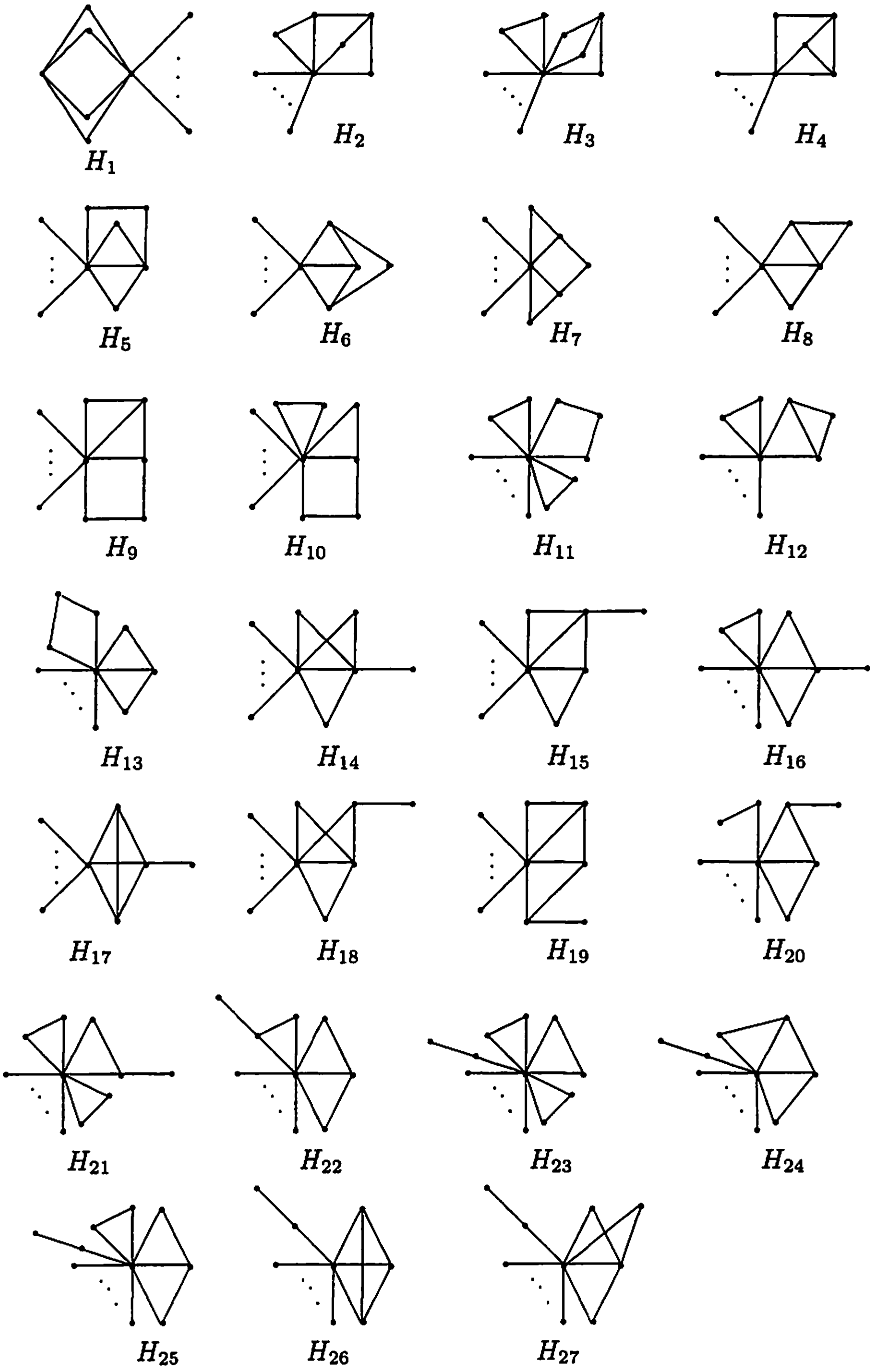
When $x \in [3, \frac{n}{2}]$, since $f'(x) = n - 2x \ge 0$, then $f(x) \ge f(3) = 3(n-3) \ge 2(n+c-1) - 2 = 2m-2$.

When $x \in [\frac{n}{2}, n-3]$, since $f'(x) = n-2x \le 0$, then $f(x) \ge f(n-3) = 3(n-3) \ge 2(n+c-1) - 2 = 2m-2$.

By combining the above discussion, the conclusion follows.

Corollary 2.1 *Suppose $G \in \mathcal{T}_n$. If $n \ge 11$ and $\Delta \le n-3$, then $\mu(G) \le n-1$.*

Let $H_1 - H_{27}$ be the tricyclic graphs on $n \ge 11$ vertices as shown in Fig. 2. For convenience, $V\backslash\{v\}$ is written as $V - v$.



H_1
H_2
H_3
H_4
H_5
H_6
H_7
H_8
H_9
H_{10}
H_{11}
H_{12}
H_{13}
H_{14}
H_{15}
H_{16}
H_{17}
H_{18}
H_{19}
H_{20}
H_{21}
H_{22}
H_{23}
H_{24}
H_{25}
H_{26}
H_{27}

Fig. 2.

Lemma 2.4 *Suppose $G \in \mathcal{T}_n$ and $n \geq 11$. If G has the property $n-1 < \mu(G) < n$, then G should be the graphs $H_1 - H_{27}$.*

Proof. Suppose $\Delta = n-1$, Lemma 2.3 implies that $\mu(G) = n$. Combining with Corollary 2.1, if G has the property $n-1 < \mu(G) < n$, then $\Delta = n-2$. Next we shall show that $G \cong H_i$, where $1 \leq i \leq 27$.

Assume that $u_0 \in V(G)$ such that $d(u_0) = n-2$ and $v_0 \notin N(u_0)$. Note that $N(v_0) \subseteq N(u_0)$. It is easy to see that $1 \leq d(v_0) \leq 4$ because $G \in \mathcal{T}_n$. We consider the next four cases.

Case 1. $d(v_0) = 1$. Three subcases should be considered.

Subcase 1. $max\{|N(u_0) \cap N(v)| : v \in V - u_0\} = 1$. It is easy to see that $G \cong H_{21}$ or H_{23}.

Subcase 2. $max\{|N(u) \cap N(v)| : v \in V - u_0\} = 2$. It is easy to see that $G \cong H_{15}$ or H_{16} or H_{17} or H_{19} or H_{20} or H_{22} or H_{24} or H_{25} or H_{26}.

Subcase 3. $max\{|N(u) \cap N(v)| : v \in V - u_0\} = 3$. It is easy to see that $G \cong H_{14}$ or H_{18} or H_{27}.

Case 2. $d(v_0) = 2$. Two subcase should be considered.

Subcase 1. $max\{|N(u_0) \cap N(v)| : v \in V - u_0\} = 1$. It is easy to see that $G \cong H_7$ or H_{10} or H_{11} or H_{12}.

Subcase 2. $max\{|N(u) \cap N(v)| : v \in V - u_0\} = 2$. It is easy to see that $G \cong H_5$ or H_6 or H_8 or H_9 or H_{13}.

Case 3. $d(v_0) = 3$. It is easy to see that $G \cong H_2$ or H_3 or H_4.

Case 4. $d(v_0) = 4$. It is easy to see that $G \cong H_1$.

Lemma 2.5 *Suppose $n \geq 11$, then $n-1 < \mu(H_i) < n$ holds for $1 \leq i \leq 27$.*

Proof. By Lemma 2.1, we have

(1a) $\Phi(H_1, \lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)^3(\lambda^3 - (n+5)\lambda^2 + (6n-2)\lambda - 4n)$.

(2a) $\Phi(H_2, \lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)(\lambda^5 - (n+9)\lambda^4 + (10n+23)\lambda^3 - (33n+11)\lambda^2 + (41n-10)\lambda - 16n)$.

(3a) $\Phi(H_3, \lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)^2(\lambda-3)(\lambda^3 - (n+4)\lambda^2 + (5n-2)\lambda - 3n)$.

(4a) $\Phi(H_4, \lambda) = \lambda(\lambda-1)^{n-6}(\lambda-2)(\lambda-4)(\lambda^3 - (n+4)\lambda^2 + (5n-2)\lambda - 3n)$.

(5a) $\Phi(H_5, \lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)(\lambda^5 - (n+9)\lambda^4 + (10n+22)\lambda^3 - (32n+8)\lambda^2 + (38n-12)\lambda - 14n)$.

(6a) $\Phi(H_6, \lambda) = \lambda(\lambda-1)^{n-6}(\lambda-3)(\lambda^4 - (n+7)\lambda^3 + (8n+9)\lambda^2 - (17n-7)\lambda + 8n)$.

(7a) $\Phi(H_7, \lambda) = \lambda(\lambda-1)^{n-7}(\lambda^2 - 5\lambda + 5)(\lambda^4 - (n+6)\lambda^3 + (7n+6)\lambda^2 - (13n-5)\lambda + 6n)$.

(8a) $\Phi(H_8,\lambda) = \lambda(\lambda-1)^{n-6}(\lambda^5-(n+10)\lambda^4+(11n+29)\lambda^3-(40n+16)\lambda^2+(54n-19)\lambda-21n)$.

(9a) $\Phi(H_9,\lambda) = \lambda(\lambda-1)^{n-7}(\lambda^6-(n+11)\lambda^5+(12n+41)\lambda^4-(53n+55)\lambda^3+(106n+4)\lambda^2-(94n-26)\lambda+29n)$.

(10a) $\Phi(H_{10},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-3)(\lambda^5-(n+8)\lambda^4+(9n+18)\lambda^3-(27n+6)\lambda^2+(31n-10)\lambda-11n)$.

(11a) $\Phi(H_{11},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)(\lambda-3)^2(\lambda^3-(n+3)\lambda^2+(4n-2)\lambda-2n)$.

(12a) $\Phi(H_{12},\lambda) = \lambda(\lambda-1)^{n-6}(\lambda-3)(\lambda-4)(\lambda^3-(n+3)\lambda^2+(4n-2)\lambda-2n)$.

(13a) $\Phi(H_{13},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)^2(\lambda-4)(\lambda^3-(n+3)\lambda^2+(4n-2)\lambda-2n)$.

(14a) $\Phi(H_{14},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)^2(\lambda^4-(n+7)\lambda^3+(8n+5)\lambda^2-(13n-7)\lambda+5n)$.

(15a) $\Phi(H_{15},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda^6-(n+11)\lambda^5+(12n+39)\lambda^4-(51n+45)\lambda^3+(95n-9)\lambda^2-(77n-31)\lambda+21n)$.

(16a) $\Phi(H_{16},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)(\lambda-3)(\lambda^4-(n+6)\lambda^3+(7n+4)\lambda^2-(11n-6)\lambda+4n)$.

(17a) $\Phi(H_{17},\lambda) = \lambda(\lambda-1)^{n-6}(\lambda-4)(\lambda^4-(n+6)\lambda^3+(7n+4)\lambda^2-(11n-6)\lambda+4n)$.

(18a) $\Phi(H_{18},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)(\lambda^5-(n+9)\lambda^4+(10n+21)\lambda^3-(31n+3)\lambda^2+(33n-16)\lambda-10n)$.

(19a) $\Phi(H_{19},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda^6-(n+11)\lambda^5+(12n+40)\lambda^4-(52n+48)\lambda^3+(99n-10)\lambda^2-(80n-34)\lambda+21n)$.

(20a) $\Phi(H_{20},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-3)(\lambda^5-(n+8)\lambda^4+(9n+17)\lambda^3-(26n+2)\lambda^2+(27n-13)\lambda-8n)$.

(21a) $\Phi(H_{21},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-3)^2(\lambda^4-(n+5)\lambda^3+(6n+3)\lambda^2-(9n-5)\lambda+3n)$.

(22a) $\Phi(H_{22},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)(\lambda-4)(\lambda^4-(n+5)\lambda^3+(6n+3)\lambda^2-(9n-5)\lambda+3n)$.

(23a) $\Phi(H_{23},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-3)^3(\lambda^3-(n+2)\lambda^2+(3n-2)\lambda-n)$.

(24a) $\Phi(H_{24},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-3)(\lambda^2-6\lambda+7)(\lambda^3-(n+2)\lambda^2+(3n-2)\lambda-n)$.

(25a) $\Phi(H_{25},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)(\lambda-3)(\lambda-4)(\lambda^3-(n+2)\lambda^2+(3n-2)\lambda-n)$.

(26a) $\Phi(H_{26},\lambda) = \lambda(\lambda-1)^{n-6}(\lambda-4)^2(\lambda^3-(n+2)\lambda^2+(3n-2)\lambda-n)$.

(27a) $\Phi(H_{27},\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)^2(\lambda-5)(\lambda^3-(n+2)\lambda^2+(3n-2)\lambda-n)$.

By Lemma 2.3, $n-1 < \mu(H_i) \leq n$ holds for $1 \leq i \leq 27$. Moreover, when $1 \leq i \leq 27$, since $\Phi(H_i, n) \neq 0$, thus $n-1 < \mu(H_i) < n$ follows.

Let $S(n,3)$ denote the class of graphs obtained by adding 3 edges to pendants of $K_{1,n-1}$. By combining Lemmas 2.3-2.5 and Corollary 2.1, we have

Corollary 2.2 *If $G \in \mathcal{T}_n$ and $n \geq 11$, then $\mu(G) = n$ if and only if $G \in S(n,3)$.*

Lemma 2.6 *Suppose $G \in \mathcal{T}_n$ and $n \geq 11$, then $\mu(H_1) > \mu(H_2) > \mu(H_3) = \mu(H_4) > \mu(H_5) > \mu(H_6) > \mu(H_7) > \mu(H_8) > \mu(H_9) > \mu(H_{10}) > \mu(H_{11}) = \mu(H_{12}) = \mu(H_{13}) > \mu(H_{14}) > \mu(H_{15}) > \mu(H_{16}) = \mu(H_{17}) > \mu(H_{18}) > \mu(H_{19}) > \mu(H_{20}) > \mu(H_{21}) = \mu(H_{22}) > \mu(H_{23}) = \mu(H_{24}) = \mu(H_{25}) = \mu(H_{26}) = \mu(H_{27})$.*

Proof. By Lemma 2.5, $n-1 < \mu(H_i) < n$ holds for $1 \leq i \leq 27$. Next we shall divide the proof into the next 17 process.

(1) $\mu(H_1) > \mu(H_2)$. Rewrite equality (1a) as $\Phi(H_1,\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)f_1(\lambda)$, where $f_1(\lambda) = \lambda^5-(n+9)\lambda^4+(10n+22)\lambda^3-(32n+12)\lambda^2+(40n-8)\lambda-16n$. Let $f_2(\lambda) = \lambda^5-(n+9)\lambda^4+(10n+23)\lambda^3-(33n+11)\lambda^2+(41n-10)\lambda-16n$. Thus, $\mu(H_1)$ and $\mu(H_2)$ equals the maximum root of the equation $f_1(\lambda)=0$ and $f_2(\lambda)=0$, respectively. When $\lambda \geq n-1$, since $f_2(\lambda)-f_1(\lambda) = \lambda(\lambda(\lambda+1-n)+n-2) > 0$, thus $\mu(H_1) > \mu(H_2)$.

(2) $\mu(H_2) > \mu(H_3) = \mu(H_4)$. By equalities (3a) and (4a), it follows that $\mu(H_3) = \mu(H_4)$. Rewrite equality (3a) as $\Phi(H_3,\lambda) = \lambda(\lambda-1)^{n-7}(\lambda-2)f_3(\lambda)$, where $f_3(\lambda) = \lambda^5-(n+9)\lambda^4+(10n+24)\lambda^3-(34n+14)\lambda^2+(45n-12)\lambda-18n$. Thus, $\mu(H_3)$ equals the maximum root of the equation $f_3(\lambda)=0$. Let $\psi_1(\lambda) = f_3(\lambda)-f_2(\lambda) = \lambda^3-(n+3)\lambda^2+(4n-2)\lambda-2n$, and α_1 denote the maximum root of $\psi_1(\lambda)=0$. Since $\psi_1(0) = -2n < 0$, $\psi_1(1) = n-4 > 0$, $\psi_1(n-1) = -2 < 0$ and $\psi_1(n) = n(n-4) > 0$, then $n-1 < \alpha_1 < n$.

It is easy to see that $f_2(\lambda) = (\lambda^2-6\lambda+7)\psi_1(\lambda)+\gamma_1(\lambda)$ and $f_3(\lambda) = (\lambda^2-6\lambda+8)\psi_1(\lambda)+\gamma_1(\lambda)$, where $\gamma_1(\lambda) = -2\lambda^2+(n+4)\lambda-2n$. Since $f_2(\alpha_1) = f_3(\alpha_1) = \gamma_1(\alpha_1) < \alpha_1(6-n)-2n < 0$, thus $\mu(H_2), \mu(H_3) \in (\alpha_1, n)$. Moreover, since $\lim\limits_{\lambda \to +\infty} \psi_1(\lambda) = +\infty$, thus $\psi_1(\lambda) > \psi_1(\alpha_1) = 0$ when $\lambda > \alpha_1$. This implies that $f_3(\lambda) > f_2(\lambda)$ when $\lambda > \alpha_1$. Therefore, $\mu(H_2) > \mu(H_3)$.

(3) $\mu(H_4) > \mu(H_5)$. Rewrite equality (4a) as $\Phi(H_4,\lambda) = \lambda(\lambda-1)^{n-6}f_4(\lambda)$, where $f_4(\lambda) = \lambda^5-(n+10)\lambda^4+(11n+30)\lambda^3-(41n+20)\lambda^2+(58n-16)\lambda-$

$24n$. Thus, $\mu(H_4)$ equals the maximum root of the equation $f_4(\lambda) = 0$. Let $f_5(\lambda) = \lambda^5 - (n+9)\lambda^4 + (10n+22)\lambda^3 - (32n+8)\lambda^2 + (38n-12)\lambda - 14n$. By equality (5a), $\mu(H_5)$ equals the maximum root of the equation $f_5(\lambda) = 0$. Let $\psi_2(\lambda) = f_5(\lambda) - f_4(\lambda) = \lambda^4 - (n+8)\lambda^3 + (9n+12)\lambda^2 - (20n-4)\lambda + 10n$. When $\lambda \geq n-1$, since $\psi_2''(\lambda) = 12\lambda^2 - (6n+48)\lambda + (18n+24) > 0$, thus $\psi_2'(\lambda) = 4\lambda^3 - (3n+24)\lambda^2 + (18n+24)\lambda - (20n-4) \geq \psi_2'(n-1) = n^3 - 12n^2 + 43n - 48 > 0$. Therefore, when $\lambda \geq n-1$, $\psi_2(\lambda) \geq \psi_2(n-1) = n^2 - 8n + 17 > 0$. Thus, $f_5(\lambda) > f_4(\lambda)$ when $\lambda \geq n-1$, this implies that $\mu(H_4) > \mu(H_5)$.

(4) $\mu(H_5) > \mu(H_6)$. Let $f_6(\lambda) = \lambda^4 - (n+7)\lambda^3 + (8n+9)\lambda^2 - (17n-7)\lambda + 8n$. By equality (6a), $\mu(H_6)$ equals the maximum root of the equation $f_6(\lambda) = 0$. It is easy to see that $f_5(\lambda) = f_6(\lambda)(\lambda - 2) + \gamma_2(\lambda)$, where $\gamma_2(\lambda) = -\lambda^3 + (n+3)\lambda^2 - (4n-2)\lambda + 2n$. Let α_2 denote the maximum root of $\gamma_2(\lambda) = 0$. Since $\gamma_2(0) = 2n > 0$, $\gamma_2(1) = 4 - n < 0$, $\gamma_2(n-1) = 2 > 0$, $\gamma_2(n) = n(4-n) < 0$, thus $n - 1 < \alpha_2 < n$.

Note that $f_6(\lambda) = \gamma_2(\lambda)(4-\lambda) + \lambda(n-1-\lambda)$, then $f_6(\alpha_2) = \alpha_2(n-1-\alpha_2) < 0$. Moreover since $\lim_{\lambda \longrightarrow +\infty} f_6(\lambda) = +\infty$, thus $\mu(H_6) > \alpha_2$. Note that $\lim_{\lambda \longrightarrow +\infty} \gamma_2(\lambda) = -\infty$, thus $f_5(\mu(H_6)) = \gamma_2(\mu(H_6)) < 0$. This implies that $\mu(H_5) > \mu(H_6)$ because $\lim_{\lambda \longrightarrow +\infty} f_5(\lambda) = +\infty$.

(5) $\mu(H_6) > \mu(H_7)$. Let $f_7(\lambda) = \lambda^4 - (n+6)\lambda^3 + (7n+6)\lambda^2 - (13n-5)\lambda + 6n$, by equality (7a), $\mu(H_7)$ equals the maximum root of the equation $f_7(\lambda) = 0$.

Note that $f_7(\lambda) - f_6(\lambda) = \lambda^3 - (n+3)\lambda^2 + (4n-2)\lambda - 2n = \psi_1(\lambda)$, by the proof of $\mu(H_2) > \mu(H_3)$, $n - 1 < \alpha_1 < n$, where α_1 is the maximum root of $\psi_1(\lambda) = 0$.

It is easy to see that $f_6(\lambda) = (\lambda - 4)\psi_1(\lambda) + \gamma_3(\lambda)$ and $f_7(\lambda) = (\lambda - 3)\psi_2(\lambda) + \gamma_3(\lambda)$, where $\gamma_3(\lambda) = -\lambda^2 + (n-1)\lambda$. Note that $n-1 < \alpha_1 < n$, thus $f_7(\alpha_1) = f_6(\alpha_1) = \gamma_3(\alpha_1) = \alpha_1(n-1-\alpha_1) < 0$, which implies that $\mu(H_6), \mu(H_7) \in (\alpha_1, n)$. Moreover, since $\lim_{\lambda \longrightarrow +\infty} \psi_1(\lambda) = +\infty$, thus $\psi_1(\lambda) > \psi_1(\alpha_1) = 0$ when $\lambda > \alpha_1$. This implies that $f_7(\lambda) > f_6(\lambda)$ when $\lambda > \alpha_1$. Therefore, $\mu(H_6) > \mu(H_7)$.

(6) $\mu(H_7) > \mu(H_8)$. Rewrite equality (7a) as $\Phi(H_7, \lambda) = \lambda(\lambda-1)^{n-7} f_8(\lambda)$, where $f_8(\lambda) = \lambda^6 - (n+11)\lambda^5 + (12n+41)\lambda^4 - (53n+55)\lambda^3 + (106n+5)\lambda^2 - (95n-25)\lambda + 30n$. Thus, $\mu(H_7)$ equals the maximum root of the equation $f_8(\lambda) = 0$. Rewrite equality (8a) as $\Phi(H_8, \lambda) = \lambda(\lambda-1)^{n-7} f_9(\lambda)$, where $f_9(\lambda) = \lambda^6 - (n+11)\lambda^5 + (12n+39)\lambda^4 - (51n+45)\lambda^3 + (94n-3)\lambda^2 - (75n-19)\lambda + 21n$. Thus, $\mu(H_8)$ equals the maximum root of the equation $f_9(\lambda) = 0$. Let $\psi_3(\lambda) = f_8(\lambda) - f_9(\lambda) = 2\lambda^4 - (2n+10)\lambda^3 + (12n+8)\lambda^2 - (20n-6)\lambda + 9n$, α_3 and α_4 denote the maximum and the second largest root of $\psi_3(\lambda) = 0$, respectively. Since $\psi_3(0) = 9n > 0$, $\psi_3(1) = 6 - n < 0$, $\psi_3(2) = n - 4 > 0$, $\psi_3(n-1) = 14 - 5n < 0$, and

$\psi_3(n) = 2n^2(n-6)+15n > 0$, thus $n-1 < \alpha_3 < n$, $2 < \alpha_4 < n-1$.

It is easy to see that $f_8(\lambda) = \frac{\lambda^2-6\lambda+7}{2}\psi_3(\lambda) + \gamma_4(\lambda)$ and $f_9(\lambda) = \frac{\lambda^2-6\lambda+5}{2}\psi_3(\lambda)+\gamma_4(\lambda)$, where $\gamma_4(\lambda) = \frac{2\lambda^3-(n+10)\lambda^2+(4n+8)\lambda-3n}{2}$. By straightly computation, $\frac{n+10+\sqrt{n^2-4n+52}}{6}$ is the maximum root of $\gamma_4'(\lambda) = 3\lambda^2-(n+10)\lambda+2n+4 = 0$. Note that $\frac{n+10+\sqrt{n^2-4n+52}}{6} < n-1$ and $\lim_{\lambda\longrightarrow+\infty}\gamma_4'(\lambda) = +\infty$, thus when $\lambda \geq \alpha_3$, $f_8(\lambda) \geq \gamma_4(\alpha_3) > \gamma_4(n-1) = \frac{1}{2}((n-1)(n-9)n+17n-20) > 0$. Thus, $\mu(H_7) \in (n-1,\alpha_3)$. With the same reason, we have $\mu(H_8) \in (n-1,\alpha_3)$.

Moreover, since $\alpha_4 \in (2,n-1)$, $\psi_3(n-1) < 0$, and $\lim_{\lambda\longrightarrow+\infty}\psi_3(\lambda) = +\infty$, thus $\psi_3(\lambda) < \psi_3(\alpha_3) = 0$ when $\lambda \in (n-1,\alpha_3)$. Note that $\mu(H_8) \in (n-1,\alpha_3)$, thus $f_8(\mu(H_8)) = \psi_3(\mu(H_8)) < 0$. Moreover, since $\lim_{\lambda\longrightarrow+\infty} f_8(\lambda) = +\infty$, then $\mu(H_7) > \mu(H_8)$.

(7) $\mu(H_8) > \mu(H_9)$. Let $f_{10}(\lambda) = \lambda^6-(n+11)\lambda^5+(12n+41)\lambda^4-(53n+55)\lambda^3+(106n+4)\lambda^2-(94n-26)\lambda+29n$, by equality (9a), $\mu(H_9)$ equals the maximum root of the equation $f_{10}(\lambda) = 0$. Let $\psi_4(\lambda) = f_{10}(\lambda) - f_9(\lambda) = 2\lambda^4-(2n+10)\lambda^3+(12n+7)\lambda^2-(19n-7)\lambda+8n$, α_5 denote the maximum root of $\psi_4(\lambda) = 0$. Since $\psi_4(0) = 8n > 0$, $\psi_4(1) = 6-n < 0$, $\psi_4(2) = 2n-6 > 0$, $\psi_4(n-1) = 12-4n < 0$, and $\psi_4(n) = n(2n^2-12n+15) > 0$, thus $n-1 < \alpha_5 < n$.

It is easy to see that $f_9(\lambda) = \frac{\lambda^2-6\lambda+5.5}{2}\psi_4(\lambda) + \gamma_5(\lambda)$ and $f_{10}(\lambda) = \frac{\lambda^2-6\lambda+7.5}{2}\psi_4(\lambda) + \gamma_5(\lambda)$, where $\gamma_5(\lambda) = \frac{-5\lambda^2+(5n-1)\lambda-4n}{4}$. When $\lambda \geq n-1$, since $\gamma_5'(\lambda) = \frac{-10\lambda+(5n-1)}{4} < 0$, thus $\gamma_5(\lambda) \leq \gamma_5(n-1) = -1 < 0$. Therefore, $f_9(\alpha_5) = f_{10}(\alpha_5) = \gamma_5(\alpha_5) < 0$. Thus, $\mu(H_8), \mu(H_9) \in (\alpha_5, n)$.

Moreover, since $\lim_{\lambda\longrightarrow+\infty}\psi_4(\lambda) = +\infty$, thus $\psi_4(\lambda) > \psi_4(\alpha_5) = 0$ when $\lambda > \alpha_5$. This implies that $f_{10}(\lambda) > f_9(\lambda)$ when $\lambda > \alpha_5$. Therefore, $\mu(H_8) > \mu(H_9)$.

(8) $\mu(H_9) > \mu(H_{10})$. Rewrite equality (10a) as $\Phi(H_{10},\lambda) = \lambda(\lambda-1)^{n-7}f_{11}(\lambda)$, where $f_{11}(\lambda) = \lambda^6-(n+11)\lambda^5+(12n+42)\lambda^4-(54n+60)\lambda^3+(112n+8)\lambda^2-(104n-30)\lambda+33n$. Thus, $\mu(H_{10})$ equals the maximum root of the equation $f_{11}(\lambda) = 0$. Let $\psi_5(\lambda) = f_{11}(\lambda) - f_{10}(\lambda) = \lambda^4-(n+5)\lambda^3+(6n+4)\lambda^2-(10n-4)\lambda+4n$, and α_6 denote the maximum root of $\psi_5(\lambda) = 0$. Since $\psi_5(0) = 4n > 0$, $\psi_5(1) = 4-n < 0$, $\psi_5(3) = n-6 > 0$, $\psi_5(n-1) = 6-2n < 0$ and $\psi_5(n) = n(n^2-6n+8) > 0$, then $n-1 < \alpha_6 < n$.

It is easy to see that $f_{10}(\lambda) = (\lambda^2-6\lambda+7)\psi_5(\lambda)+\gamma_6(\lambda)$ and $f_{11}(\lambda) = (\lambda^2-6\lambda+8)\psi_5(\lambda)+\gamma_6(\lambda)$, where $\gamma_6(\lambda) = n-2\lambda$. Thus, $f_{11}(\alpha_6) = f_{10}(\alpha_6) = \gamma_6(\alpha_6) = n-2\alpha_6 < 0$, which implies that $\mu(H_9), \mu(H_{10}) \in (\alpha_6, n)$. Moreover, since $\lim_{\lambda\longrightarrow+\infty}\psi_5(\lambda) = +\infty$, thus $\psi_5(\lambda) > \psi_5(\alpha_6) = 0$ when $\lambda > \alpha_6$. This implies that $f_{11}(\lambda) > f_{10}(\lambda)$ when $\lambda > \alpha_6$. Therefore, $\mu(H_9) > \mu(H_{10})$.

(9) $\mu(H_{10}) > \mu(H_{11}) = \mu(H_{12}) = \mu(H_{13})$. By equalities (11a), (12a) and (13a), it follows that $\mu(H_{11}) = \mu(H_{12}) = \mu(H_{13})$. Rewrite equality (11a) as $\Phi(H_{11}, \lambda) = \lambda(\lambda-1)^{n-7} f_{12}(\lambda)$, where $f_{12}(\lambda) = \lambda^6 - (n+11)\lambda^5 + (12n+43)\lambda^4 - (55n+65)\lambda^3 + (118n+12)\lambda^2 - (114n-36)\lambda + 36n$. Thus, $\mu(H_{11})$ equals the maximum root of the equation $f_{12}(\lambda) = 0$. Let $\psi_6(\lambda) = f_{12}(\lambda) - f_{11}(\lambda) = \lambda^4 - (n+5)\lambda^3 + (6n+4)\lambda^2 - (10n-6)\lambda + 3n$, and α_7 denote the maximum root of $\psi_6(\lambda) = 0$. Since $\psi_6(0) = 3n > 0$, $\psi_6(1) = 6 - 2n < 0$, $\psi_6(2.8) = 0.088n - 0.1344 > 0$, $\psi_6(n-1) = 4 - n < 0$ and $\psi_6(n) = n(n^2 - 6n + 9) > 0$, then $n - 1 < \alpha_7 < n$.

It is easy to see that $f_{11}(\lambda) = (\lambda^2 - 6\lambda + 8)\psi_6(\lambda) + \gamma_7(\lambda)$ and $f_{12}(\lambda) = (\lambda^2 - 6\lambda + 9)\psi_6(\lambda) + \gamma_7(\lambda)$, where $\gamma_7(\lambda) = -2\lambda^3 + (n+12)\lambda^2 - (6n+18)\lambda + 9n$. Note that the maximum root of $\gamma_7'(\lambda) = -6\lambda^2 + (2n+24)\lambda - (6n+18)$ is $\frac{n}{3} + 1$. Moreover, since $\frac{n}{3} + 1 < n - 1$ and $\lim_{\lambda \longrightarrow +\infty} \gamma_7'(\lambda) = -\infty$, thus $\gamma_7(\lambda) \leq \gamma_7(n-1) = (n-1)n(9-n) - 23n + 32 < 0$ when $\lambda \geq n-1$. Therefore, $f_{11}(\alpha_7) = f_{12}(\alpha_7) = \gamma_7(\alpha_7) < 0$. Thus, $\mu(H_{10}), \mu(H_{11}) \in (\alpha_7, n)$.

Moreover, since $\lim_{\lambda \longrightarrow +\infty} \psi_6(\lambda) = +\infty$, thus $\psi_6(\lambda) > \psi_6(\alpha_7) = 0$ when $\lambda > \alpha_7$. This implies that $f_{12}(\lambda) > f_{11}(\lambda)$ when $\lambda > \alpha_7$. Therefore, $\mu(H_{10}) > \mu(H_{11})$.

(10) $\mu(H_{13}) > \mu(H_{14})$. Let $f_{13}(\lambda) = \lambda^3 - (n+3)\lambda^2 + (4n-2)\lambda - 2n$, by equality (13a), $\mu(H_{13})$ equals the maximum root of the equation $f_{13}(\lambda) = 0$. Let $f_{14}(\lambda) = \lambda^4 - (n+7)\lambda^3 + (8n+5)\lambda^2 - (13n-7)\lambda + 5n$, by equality (14a), $\mu(H_{14})$ equals the maximum root of the equation $f_{14}(\lambda) = 0$. It is easy to see that $f_{14}(\lambda) = (\lambda - 5)f_{13}(\lambda) + \gamma_8(\lambda)$, where $\gamma_8(\lambda) = \lambda^3 - (n+8)\lambda^2 + (9n-3)\lambda - 5n$. Note that $\gamma_8'(\lambda) = 3\lambda^2 - (2n+16)\lambda + 9n - 3$, and the maximum root of $\gamma_8'(\lambda) = 0$ is $\frac{n+8+\sqrt{n^2-11n+73}}{3}$. When $\lambda > n-1$, since $\frac{n+8+\sqrt{n^2-11n+73}}{3} < n - 1$ and $\lim_{\lambda \longrightarrow +\infty} \gamma_8'(\lambda) = +\infty$, then $\gamma_8'(\lambda) > 0$. Thus, when $\lambda \geq \mu(H_{13})$, $f_{14}(\lambda) \geq \gamma_8(\mu(H_{13})) > \gamma_8(n-1) = n - 6 > 0$. This implies that $\mu(H_{13}) > \mu(H_{14})$.

(11) $\mu(H_{14}) > \mu(H_{15})$. Let $f_{15}(\lambda) = \lambda^6 - (n+11)\lambda^5 + (12n+39)\lambda^4 - (51n+45)\lambda^3 + (95n-9)\lambda^2 - (77n-31)\lambda + 21n)$, by equality (15a), $\mu(H_{15})$ equals the maximum root of the equation $f_{15}(\lambda) = 0$. It is easy to see that $f_{15}(\lambda) = (\lambda^2 - 4\lambda + 6)f_{14}(\lambda) + \gamma_9(\lambda)$, where $\gamma_9(\lambda) = 10\lambda^3 - (10n+11)\lambda^2 + (21n-11)\lambda - 9n$. By straightly computation, $\frac{10n+11+\sqrt{100n^2-410n+451}}{30}$ is the maximum root of $\gamma_9'(\lambda) = 30\lambda^2 - (20n+22)\lambda + 21n - 11 = 0$. When $\lambda > n-1$, since $\frac{10n+11+\sqrt{100n^2-410n+451}}{30} < n - 1$ and $\lim_{\lambda \longrightarrow +\infty} \gamma_9'(\lambda) = +\infty$, then $\gamma_9'(\lambda) > 0$. Thus, when $\lambda \geq \mu(H_{14})$, $f_{15}(\lambda) \geq \gamma_9(\mu(H_{14})) > \gamma_9(n-1) = n - 10 > 0$. This implies that $\mu(H_{14}) > \mu(H_{15})$.

(12) $\mu(H_{15}) > \mu(H_{16}) = \mu(H_{17})$. By equalities (16a) and (17a), it follows that $\mu(H_{16}) = \mu(H_{17})$. Rewrite equality (16a) as $\Phi(H_{16}, \lambda) = \lambda(\lambda-1)^{n-7} f_{16}(\lambda)$, where $f_{16}(\lambda) = \lambda^6 - (n+11)\lambda^5 + (12n+40)\lambda^4 - (52n+$

$50)\lambda^3+(101n-6)\lambda^2-(86n-36)\lambda+24n$. Thus, $\mu(H_{16})$ equals the maximum root of the equation $f_{16}(\lambda)=0$. Let $\psi_7(\lambda)=f_{16}(\lambda)-f_{15}(\lambda)=\lambda^4-(n+5)\lambda^3+(6n+3)\lambda^2-(9n-5)\lambda+3n$, and α_8 denote the maximum root of $\psi_7(\lambda)=0$. Since $\psi_7(0)=3n>0$, $\psi_7(1)=4-n<0$, $\psi_7(2)=n-2>0$, $\psi_7(n-1)=4-n<0$ and $\psi_7(n)=n(n^2-6n+8)>0$, then $n-1<\alpha_8<n$.

It is easy to see that $f_{15}(\lambda)=(\lambda^2-6\lambda+6)\psi_7(\lambda)+\gamma_{10}(\lambda)$ and $f_{16}(\lambda)=(\lambda^2-6\lambda+7)\psi_7(\lambda)+\gamma_{10}(\lambda)$, where $\gamma_{10}(\lambda)=-2\lambda^3+(2n+3)\lambda^2-(5n-1)\lambda+3n$. By straightly computation, $\frac{2n+3+\sqrt{4n^2-18n+15}}{6}$ is the maximum root of $\gamma_{10}'(\lambda)=-6\lambda^2+(4n+6)\lambda-(5n-1)=0$. When $\lambda\geq n-1$, since $\frac{2n+3+\sqrt{4n^2-18n+15}}{6}<n-1$ and $\lim\limits_{\lambda\longrightarrow+\infty}\gamma_{10}'(\lambda)=-\infty$, then $\gamma_{10}(\lambda)\leq\gamma_{10}(n-1)=4-n<0$. Thus, $f_{15}(\alpha_8)=f_{16}(\alpha_8)=\gamma_{10}(\alpha_8)<0$, which yields that $\mu(H_{15}),\mu(H_{16})\in(\alpha_8,n)$. Moreover, since $\lim\limits_{\lambda\longrightarrow+\infty}\psi_7(\lambda)=+\infty$, thus $\psi_7(\lambda)>\psi_7(\alpha_8)=0$ when $\lambda>\alpha_8$. This implies that $f_{16}(\lambda)>f_{15}(\lambda)$ when $\lambda>\alpha_8$. Therefore, $\mu(H_{15})>\mu(H_{16})$.

(13) $\mu(H_{17})>\mu(H_{18})$. Rewrite equality (17a) as $\Phi(H_{17},\lambda)=\lambda(\lambda-1)^{n-7}f_{17}(\lambda)$, where $f_{17}(\lambda)=\lambda^6-(n+11)\lambda^5+(12n+38)\lambda^4-(50n+38)\lambda^3+(87n-14)\lambda^2-(64n-24)\lambda+16n$. Thus, $\mu(H_{17})$ equals the maximum root of the equation $f_{17}(\lambda)=0$. Rewrite equality (18a) as $\Phi(H_{18},\lambda)=\lambda(\lambda-1)^{n-7}f_{18}(\lambda)$, where $f_{18}(\lambda)=\lambda^6-(n+11)\lambda^5+(12n+39)\lambda^4-(51n+45)\lambda^3+(95n-10)\lambda^2-(76n-32)\lambda+20n$. Thus, $\mu(H_{18})$ equals the maximum root of the equation $f_{18}(\lambda)=0$. Let $\psi_8(\lambda)=f_{18}(\lambda)-f_{17}(\lambda)=\lambda^4-(n+7)\lambda^3+(8n+4)\lambda^2-(12n-8)\lambda+4n$, then $\psi_8'(\lambda)=4\lambda^3-(3n+21)\lambda^2+(16n+8)\lambda-(12n-8)$. When $\lambda\geq n-1$, since $\psi_8''(\lambda)=12\lambda^2-(6n+42)\lambda+(16n+8)\geq(n-1)(6n-54)+(16n+8)>0$, thus $\psi_8'(\lambda)\geq\psi_8'(n-1)=(n-1)(n^2-10n)+21n-25>0$. So when $\lambda\geq n-1$, $f_{18}(\lambda)-f_{17}(\lambda)=\psi_8(\lambda)\geq\psi_8(n-1)=4>0$. Thus, $\mu(H_{17})>\mu(H_{18})$.

(14) $\mu(H_{18})>\mu(H_{19})$. Let $f_{19}(\lambda)=\lambda^6-(n+11)\lambda^5+(12n+40)\lambda^4-(52n+48)\lambda^3+(99n-10)\lambda^2-(80n-34)\lambda+21n$. By equality (19a), $\mu(H_{19})$ equals the maximum root of the equation $f_{19}(\lambda)=0$. Let $\psi_9(\lambda)=f_{19}(\lambda)-f_{18}(\lambda)=\lambda^4-(n+3)\lambda^3+4n\lambda^2-(4n-2)\lambda+n$, and α_9 denote the maximum root of $\psi_9(\lambda)=0$. Since $\psi_9(0)=n>0$, $\psi_9(\frac{1}{2})=\frac{11-2n}{16}<0$, $\psi_9(2)=n-4>0$, $\psi_9(n-1)=2-n<0$ and $\psi_9(n)=n(n^2-4n+3)>0$, then $n-1<\alpha_9<n$.

It is easy to see that $f_{18}(\lambda)=(\lambda^2-8\lambda+15)\psi_9(\lambda)+\gamma_{11}(\lambda)$ and $f_{19}(\lambda)=(\lambda^2-8\lambda+16)\psi_9(\lambda)+\gamma_{11}(\lambda)$, where $\gamma_{11}(\lambda)=-2\lambda^3+(2n+6)\lambda^2-(8n-2)\lambda+5n$. By straightly computation, $\frac{n+3+\sqrt{n^2-6n+12}}{3}$ is the maximum root of $\gamma_{11}'(\lambda)=-6\lambda^2+(4n+12)\lambda-(8n-2)=0$. When $\lambda\geq n-1$, since $\frac{n+3+\sqrt{n^2-6n+12}}{3}<n-1$ and $\lim\limits_{\lambda\longrightarrow+\infty}\gamma_{11}'(\lambda)=-\infty$, then $\gamma_{11}(\lambda)\leq\gamma_{11}(n-1)=6-n<0$. Thus, $f_{18}(\alpha_9)=f_{19}(\alpha_9)=\gamma_{11}(\alpha_9)<0$, which implies that $\mu(H_{18}),\mu(H_{19})\in(\alpha_9,n)$. Moreover, since $\lim\limits_{\lambda\longrightarrow+\infty}\psi_9(\lambda)=+\infty$, thus

$\psi_9(\lambda) > \psi_9(\alpha_9) = 0$ when $\lambda > \alpha_9$. This implies that $f_{19}(\lambda) > f_{18}(\lambda)$ when $\lambda > \alpha_9$. Therefore, $\mu(H_{18}) > \mu(H_{19})$.

(15) $\mu(H_{19}) > \mu(H_{20})$. Rewrite equality (20a) as $\Phi(H_{20}, \lambda) = \lambda(\lambda - 1)^{n-7} f_{20}(\lambda)$, where $f_{20}(\lambda) = \lambda^6 - (n+11)\lambda^5 + (12n+41)\lambda^4 - (53n+53)\lambda^3 + (105n-7)\lambda^2 - (89n-39)\lambda + 24n$. Thus, $\mu(H_{20})$ equals the maximum root of the equation $f_{20}(\lambda) = 0$. Note that $f_{20}(\lambda) - f_{19}(\lambda) = \lambda^4 - (n+5)\lambda^3 + (6n+3)\lambda^2 - (9n-5)\lambda + 3n = \psi_7(\lambda)$. By the proof of $\mu(H_{15}) > \mu(H_{16})$, we have $n - 1 < \alpha_8 < n$, where α_8 is the maximum root of $\psi_7(\lambda) = 0$.

It is easy to see that $f_{19}(\lambda) = (\lambda^2 - 6\lambda + 7)\psi_7(\lambda) + \gamma_3(\lambda)$ and $f_{20}(\lambda) = (\lambda^2 - 6\lambda + 8)\psi_7(\lambda) + \gamma_3(\lambda)$, where $\gamma_3(\lambda) = -\lambda^2 + (n-1)\lambda$. Since $f_{19}(\alpha_8) = f_{20}(\alpha_8) = \gamma_3(\alpha_8) < 0$, thus $\mu(H_{19}), \mu(H_{20}) \in (\alpha_8, n)$. Moreover, since $\lim_{\lambda \longrightarrow +\infty} \psi_7(\lambda) = +\infty$, thus $\psi_7(\lambda) > \psi_7(\alpha_8) = 0$ when $\lambda > \alpha_8$. This implies that $f_{20}(\lambda) > f_{19}(\lambda)$ when $\lambda > \alpha_8$. Therefore, $\mu(H_{19}) > \mu(H_{20})$.

(16) $\mu(H_{20}) > \mu(H_{21}) = \mu(H_{22})$. By equalities (21a) and (22a), it follows that $\mu(H_{21}) = \mu(H_{22})$. Rewrite equality (22a) as $\Phi(H_{22}, \lambda) = \lambda(\lambda-1)^{n-7} f_{21}(\lambda)$, where $f_{21}(\lambda) = \lambda^6 - (n+11)\lambda^5 + (12n+41)\lambda^4 - (53n+53)\lambda^3 + (105n-6)\lambda^2 - (90n-40)\lambda + 24n$. Thus, $\mu(H_{22})$ equals the maximum root of the equation $f_{21}(\lambda) = 0$. When $\lambda > n-1$, since $f_{21}(\lambda) - f_{20}(\lambda) = \lambda(\lambda + 1 - n) > 0$, thus $f_{21}(\lambda) > f_{20}(\lambda)$. Recall that $\mu(H_{20})$, $\mu(H_{22}) \in (n-1, n)$, then $\mu(H_{20}) > \mu(H_{22})$.

(17) $\mu(H_{22}) > \mu(H_{23}) = \mu(H_{24}) = \mu(H_{25}) = \mu(H_{26}) = \mu(H_{27})$. By equalities (23a)-(27a), it follows that $\mu(H_{23}) = \mu(H_{24}) = \mu(H_{25}) = \mu(H_{26}) = \mu(H_{27})$. Let $f_{22}(\lambda) = \lambda^4 - (n+5)\lambda^3 + (6n+3)\lambda^2 - (9n-5)\lambda + 3n$, $f_{23}(\lambda) = \lambda^3 - (n+2)\lambda^2 + (3n-2)\lambda - n$. By equalities (22a) and (23a), $\mu(H_{22})$ and $\mu(H_{23})$ equals the maximum root of the equation $f_{22}(\lambda) = 0$ and $f_{23}(\lambda) = 0$, respectively. Clearly, $f_{22}(\lambda) = (\lambda - 3)f_{23}(\lambda) + \gamma_3(\lambda)$, where $\gamma_3(\lambda) = -\lambda^2 + (n-1)\lambda$. Since $\mu(H_{23}) > n-1$, thus $f_{22}(\mu(H_{23})) = \gamma_3(\mu(H_{23})) = -\mu(H_{23})(\mu(H_{23}) + 1 - n) < 0$. Moreover, since $\lim_{\lambda \longrightarrow +\infty} f_{22}(\lambda) = +\infty$, thus $\mu(H_{22}) > \mu(H_{23})$.

By combining equalities (1a)-(27a), Lemmas 2.4-2.6 and Corollaries 2.1-2.2, we have

Theorem 2.1 *Suppose $G \in \mathcal{T}_n$ and $n \geq 11$.*

(1) *Then $\mu(G) \leq n$, the equality holds if and only if $G \in S(n, 3)$.*

(2) *If $G \notin S(n,3)$, then $\mu(G) \leq \mu(H_1)$, the equality holds if and only if $G \cong H_1$, where $\mu(H_1)$ is the maximum root of the equation $\lambda^3 - (n+5)\lambda^2 + (6n-2)\lambda - 4n = 0$.*

(3) *If $G \notin S(n,3)$ and $G \ncong H_1$, then $\mu(G) \leq \mu(H_2)$, the equality holds if and only if $G \cong H_2$, where $\mu(H_2)$ is the maximum root of the equation $\lambda^5 - (n+9)\lambda^4 + (10n+23)\lambda^3 - (33n+11)\lambda^2 + (41n-10)\lambda - 16n = 0$.*

(4) *If $G \notin S(n,3)$ and $G \ncong H_i$ $(1 \leq i \leq 2)$, then $\mu(G) \leq \mu(H_3)$,*

the equality holds if and only if $G \cong H_3$ *or* $G \cong H_4$*, where* $\mu(H_3)$ *is the maximum root of the equation* $\lambda^3 - (n+4)\lambda^2 + (5n-2)\lambda - 3n = 0$.

(5) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \le i \le 4)$*, then* $\mu(G) \le \mu(H_5)$*, the equality holds if and only if* $G \cong H_5$*, where* $\mu(H_5)$ *is the maximum root of the equation* $\lambda^5-(n+9)\lambda^4+(10n+22)\lambda^3-(32n+8)\lambda^2+(38n-12)\lambda-14n = 0$.

(6) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \le i \le 5)$*, then* $\mu(G) \le \mu(H_6)$*, the equality holds if and only if* $G \cong H_6$*, where* $\mu(H_6)$ *is the maximum root of the equation* $\lambda^4 - (n+7)\lambda^3 + (8n+9)\lambda^2 - (17n-7)\lambda + 8n = 0$.

(7) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \le i \le 6)$*, then* $\mu(G) \le \mu(H_7)$*, the equality holds if and only if* $G \cong H_7$*, where* $\mu(H_7)$ *is the maximum root of the equation* $\lambda^4 - (n+6)\lambda^3 + (7n+6)\lambda^2 - (13n-5)\lambda + 6n = 0$.

(8) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \le i \le 7)$*, then* $\mu(G) \le \mu(H_8)$*, the equality holds if and only if* $G \cong H_8$*, where* $\mu(H_8)$ *is the maximum root of the equation* $\lambda^5-(n+10)\lambda^4+(11n+29)\lambda^3-(40n+16)\lambda^2+(54n-19)\lambda-21n = 0$.

(9) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \le i \le 8)$*, then* $\mu(G) \le \mu(H_9)$*, the equality holds if and only if* $G \cong H_9$*, where* $\mu(H_9)$ *is the maximum root of the equation* $\lambda^6 - (n+11)\lambda^5 + (12n+41)\lambda^4 - (53n+55)\lambda^3 + (106n+4)\lambda^2 - (94n-26)\lambda + 29n = 0$.

(10) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \le i \le 9)$*, then* $\mu(G) \le \mu(H_{10})$*, the equality holds if and only if* $G \cong H_{10}$*, where* $\mu(H_{10})$ *is the maximum root of the equation* $\lambda^5-(n+8)\lambda^4+(9n+18)\lambda^3-(27n+6)\lambda^2+(31n-10)\lambda-11n = 0$.

(11) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \le i \le 10)$*, then* $\mu(G) \le \mu(H_{11})$*, the equality holds if and only if* $G \cong H_{11}$ *or* $G \cong H_{12}$ *or* $G \cong H_{13}$*, where* $\mu(H_{11})$ *is the maximum root of the equation* $\lambda^3 - (n+3)\lambda^2 + (4n-2)\lambda - 2n = 0$.

(12) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \le i \le 13)$*, then* $\mu(G) \le \mu(H_{14})$*, the equality holds if and only if* $G \cong H_{14}$*, where* $\mu(H_{14})$ *is the maximum root of the equation* $\lambda^4 - (n+7)\lambda^3 + (8n+5)\lambda^2 - (13n-7)\lambda + 5n = 0$.

(13) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \le i \le 14)$*, then* $\mu(G) \le \mu(H_{15})$*, the equality holds if and only if* $G \cong H_{15}$*, where* $\mu(H_{15})$ *is the maximum root of the equation* $\lambda^6 - (n+11)\lambda^5 + (12n+39)\lambda^4 - (51n+45)\lambda^3 + (95n-9)\lambda^2 - (77n-31)\lambda + 21n = 0$.

(14) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \le i \le 15)$*, then* $\mu(G) \le \mu(H_{16})$*, the equality holds if and only if* $G \cong H_{16}$ *or* $G \cong H_{17}$*, where* $\mu(H_{16})$ *is the maximum root of the equation* $\lambda^4-(n+6)\lambda^3+(7n+4)\lambda^2-(11n-6)\lambda+4n = 0$.

(15) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \le i \le 17)$*, then* $\mu(G) \le \mu(H_{18})$*, the equality holds if and only if* $G \cong H_{18}$*, where* $\mu(H_{18})$ *is the maximum root of the equation* $\lambda^5-(n+9)\lambda^4+(10n+21)\lambda^3-(31n+3)\lambda^2+(33n-16)\lambda-10n = 0$.

(16) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \le i \le 18)$*, then* $\mu(G) \le \mu(H_{19})$*, the equality holds if and only if* $G \cong H_{19}$*, where* $\mu(H_{19})$ *is the maximum root of the equation* $\lambda^6 - (n+11)\lambda^5 + (12n+40)\lambda^4 - (52n+48)\lambda^3 + (99n-10)\lambda^2 - (80n-34)\lambda + 21n = 0$.

(17) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \leq i \leq 19)$, *then* $\mu(G) \leq \mu(H_{20})$, *the equality holds if and only if* $G \cong H_{20}$, *where* $\mu(H_{20})$ *is the maximum root of the equation* $\lambda^5-(n+8)\lambda^4+(9n+17)\lambda^3-(26n+2)\lambda^2+(27n-13)\lambda-8n=0$.

(18) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \leq i \leq 20)$, *then* $\mu(G) \leq \mu(H_{21})$, *the equality holds if and only if* $G \cong H_{21}$ *or* $G \cong H_{22}$, *where* $\mu(H_{21})$ *is the maximum root of the equation* $\lambda^4-(n+5)\lambda^3+(6n+3)\lambda^2-(9n-5)\lambda+3n=0$.

(19) *If* $G \notin S(n,3)$ *and* $G \ncong H_i$ $(1 \leq i \leq 22)$, *then* $\mu(G) \leq \mu(H_{23})$, *the equality holds if and only if* $G \cong H_{23}$ *or* $G \cong H_{24}$ *or* $G \cong H_{25}$ *or* $G \cong H_{26}$ *or* $G \cong H_{27}$, *where* $\mu(H_{23})$ *is the maximum root of the equation* $\lambda^3-(n+2)\lambda^2+(3n-2)\lambda-n=0$.

Acknowledgements

The authors would like to thank the referee's many suggestions and comments, which have improved vision of the original manuscript.

References

[1] J. M. Guo, On the Laplacian spectral radius of a tree, *Lin. Algebra Appl.*, 368(2003): 379-385.

[2] A. Yu, M. Lu, F. Tian, Ordering trees by their Laplacian spectral radii, *Lin. Algebra Appl.*, 405(2005): 45-59.

[3] M. H. Liu, Bolian Liu, F. Y. Wei, The ordering of the Laplacian spectral radii of trees, *Lin. Algebra Appl.*, submitted.

[4] S. G. Guo, On the largest Laplacian eigenvalues of unicyclic graph (Chinese), *Appl. Math. J. Chinese Univ. Ser. A.*, 16(2)(2001): 131-135.

[5] Y. Liu, J. Y. Shao, X. Y. Yuan, Some results on the ordering of the Laplacian spectral radii of unicyclic graphs, *Disc. Appl. Math.*, 156(14)(2008): 2679-2697.

[6] Y. Liu, Y. Liu, Ordering of unicyclic graphs with Laplacian spectral radii, *J. Tong Ji Univ. Natu. Scie.*, 36(2008): 841-843.

[7] C. X. He, Y. Liu, J. Y. Shao, On the Laplacian spectral radii of bicyclic graphs, *Disc. Math.*, 308(24)(2008): 5981-5995.

[8] F. Y. Wei, M. H. Liu, The spectral radius of Lapacian matrix of bicyclic graphs, *Pure Appl. Math.*, preprinted (Chinese).

[9] M. H. Liu, Bolian Liu, The characteristic polynomials of a graph, prepared.

[10] R. Merris, A note on the Laplacian graph eigenvalues, *Lin. Algebra Appl.*, 285(1998): 33-35.

[11] R. Merris, Laplacian matrices of graphs: a survey, *Lin. Algebra Appl.*, 197-198(1994): 143-176.