

The Maximal Total Irregularity of Unicyclic Graphs *

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Abstract

In [1], Hosam Abdo and Darko Dimitrov introduced the total irregularity of a graph. For a graph G , it is defined as

$$\text{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V} |d_G(u) - d_G(v)|,$$

where $d_G(u)$ denotes the vertex degree of a vertex $u \in V(G)$. In this paper, we introduce two transformations to study the total irregularity of unicyclic graphs, and determine the graph with the maximal total irregularity among all unicyclic graphs with n vertices.

Keywords: total irregularity of a graph; irregularity of a graph; unicyclic graph.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph with vertex set V and edge set E . For any vertices $u, v \in V$, the degree of v is denoted by $d_G(v)$, the distance $d_G(u, v)$ is defined as the length of the shortest path between u and v in G . Let P_n , C_n and S_n be the path, cycle and star on n vertices, respectively.

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A graph is *regular* if all its vertices have the same degree, otherwise it is *irregular*. Several approaches that characterize how irregular a graph is have been proposed. In [4], Albersson defined the *imbalance* of an edge $e = uv \in E$ as $|d_G(u) - d_G(v)|$ and the irregularity of G as

$$\text{irr}(G) = \sum_{uv \in E} |d_G(u) - d_G(v)|. \quad (1)$$

More results on imbalance, the irregularity of a graph G can be found in [2]-[6], [8], [10]-[13].

Inspired by the structure and meaning of the equation (1), Abdo and Dimitov [1] introduced a new irregularity measure, called the total irregularity. For a graph G , it is defined as

$$\text{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V} |d_G(u) - d_G(v)|. \quad (2)$$

Although two irregularity measures capture the irregularity only by a single parameter, namely the degree of a vertex, the new measure is more superior than the old one in some aspects. For example, (2) has an expected property of an irregularity measure that graphs with the same degree sequences have the same total irregularity, while (1) does not have. Both measures also have common properties, including that they are zero if and only if G is regular. Obviously, $\text{irr}_t(G)$ is an upper bound of $\text{irr}(G)$. In [9], the authors derived a relation between $\text{irr}_t(G)$ and $\text{irr}(G)$ for a connected graph G with n vertices, that is,

$$\text{irr}_t(G) \leq n^2 \text{irr}(G)/4.$$

Furthermore, they showed that for any tree T

$$\text{irr}_t(T) \leq (n - 2) \cdot \text{irr}(T).$$

In [1], the authors obtained the upper bound of the total irregularity among all graphs with n vertices, and they show the star graph S_n is the tree with the maximal total irregularity among all trees with n vertices.

Theorem 1.A. ([1]) *Let G be a simple, undirected graph on n vertices. Then $\text{irr}_t(G) \leq \frac{1}{12}(2n^3 - 3n^2 - 2n + 3)$.*

Theorem 1.B. ([1]) *Let G be a tree on n vertices. Then*

$$\text{irr}_t(G) \leq (n - 1)(n - 2),$$

with equality holds if and only if $G \cong S_n$.

In this paper, we will consider the total irregularity of unicyclic graphs by introducing two transformations in Section 2, and determine the graph with the maximal total irregularity among unicyclic graphs with n vertices in Section 3.

2 Transformations

In this section, we introduce two transformations which are important to our main results.

Let n, r be integers and $3 \leq r \leq n$. Let $\mathbb{G}(n, r)$ be the set of all unicyclic graphs with n vertices in which the fundamental cycle C_r has r vertices. A rooted graph has one of its vertices, called the root, distinguished from the others. Let T_1, T_2, \dots, T_k be k rooted trees with $|V(T_i)| \geq 2$ ($1 \leq i \leq k$) and roots w_1, w_2, \dots, w_k .

Define $G(n, r, 0) = C_r$. For $1 \leq k \leq r \leq n$, define $G(n, r, k)$ be an unicyclic graph on n vertices obtained from $C_r, T_1, T_2, \dots, T_k$ by attaching k rooted trees T_1, T_2, \dots, T_k to k distinct vertices of the cycle C_r , that is, $G(n, r, k)$ is an unicyclic graph on n vertices by identifying some vertex of C_r with the root w_i of T_i for each i ($1 \leq i \leq k$).

Let $\mathbb{P}^* = \{P | P \text{ is a rooted path and the root is its starting vertex}\}$, $\mathbb{S}^* = \{S | S \text{ is a rooted star and the root is its center}\}$. For a path $P \in \mathbb{P}^*$ and a star S , the rooted graph $P + S$ is obtained by identifying the end vertex of P with the center of S and the root of $P + S$ is the root of P . Define $\mathbb{PS}^* = \{P + S | P \in \mathbb{P}^* \text{ and } S \text{ is a star}\}$.

2.1 α -transformation

Let n, r, k be integers with $r \geq 3$ and $1 \leq k \leq r \leq n$. Let $\mathbb{G}_1(n, r, k)$ be the set of $G(n, r, k)$ that obtained from C_r and rooted trees T_1, T_2, \dots, T_k by identifying the roots of $T_1, T_2, \dots, T_k, w_1, w_2, \dots, w_k$, with k distinct vertices of C_r (see Fig.1), where $T_i \in \mathbb{PS}^* \cup \mathbb{S}^*$ for any $i \in \{1, 2, \dots, k\}$.

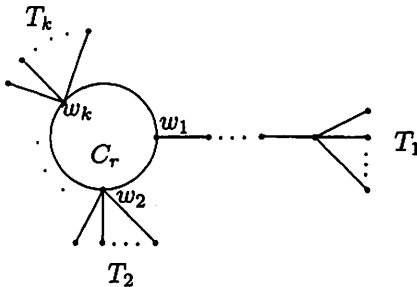


Fig.1. A graph in $\mathbb{G}_1(n, r, k)$ with rooted tree $T_1 \in \mathbb{PS}^*, T_2, \dots, T_k \in \mathbb{S}^*$

α -transformation: Let $G(n, r, k)$ be defined as above, if $G(n, r, k) \notin \mathbb{G}_1(n, r, 1)$. Without loss of generality, let $u \in V(T_1)$ be one of the maximal degree vertices of $G(n, r, k)$ and x be any pendent vertex of $G(n, r, k)$ which is adjacent to vertex y ($y \neq u$). G' is obtained from $G(n, r, k)$ by deleting the pendent edge xy and adding a pendent edge ux (see Fig.2).

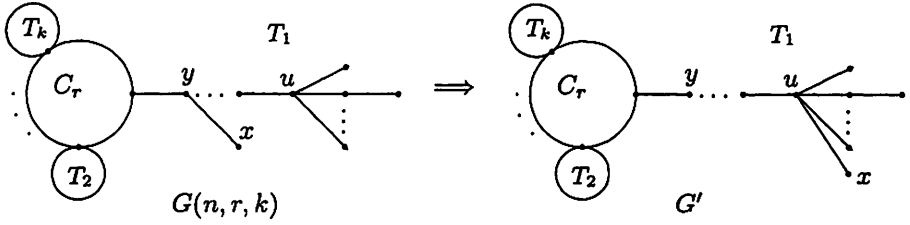


Fig.2. α -transformation

Lemma 2.1. Let G' be the graph obtained from $G(n, r, k)$ by α -transformation. Then $\text{irr}_t(G(n, r, k)) < \text{irr}_t(G')$.

Proof. For convenience, let $G = G(n, r, k)$. Note that after α -transformation, only the degrees of u and y have been changed, namely, $d_{G'}(u) = d_G(u) + 1$, $d_{G'}(y) = d_G(y) - 1$ and $d_{G'}(x) = d_G(x)$ for any $x \in V \setminus \{u, y\}$. Thus, we have

$$\begin{aligned}
 & \text{irr}_t(G') - \text{irr}_t(G) \\
 &= |d_{G'}(u) - d_{G'}(y)| + \sum_{x \in V \setminus \{u, y\}} |d_{G'}(u) - d_{G'}(x)| + \sum_{x \in V \setminus \{u, y\}} |d_{G'}(y) - d_{G'}(x)| \\
 & - (|d_G(u) - d_G(y)| + \sum_{x \in V \setminus \{u, y\}} |d_G(u) - d_G(x)| + \sum_{x \in V \setminus \{u, y\}} |d_G(y) - d_G(x)|) \\
 &= (|d_{G'}(u) - d_{G'}(y)| - |d_G(u) - d_G(y)|) \\
 & + \sum_{x \in V \setminus \{u, y\}} (|d_{G'}(u) - d_{G'}(x)| - |d_G(u) - d_G(x)|) \\
 & + \sum_{x \in V \setminus \{u, y\}} (|d_{G'}(y) - d_{G'}(x)| - |d_G(y) - d_G(x)|).
 \end{aligned}$$

Note that

$$\begin{aligned}
 & |d_{G'}(u) - d_{G'}(y)| - |d_G(u) - d_G(y)| = 2, \\
 & \sum_{x \in V \setminus \{u, y\}} (|d_{G'}(u) - d_{G'}(x)| - |d_G(u) - d_G(x)|) = n - 2, \\
 & \sum_{x \in V \setminus \{u, y\}} (|d_{G'}(y) - d_{G'}(x)| - |d_G(y) - d_G(x)|) \geq -(n - 2)
 \end{aligned}$$

because of for any integer a , $|a - 1| - |a| = \begin{cases} -1, & \text{if } a > 0, \\ 1, & \text{if } a \leq 0. \end{cases}$

Thus, we have $\text{irr}_t(G') - \text{irr}_t(G) \geq 2 + (n - 2) + (-n + 2) = 2 > 0$. \square

By the proof of Lemma 2.1 and the definition of α -transformation, we have

Lemma 2.2. Let $G(n, r, k)$ be defined as above, G_1 be the graph obtained from $G(n, r, k)$ by repeating α -transformation, and we cannot get other graph from G_1 by repeating α -transformation. Then

- (1) $G_1 \in \mathbb{G}_1(n, r, 1)$.
- (2) $\text{irr}_t(G) \leq \text{irr}_t(G_1)$, and the equality holds if and only if $G \cong G_1$.

2.2 β - transformation

Define $\mathbb{G}_{11}(n, r, 1) = \{G(n, r, 1) \in \mathbb{G}_1(n, r, 1) \mid \text{the unique rooted tree } T \text{ of } G(n, r, 1) \text{ belongs to } \mathbb{PS}^*\}$ and $\mathbb{G}_{12}(n, r, 1) = \{G(n, r, 1) \in \mathbb{G}_1(n, r, 1) \mid \text{the unique rooted tree } T \text{ of } G(n, r, 1) \text{ belongs to } \mathbb{S}^*\}$.

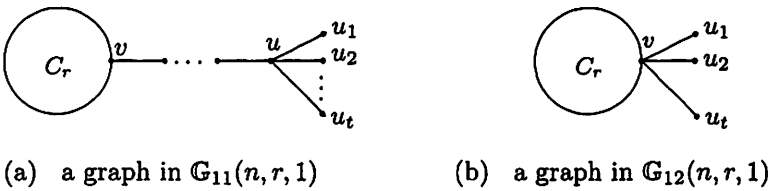


Fig.3. Two graphs in $\mathbb{G}_1(n, r, 1)$

β -transformation: Let $G(n, r, 1) \in \mathbb{G}_{11}(n, r, 1)$, v be the root of T , u be the maximal degree vertex of $G(n, r, 1)$ and u_1, u_2, \dots, u_t ($t \geq 2$) be the pendent vertices adjacent to u , where $t = \text{deg}_{G(n, r, 1)}(u) - 1$. $G'(n, r, 1)$ is obtained from $G(n, r, 1)$ by deleting the pendent edges uu_1, uu_2, \dots, uu_t and adding pendent edges vu_1, vu_2, \dots, vu_t (see Fig.4).

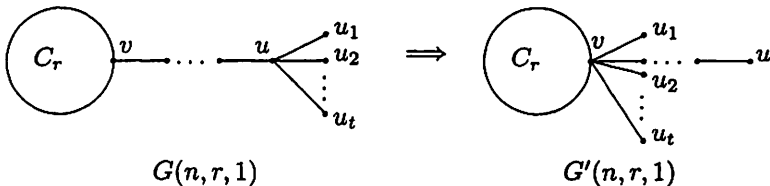


Fig.4. β -transformation

Lemma 2.3. Let $G(n, r, 1)$ be defined as above, $G'(n, r, 1)$ be the graph obtained from $G(n, r, 1)$ by β -transformation as above. Then

$$\text{irr}_t(G(n, r, 1)) < \text{irr}_t(G'(n, r, 1)).$$

Proof. For convenience, let $G = G(n, r, 1)$ and $G' = G'(n, r, 1)$. Note that only the degrees of u and v have been changed after β -transformation, namely, $d_{G'}(u) = 1$, $d_{G'}(v) = d_G(v) + d_G(u) - 1$ and $d_{G'}(x) = d_G(x)$ for any vertex $x \in V \setminus \{u, v\}$. Thus, we have

$$|d_{G'}(u) - d_{G'}(v)| - |d_G(u) - d_G(v)| = 2d_G(v) - 2, \quad (2.1)$$

$$\begin{aligned} & \sum_{x \in V \setminus \{u, v\}} |d_{G'}(u) - d_{G'}(x)| - \sum_{x \in V \setminus \{u, v\}} |d_G(u) - d_G(x)| \\ &= \sum_{x \in V \setminus \{u, v\}} [(d_G(x) - 1) - (d_G(u) - d_G(x))] \\ &= 2 \sum_{x \in V \setminus \{u, v\}} d_G(x) - (n-2)(d_G(u) + 1), \end{aligned} \quad (2.2)$$

and for any $x \in V \setminus \{u, v\}$, $d_{G'}(v) > d_G(x)$, therefore

$$\begin{aligned} & \sum_{x \in V \setminus \{u, v\}} |d_{G'}(v) - d_{G'}(x)| - \sum_{x \in V \setminus \{u, v\}} |d_G(v) - d_G(x)| \\ &= \sum_{x \in V \setminus \{u, v\}} [(d_G(v) + d_G(u) - 1 - d_G(x)) - (d_G(v) - d_G(x))] \\ &= (n-2)(d_G(u) - 1). \end{aligned} \quad (2.3)$$

From (2.1)-(2.3), we have

$$\begin{aligned} & \text{irr}_t(G') - \text{irr}_t(G) \\ &= 2d_G(v) - 2 + 2 \sum_{x \in V \setminus \{u, v\}} d_G(x) - (n-2)(d_G(u) + 1) + (n-2)(d_G(u) - 1) \\ &= 2 \left[\sum_{x \in V \setminus \{u\}} d_G(x) - (n-1) \right] \\ &> 0. \end{aligned} \quad \square$$

3 The maximal total irregularity of unicyclic graphs

In this section, the maximal total irregularity of unicyclic graphs and the extremal graph are determined.

Let $G \in \mathbb{G}_{12}(n, r, 1)$ and u be the maximal degree vertices of G . By simple calculation, we have

$$\begin{aligned} \text{irr}_t(G) &= \sum_{x \in V \setminus \{u\}} |d_G(u) - d_G(x)| \\ &+ \sum_{v \in V(C_r) \setminus \{u\}} \sum_{x \in V \setminus \{u, v\}} |d_G(v) - d_G(x)| \\ &= (n-r+2-2)(r-1) + (n-r+2-1)(n-r) + (2-1)(n-r)(r-1) \\ &= (n-r)(n+r-1). \end{aligned}$$

Theorem 3.1. *Let n, r be positive integers with $3 \leq r \leq n-1$, $G \in \mathbb{G}(n, r)$. Then $\text{irr}_t(G) \leq (n-r)(n+r-1)$, the equality holds if and only if $G \in \mathbb{G}_{12}(n, r, 1)$.*

Proof. If $G \notin \mathbb{G}_{12}(n, r, 1)$, G_1 is obtained from G by repeating α -transformation and we cannot get other graph from G_1 by repeating α -transformation, then $G_1 \in \mathbb{G}_1(n, r, 1)$ and $\text{irr}_t(G) < \text{irr}_t(G_1)$ by Lemma 2.2. Let u be the maximal degree vertices of G_1 .

Case 1: $u \in V(C_r)$.

Then $G_1 \in \mathbb{G}_{12}(n, r, 1)$, and $\text{irr}_t(G) < \text{irr}_t(G_1) = (n - r)(n + r - 1)$.

Case 2: $u \notin V(C_r)$.

Then $G_1 \in \mathbb{G}_{11}(n, r, 1)$ and $d_{G_1}(u) \geq 4$. Let G_2 be the graph obtained from G_1 by β -transformation, therefore $\text{irr}_t(G_1) < \text{irr}_t(G_2)$ by Lemma 2.3. If $d_{G_1}(v, u) = 1$, then $G_2 \in \mathbb{G}_{12}(n, r, 1)$ and $\text{irr}_t(G) < \text{irr}_t(G_2) = (n - r)(n + r - 1)$. If $d_{G_1}(v, u) > 1$, then let G_3 be the graph obtained from G_2 by repeating $(d_{G_1}(v, u) - 1)$ times α -transformations. Obviously, $G_3 \in \mathbb{G}_{12}(n, r, 1)$ and $\text{irr}_t(G_2) < \text{irr}_t(G_3)$ by Lemma 2.2. Thus

$$\text{irr}_t(G) < \text{irr}_t(G_1) < \text{irr}_t(G_2) < \text{irr}_t(G_3) = (n - r)(n + r - 1).$$

Combining the above arguments, $\text{irr}_t(G) \leq (n - r)(n + r - 1)$ for any $G \in \mathbb{G}(n, r)$, and the equality holds if and only if $G \in \mathbb{G}_{12}(n, r, 1)$. \square

Let n be a given positive integer, and $f(r) = (n - r)(n + r - 1)$, then $f'(r) = 1 - 2r < 0$ when $3 \leq r \leq n - 1$. Hence, $f(r)$ is a decreasing function when $3 \leq r \leq n - 1$. Moreover $\text{irr}_t(C_n) = 0$, thus we immediately have

Theorem 3.2. *Let G be an unicyclic graph with n vertices. Then*

$$\text{irr}_t(G) \leq n^2 - n - 6,$$

and the equality holds if and only if $G \in \mathbb{G}_{12}(n, 3, 1)$ (see Fig.5).

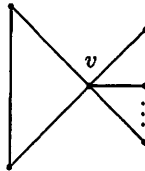


Fig.5. A graph in $\mathbb{G}_{12}(n, 3, 1)$

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References

- [1] H. Abdo, D. Dimitrov, The total irregularity of a graph, *arXiv:1207.5267v1* 24 July 2012 .

- [2] H. Abdo, N. Cohen, D. Dimitrov, Bounds and computation of irregularity of a graph, *arXiv:1207.4804v1* 19 July 2012 .
- [3] H. Abdo, N. Cohen, D. Dimitrov, The Total Irregularity of Graphs under Graph Operations, *arXiv:1304.0185v1* 31 Mar 2013 .
- [4] M. O. Albertson, The irregularity of a graph, *Ars Comb.* **46** (1997) 219–225.
- [5] Y. Alavi, A. Boals, G. Chartrand, P. Erdős, O. R. Oellermann, k -path irregular graphs, *Congr. Numer.* **65** (1988) 201–210.
- [6] Y. Alavi, G. Chartrand, F. R. K. Chung, P. Erdős, R. L. Graham, O. R. Oellermann, Highly irregular graphs, *J. Graph Theory* **11** (1987) 235–249.
- [7] J. A. Bondy, U. S. R. Murty, Graph Theory with Applications, MacMillan, London, 1976.
- [8] G. Chartrand, P. Erdős, O. R. Oellermann, How to define an irregular graph, *Coll. Math. J.* **19** (1988) 36–42.
- [9] D. Dimitrov, R. Škrekovski, Comparing the irregularity and the total irregularity of graphs, *will appear in Ars Mathematica Contemporanea*.
- [10] M. A. Henning, D. Rautenbach, On the irregularity of bipartite graphs, *Discrete Math.* **307** (2007) 1467–1472.
- [11] P. Hansen, H. Mélot, Variable neighborhood search for extremal graphs. 9. Bounding the irregularity of a graph, *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.* **69** (2005) 253–264.
- [12] D. Rautenbach, Propagation of mean degrees, *Electr. J. Comb.* **11** (2004) N11.
- [13] D. Rautenbach, L. Volkmann, How local irregularity gets global in a graph, *J. Graph Theory* **41** (2002) 18–23.