On the Dynamic Coloring of Cartesian Product Graphs *†

S. Akbari[b,a], M. Ghanbari[a], and S. Jahanbekam[a]

[a] *Department of Mathematical Sciences, Sharif University of Technology*

[b] *School of Mathematics, Institute for Research in Fundamental Sciences (IPM)*

s_akbari@sharif.edu marghanbari@gmail.com sowkam305@gmail.com

Abstract

Let G and H be two graphs. A proper vertex coloring of G is called a dynamic coloring, if for every vertex v with degree at least 2, the neighbors of v receive at least two different colors. The smallest integer k such that G has a dynamic coloring with k colors denoted by $\chi_2(G)$. We denote the cartesian product of G and H by $G\Box H$. In this paper, we prove that if G and H are two graphs and $\delta(G) \geq 2$, then $\chi_2(G\Box H) \leq \max(\chi_2(G), \chi(H))$. We show that for every two natural numbers m and n, $m, n \geq 2$, $\chi_2(P_m\Box P_n) = 4$. Also, among other results it is shown that if $3|mn$, then $\chi_2(C_m\Box C_n) = 3$ and otherwise $\chi_2(C_m\Box C_n) = 4$.

1. Introduction

Let G be a graph. We denote the edge set and the vertex set of G, by

* *Key Words*: Dynamic coloring, Cartesian product of graphs.

† 2000 *Mathematics Subject Classification*: 05C15, 05C38.

$E(G)$ and $V(G)$, respectively. The number of vertices of G is called the *order* of G. A *proper vertex coloring* of G is a function $c : V(G) \longrightarrow L$, with this property: if $u, v \in V(G)$ are adjacent, then $c(u)$ and $c(v)$ are different. A *vertex k-coloring* is a proper vertex coloring with $|L| = k$. The smallest integer k such that G has a vertex k-coloring is called the *chromatic number* of G and denoted by $\chi(G)$. A proper vertex k-coloring of a graph G is called *dynamic* if for every vertex v with degree at least 2, the neighbors of v receive at least two different colors. The smallest integer k such that G has a dynamic k-coloring is called the *dynamic chromatic number* of G and denoted by $\chi_2(G)$. Recently, the dynamic coloring of graphs has been studied by several authors, see [1], [2], [3]. For any $v \in V(G)$, $N_G(v)$ denotes the neighbor set of v in G. Let c be a proper vertex coloring of G. For any $v \in V(G)$, we mean $c(N_G(v))$ the set of all colors appearing in the neighbors of v in G. In this article, P_n and C_n denote the path and cycle of order n, respectively. In the proof of our results we need the following lemma.

Lemma 1. [4, p.5] *Let $n \geq 3$ be a natural number. Then we have,*

(*i*) $\chi_2(P_n) = 3$

$$(ii)\ \chi_2(C_n) = \begin{cases} 3 & 3 \mid n \\ 4 & 3 \nmid n,\ n \neq 5 \\ 5 & n = 5 \end{cases}$$

Let G and H be two graphs. We recall that the *cartesian product* of G and H, $G\Box H$, is a graph with the vertex set $V(G) \times V(H)$ such that two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. Clearly, $\Delta(G\Box H) = \Delta(G) + \Delta(H)$. For any $(u, v) \in V(G\Box H)$, $N_{G\Box H}((u, v))$ denotes the neighbor set of (u, v) in $G\Box H$.

In the next theorem, we provide an upper bound for the dynamic chromatic number of cartesian product of two graphs.

Theorem 1. *Let G and H be two graphs. If $\delta(G) \geq 2$, then* $\chi_2(G\Box H) \leq$

$\max(\chi_2(G), \chi(H))$.

Proof. Suppose that there are dynamic coloring $c_1 : V(G) \longrightarrow \{1, \ldots, \chi_2(G)\}$ and the vertex coloring $c_2 : V(H) \longrightarrow \{1, \ldots, \chi(H)\}$. Assume that $k = \max(\chi_2(G), \chi(H))$. For every $u \in V(G)$ and $v \in V(H)$, define a vertex coloring $c : V(G \Box H) \longrightarrow \{1, \ldots, k\}$, $c((u,v)) \equiv c_1(u) + c_2(v) \ (mod\ k)$. Now, we claim that c is a dynamic coloring of $G \Box H$. Clearly, c is a proper coloring. Moreover, for every vertex $u \in V(G)$, $|c_1(N_G(u))| \geq 2$. Thus for every vertex $(u,v) \in V(G \Box H)$, $|c(N_{G \Box H}((u,v)))| \geq 2$ and the proof is complete. □

Theorem 2. *For every two natural numbers m and n, $m, n \geq 2$, we have $\chi_2(P_m \Box P_n) = 4$.*

Proof. Let $V(P_m) = \{u_1, \ldots, u_m\}$, $V(P_n) = \{v_1, \ldots, v_n\}$ and $G = P_m \Box P_n$. First note that since $\Delta(G) \geq 2$, $\chi_2(G) \geq 3$. We claim that $\chi_2(G) \geq 4$. To the contrary, assume that $\chi_2(G) = 3$. Consider a dynamic 3-coloring c of G. With no loss of generality we can assume that $c((u_1, v_1)) = 1$ and $c((u_2, v_1)) = 2$. Also, since $N_G((u_1, v_1)) = \{(u_1, v_2), (u_2, v_1)\}$ and c is a dynamic coloring of G, $c((u_1, v_2)) = 3$. Now, $\{2, 3\} \subseteq c(N_G((u_2, v_2)))$ and so $c((u_2, v_2)) = 1$. Also, since $N_G((u_2, v_1)) = \{(u_1, v_1), (u_2, v_2), (u_3, v_1)\}$ and the dynamic property holds for (u_2, v_1), $c((u_3, v_1)) = 3$. Now, $\{1, 3\} \subseteq c(N_G((u_3, v_2)))$ and so $c((u_3, v_2)) = 2$. By repeating this procedure, we conclude that the colors of the vertices $(u_1, v_1), \ldots, (u_m, v_1)$ are $1, 2, 3, 1, 2, 3, \ldots$, and the colors of the vertices $(u_1, v_2), \ldots, (u_m, v_2)$ are $3, 1, 2, 3, 1, 2, \ldots$, respectively. Since $N_G((u_m, v_1)) = \{(u_{m-1}, v_1), (u_m, v_2)\}$ and also $c(u_{m-1}, v_1) = c(u_m, v_2)$ we have $|c(N_G((u_m, v_1)))| = 1$, a contradiction. So $\chi_2(G) \geq 4$. Now, we claim that the function $c : V(G) \longrightarrow \{1, 2, 3, 4\}$, $c((u_i, v_j)) \equiv i + 2j \ (mod\ 4)$ is a dynamic 4-coloring of G. Since a pair of adjacent vertices is as (u_i, v_j) and (u_{i+1}, v_j) or (u_i, v_j) and (u_i, v_{j+1}) for some i, j, c is a proper coloring of G. In order to see that c is a dynamic coloring, it suffices to show that in the vertices of each subgraph isomorphic to C_4 of G, four different colors are appeared. Clearly, the vertices of each subgraph isomorphic to C_4 of G, are $(u_i, v_j), (u_i, v_{j+1}), (u_{i+1}, v_{j+1})$ and (u_{i+1}, v_j), for some i, j. We have $c((u_i, v_j)) \equiv i + 2j$, $c((u_i, v_{j+1})) \equiv i + 2j + 2$, $c((u_{i+1}, v_j)) \equiv i + 2j + 1$ and $c((u_{i+1}, v_{j+1})) \equiv i + 2j + 3$, *mod* 4. Obviously, these four colors are different and so c is a dynamic 4-coloring of G and the claim is proved. Thus for every two natural numbers m and n, $m, n \geq 2$, $\chi_2(P_m \square P_n) = 4$. □

In the following theorem, we obtain the dynamic chromatic number of the cartesian product of C_m and P_n.

Theorem 3. *For every two natural numbers m and n ($m \geq 3$),*

$$\chi_2(C_m \square P_n) = \begin{cases} \chi_2(C_m) & n = 1 \\ 3 & 3 \mid m \\ 4 & otherwise \end{cases}$$

Proof. Let $V(C_m) = \{u_1, \ldots, u_m\}$, $V(P_n) = \{v_1, \ldots, v_n\}$ and $G = C_m \square P_n$. If $n = 1$, then $G \simeq C_m$ and the assertion is trivial. So we can assume that $n \neq 1$. Since $\Delta(G) \geq 2$, $\chi_2(G) \geq 3$. If $3|m$, then by Lemma 1 and Theorem 1, we conclude that in this case, $\chi_2(G) = 3$. Now, suppose that $3 \nmid m$ and $m \neq 5$. By Theorem 1, $\chi_2(G) \leq 4$. We claim that in this case, $\chi_2(G) = 4$. To the contrary, assume that $\chi_2(G) = 3$. Consider a dynamic 3-coloring c of G. Since $3 \nmid m$, by Lemma 1, $\chi_2(C_m) \geq 4$. Thus, there exists a vertex in the first copy of C_m in G, say (u_1, v_1), for which the dynamic property does not hold. With no loss of generality assume that $c((u_1, v_1)) = 1$ and $c((u_2, v_1)) = c((u_m, v_1)) = 2$. Since the dynamic property holds for (u_1, v_1) in G, $c((u_1, v_2)) = 3$. Also, since $\{(u_2, v_1), (u_1, v_2)\} \subseteq N_G((u_2, v_2))$ and $\{(u_m, v_1), (u_1, v_2)\} \subseteq N_G((u_m, v_2))$, $c((u_2, v_2)) = c((u_m, v_2)) = 1$. Moreover, since c is a dynamic coloring of G, $c((u_1, v_3)) = 2$. By repeating this procedure, we conclude that $|c(N_G((u_1, v_n)))| = 1$, a contradiction. So, in this case $\chi_2(G) = 4$. Now, suppose that $m = 5$. Since $n \neq 1$, then for every odd number j, $1 \leq j \leq n$, define $c((u_1, v_j)) = 1$, $c((u_2, v_j)) =$

2, $c((u_3,v_j)) = 3$, $c((u_4,v_j)) = 4$, $c((u_5,v_j)) = 2$ and for every even number j, $1 \leq j \leq n$, define $c((u_1,v_j)) = 3$, $c((u_2,v_j)) = 1$, $c((u_3,v_j)) = 2$, $c((u_4,v_j)) = 1$, $c((u_5,v_j)) = 4$. Clearly, this provides a dynamic 4-coloring of $C_5 \Box P_n$ and so $\chi_2(C_5 \Box P_n) \leq 4$. By a similar argument, as we did before, we have $\chi_2(C_5 \Box P_n) \geq 4$. Hence, $\chi_2(C_5 \Box P_n) = 4$ and the proof is complete. □

Theorem 4. *Let G be a graph and $m \geq 3$ be a natural number. Then the following hold:*

(i) If $3 \mid m$, then $\chi_2(C_m \Box G) = \max\{3, \chi(G)\}$.

(ii) If $3 \nmid m$ and $\chi_2(G) = 3$, then $\chi_2(C_m \Box G) = \begin{cases} 3 & \delta(G) \geq 2 \\ 4 & \delta(G) = 1 \end{cases}$

(iii) If $3 \nmid m$ and $\chi_2(G) > 3$, then $\chi_2(C_m \Box G) \geq 4$. Moreover, if G is a bipartite graph with no isolated vertex, then $\chi_2(C_m \Box G) = 4$.

Proof. Let $V(C_m) = \{u_1, \dots, u_m\}$, $V(G) = \{v_1, \dots, v_n\}$ and $H = C_m \Box G$. For every i, $1 \leq i \leq m$, call the *i-th* copy of G in H, by G_i.

(*i*) Note that by Theorem 1, $\chi_2(H) \leq \max(3, \chi(G))$. Moreover, since $\Delta(H) \geq 2$ and G is a subgraph of H, $\chi_2(H) \geq \max(3, \chi(G))$. So $\chi_2(H) = \max(3, \chi(G))$.

(*ii*) If $\delta(G) \geq 2$, then using Theorem 1, $\chi_2(H) = 3$. Now, assume that $\delta(G) = 1$. First we prove that $\chi_2(H) \leq 4$. If $m \neq 5$, then by Theorem 1, $\chi_2(H) \leq 4$. Now, suppose that $m = 5$. We can assume that G is a connected graph. Let $c_1 : V(G) \longrightarrow \{1, 2, 3\}$ be a dynamic 3-coloring of G. For every vertex (u_i, v_j), $1 \leq i \leq 5$ and $1 \leq j \leq n$, define the vertex 3-coloring c of H as follows:
$c((u_i, v_j)) = c_1(v_j) + i \ (mod\ 3)$. Since c_1 is a dynamic coloring of G, for every vertex (u, v) in H with $d_G(v) \geq 2$, the dynamic property holds for this vertex in H. Also, clearly for every $2 \leq i \leq 4$ and $1 \leq j \leq n$, $|c(N_H((u_i, v_j)))| \geq 2$. Now, for every j, $1 \leq j \leq n$, if $d_G(v_j) = 1$, then we change the colors of vertices (u_2, v_j) and (u_4, v_j) to 4. Since G has

no two adjacent vertices of degree one, the new coloring is still a proper coloring, moreover the dynamic property holds for every vertex of H and so $\chi_2(H) \leq 4$. Now, it suffices to prove that $\chi_2(H) \geq 4$. To the contrary, suppose that c is a dynamic 3-coloring of H with colors $\{1, 2, 3\}$. With no loss of generality let $v_1 \in V(G)$ be a vertex of G such that $N_G(v_1) = \{v_2\}$, $c((u_1, v_1)) = 1$ and $c((u_1, v_2)) = 2$. Since the dynamic property holds for (u_1, v_1) in H, with no loss of generality we may assume that $c((u_2, v_1)) = 3$. Now, $\{2, 3\} \subseteq c(N_H((u_2, v_2)))$ and so $c((u_2, v_2)) = 1$. Similarly, since the dynamic property holds for (u_2, v_1) in H, $c((u_3, v_1)) = 2$. Now, $\{1, 2\} \subseteq c(N_H((u_3, v_2)))$ and so $c((u_3, v_2)) = 3$. By repeating this procedure, we conclude that $c((u_4, v_1)) = 1, c((u_5, v_1)) = 3, c((u_6, v_1)) = 2, \ldots$ Now, if $c((u_m, v_1)) = 3$, then $c(N_H((u_m, v_1))) = \{1\}$, a contradiction. Thus, $c((u_m, v_1)) = 2$. This implies that $3 \mid m$, a contradiction. Thus, $\chi_2(H) = 4$.

(*iii*) To the contrary, suppose that c is a dynamic 3-coloring of H with colors $\{1, 2, 3\}$. Note that $\chi_2(G) > 3$ and so there exists a vertex, say (u_1, v_1), such that $c((u_1, v_1)) = 1$ and for every $v_i \in N_G(v_1)$, $c((u_1, v_i)) = 2$. Since the dynamic property holds for (u_1, v_1) in H, with no loss of generality we may assume that $c((u_2, v_1)) = 3$. Hence for every $v_i \in N_G(v_1)$, $c((u_2, v_i)) = 1$. Thus $c((u_3, v_1)) = 2$. By repeating this procedure, we conclude that $c((u_4, v_1)) = 1, c((u_5, v_1)) = 3, c((u_6, v_1)) = 2, \ldots$ Now, if $c((u_m, v_1)) = 3$, then $c(N_H((u_m, v_1))) = \{1\}$, a contradiction. Thus, $c((u_m, v_1)) = 2$. This implies that $3 \mid m$, a contradiction. Thus, $\chi_2(H) \geq 4$.

Now, assume that $G = (X, Y)$ is a bipartite graph such that $X = \{x_1, \ldots, x_s\}$ and $Y = \{y_1, \ldots, y_t\}$. If $m = 5$, then consider two vertex 4-colorings c and c' of C_5, $c(u_1) = 1$, $c(u_2) = 2$, $c(u_3) = 3$, $c(u_4) = 4$, $c(u_5) = 2$ and $c'(u_1) = 3$, $c'(u_2) = 4$, $c'(u_3) = 1$, $c'(u_4) = 2$, $c'(u_5) = 1$. Now, define the dynamic 4-coloring c'' of H as follows:
For $1 \leq i \leq 5$ and $1 \leq j \leq s$, let $c''((u_i, x_j)) = c(u_i)$ and for $1 \leq i \leq 5$ and $1 \leq k \leq t$, let $c''((u_i, y_k)) = c'(u_i)$. This shows that in this case $\chi_2(H) \leq 4$ and so $\chi_2(H) = 4$. Now, suppose that $m \neq 5$. Since $3 \nmid m$, then $\chi_2(C_m) = 4$. Consider a dynamic 4-coloring c' of C_m. Then for every vertex (u_i, x_j), $1 \leq i \leq m$ and $1 \leq j \leq s$, define $c((u_i, x_j)) = c'(u_i)$ and also for

every vertex (u_i, y_k), $1 \leq i \leq m$ and $1 \leq k \leq t$, define $c((u_i, y_k)) \equiv c'(u_i)+1$ $(mod\ 4)$. Clearly, c is a dynamic 4-coloring of H. Thus, we conclude that $\chi_2(H) \leq 4$. So, $\chi_2(H) = 4$. □

Theorem 5. *Let $m, n \geq 3$ be two natural numbers. Then*

$$\chi_2(C_m \square C_n) = \begin{cases} 3 & if\ 3 \mid mn \\ 4 & if\ 3 \nmid mn \end{cases}$$

Proof. Let $V(C_m) = \{u_1, \ldots, u_m\}$, $V(C_n) = \{v_1, \ldots, v_n\}$ and $G = C_m \square C_n$. Since $\Delta(G) \geq 2$, $\chi_2(G) \geq 3$. First suppose that $3 \mid mn$. By Theorem 1, $\chi_2(G) = 3$. Now, suppose that $3 \nmid mn$. By Lemma 1 and Theorem 4, Part *(iii)*, $\chi_2(G) \geq 4$. If one of the m and n is not 5, then by Theorem 1, $\chi_2(G) \leq 4$ and we are done. So, suppose that $m = n = 5$. Now, we define the dynamic 4-coloring c of $C_5 \square C_5$ as follows:
Consider the following 5×5 matrix A, $A = [a_{ij}]$ and define $c((u_i, v_j)) = a_{ij}$, for every i and j, $1 \leq i, j \leq 5$.

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & 3 \\ 2 & 3 & 2 & 3 & 1 \\ 3 & 1 & 3 & 1 & 2 \\ 2 & 4 & 2 & 4 & 1 \\ 4 & 1 & 4 & 1 & 2 \end{pmatrix}$$

So $\chi_2(C_5 \square C_5) \leq 4$. Thus, $\chi_2(C_5 \square C_5) = 4$ and the proof is complete.□

Acknowledgements. The research of the first author was in part supported by a grant (No. 87050211) from IPM.

References

[1] S. Akbari, M. Ghanbari, S. Jahanbekam, On the dynamic coloring of strongly regular graphs, Ars Combin., to appear.

[2] H.-J. Lai, J. Lin, B. Montgomery, T. Shui, S. Fan, Conditional colorings of graphs, Discrete Math. 306 (2006), 1997-2004.

[3] H.-J. Lai, B. Montgomery and H. Poon, Upper bounds of dynamic chromatic number, Ars Combin. 68 (2003), 193-201.

[4] B. Montgomery, Dynamic Coloring of Graphs, Ph.D Dissertation, Morgantown, West Virginia, 2001.