PEBBLING NUMBER OF THE GRAPH D_{n,C_m}

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Abstract. Given a distribution D of pebbles on the vertices of a graph G, a pebbling move consists of taking two pebbles off from a given vertex and placing one of them onto an adjacent vertex (the other one is discarded). The pebbling number of a graph, denoted by $f(G)$, is the minimal integer k such that any distribution of k pebbles on G allows one pebble to be moved to any specified vertex by a sequence of pebbling moves. In this paper, we calculate the pebbling number of the graph D_{n,C_m}, and consider the relationship of pebbling number between the graph D_{n,C_m} and the subgraph of D_{n,C_m}.

Keywords: pebbling number, $t-$pebbling number, graph D_{n,C_m}, pigeonhole principle.

1. Introduction

Graph pebbling is a mathematical game and area of interest played on a graph with pebbles on the vertices. The game of pebbling was first suggested by Lagarias and Saks, as a tool for solving a particular problem in number theory. The pebbling number of a graph was first introduced into the literature by Chung [1]. A pebbling move consists of removing two pebbles from one vertex, throwing one away, and putting the other pebble on an adjacent vertex. The pebbling number of a specified vertex v in a graph G is the smallest number $f(G,v)$ with the property that from any distribution of $f(G,v)$ pebbles on G, it is possible to move a pebble to v by a sequence of pebbling moves. The pebbling number of a graph G, denoted by $f(G)$, is the maximum of $f(G,v)$ over all the vertices of graph G. The $t-$pebbling number of a connected graph G, denoted by $f_t(G)$, is the smallest positive integer such that no matter how $f_t(G)$ pebbles are placed on the vertices of G, t pebbles can be moved to any vertex by a sequence of pebbling moves.

There are some basic results regarding $f(G)$ (see[2,3,5,8]). If at most one pebble is placed on each vertex other than the vertex v, then no pebble can be moved to v. Moreover, if the vertex u is at a distance d from the target vertex v, and only 2^d-1 pebbles are placed on u, then no pebble can be moved to v. Obviously, $f(G) \geq max\{|V(G)|, 2^d\}$, where $|V(G)|$ is the number of vertices of G, and d is the diameter of G (see [6]). Furthermore, the pebbling numbers of some graphs are given as follows.

Theorem 1.1. [1] *Let P_n be a path, then $f(P_n) = 2^{n-1}$.*

Theorem 1.2. [4] *Let C_n denote a simple cycle with n vertices, where $n \geq 3$, then*
(i)$f(C_{2m}) = 2^m$. (ii)$f(C_{2m+1}) = 2\lfloor \frac{2^{m+1}}{3} \rfloor + 1 = \frac{2^{m+2}-(-1)^m}{3}$.

Theorem 1.3. [7] *Let C_n denote a simple cycle with n vertices, where $n \geq 3$, then*
(i)$f_t(C_{2m}) = t \cdot 2^m$.
(ii)$f_t(C_{2m+1}) = 2\lfloor \frac{2^{m+1}}{3} \rfloor + 1 + 2^m(t-1) = \frac{2^{m+2}-(-1)^m}{3} + 2^m(t-1)$.

In this paper, G denotes a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $p(G)$ be the number of pebbles on a graph G and $p(v)$ be the number of pebbles on a vertex v. For u, $v \in V(G)$, the distance between u and v in G denoted by $d(u, v)$. As shown in Fig.1, the graph D_{n,C_m} consists of n cycles with one common vertex, which denoted by u, and each cycle has m vertices besides the center point u.

This paper is organized as follows. In Section 2 and 3, we start with showing some preliminary lemmas and theorems relies on the pigeonhole principle, and then, we calculate the pebbling number of the graph D_{n,C_m} by considering the parity of m. Finally, we mention possibilities for further research, in Section 4.

2. THE PEBBLING NUMBER OF $D_{n,C_{2m}}$

This section studies the pebbling number of $D_{n,C_{2m}}$. First we introduce the following lemmas, which is necessary for the proof of the main theorems.

Lemma 2.1. *Let x_i be the number of objects in the ith box, and let $(p-1)n + pa$ objects be in n boxes, where p, n are positive integers, then*
(i) $\sum_{i=1}^{n} \left\lfloor \frac{x_i}{p} \right\rfloor \geq a$
(ii) If there exists $i_0 \in \{1, 2, \cdots, n\}$ with $x_{i_0} \neq pt + (p-1)$, where t is a non-negative integer, then $\sum_{i=1}^{n} \left\lfloor \frac{x_i}{p} \right\rfloor \geq a+1$.

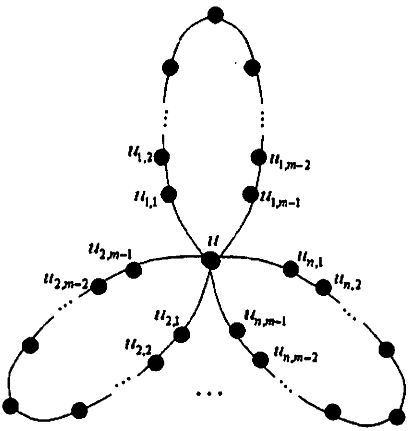

FIGURE 1. The graph D_{n,C_m}.

Proof. The first part of Lemma 2.1 and the cases $a = 1$ is easy to be proved, so we will prove the second part of Lemma 2.1 by discussing the range x_i of when $a > 1$.

If there exists $i_1 \in \{1, 2, \cdots, n\}$ with $x_{i_1} \neq ap + p$, then $\sum_{i=1}^{n} \left\lfloor \frac{x_i}{p} \right\rfloor \geq \left\lfloor \frac{x_{i_1}}{p} \right\rfloor \geq a + 1$. Otherwise, for every $i \in \{1, 2, \cdots, n\}$ with $0 \leq x_i \leq ap + (p-1)$, all cases are as follows:

Case 1: If there exists $i_2 \in \{1, 2, \cdots, n\}$ with $x_{i_2} = ap + (p-1)$, then $(n-1)(p-1)$ objects are demanded to be put into the rest $(n-1)$ boxes. Thus, not every box of these $(n-1)$ boxes has $(p-1)$ objects inside, since there exists $i_0 \in \{1, 2, \cdots, n\}$ with $x_{i_0} \neq pt + (p-1)$. According to the pigeonhole principle, there exists $i'_2 \in \{1, 2, \cdots, n\} - \{i_2\}$ with $x_{i'_2} \geq p$. Thus, $\sum_{i=1}^{n} \left\lfloor \frac{x_i}{p} \right\rfloor \geq \left\lfloor \frac{x_{i_2}}{p} \right\rfloor + \left\lfloor \frac{x_{i'_2}}{p} \right\rfloor \geq a + 1$.

Case 2: If there exists $i_3 \in \{1, 2, \cdots, n\}$ with $ap \leq x_{i_3} \leq ap + (p-1) - 1$, such that at least $(p-1)(n-1) + 1$ objects need to be put into the rest $(n-1)$ boxes. According to the pigeonhole principle, there exists $i'_3 \in \{1, 2, \cdots, n\} - i_3$ with $x_{i'_3} \geq p$, then $\sum_{i=1}^{n} \left\lfloor \frac{x_i}{p} \right\rfloor \geq \left\lfloor \frac{x_{i_3}}{p} \right\rfloor + \left\lfloor \frac{x_{i'_3}}{p} \right\rfloor \geq a + 1$.

Case 3: If there exists $i_4 \in \{1, 2, \cdots, n\}$ with $(a-1)p \leq x_{i_4} \leq (a-1)p + (p-1)$, then at least $n(p-1) + 1 = (p-1)(n-1) + p$ objects need to be

put into the rest $(n-1)$ boxes. As it comes back to the Case $a=1$, we get $\sum_{i=1}^{n}\left\lfloor\frac{x_i}{p}\right\rfloor \geq \left\lfloor\frac{x_{i_4}}{p}\right\rfloor + \sum_{i\neq i_4}\left\lfloor\frac{x_i}{p}\right\rfloor \geq a-1+2=a+1$.
Case 4: If there exists $i_k \in \{1,2,\cdots,n\}$ with $p \leq x_{i_k} \leq p+(p-1)$, then at least $(p-1)(n-1)+p(a-1)$ objects need to be put into the rest $(n-1)$ boxes, then $\sum_{i\neq i_k}^{n}\left\lfloor\frac{x_i}{p}\right\rfloor \geq (a-1)+1=a$. The case $0 \leq x_{i_{k+1}} \leq p-1$ is easy to be proved. □

For the graph D_{n,C_m}, each cycle can be regarded as a box and pebbles can be regarded as objects. Thus Lemma 2.2 is given as an inference of Lemma 2.1.

Lemma 2.2. *Let f be the pebbling number of C_m and place $(f-1)n+k$ pebbles on n cycles of D_{n,C_m} arbitrarily, then at least $\left\lfloor\frac{f-1+k}{f}\right\rfloor$ pebbles can be moved to the center point u, where k and n are positive integers.*

Proof. Lemma 2.2 is equivalent to: Let $(f-1)n+k$ objects be in n boxes, then $\sum_{i=1}^{n}\left\lfloor\frac{x_i}{f}\right\rfloor \geq \left\lfloor\frac{f-1+k}{f}\right\rfloor$, where x_i is the number of objects in the *ith* box. The mathematical induction is used to prove the Lemma 2.2.
Base case: If $k=1$, according to the pigeonhole principle, there exists a box having at least f objects. It is easy to get $\sum_{i=1}^{n}\left\lfloor\frac{x_i}{f}\right\rfloor \geq \left\lfloor\frac{f-1+k}{f}\right\rfloor$. Hence Lemma 2.2 is proven when $k=1$.
Inductive case: Assume that Lemma 2.2 is true for k.

Subcase 1: When k is not a multiple of f, denoted by $k \neq af$, then $\left\lfloor\frac{f+k}{f}\right\rfloor = \left\lfloor\frac{f-1+k}{f}\right\rfloor$. Let x_i' be the number of objects in the *ith* box for the case $(k+1)$. Thus $\sum_{i=1}^{n}\left\lfloor\frac{x_i'}{f}\right\rfloor \geq \sum_{i=1}^{n}\left\lfloor\frac{x_i}{f}\right\rfloor \geq \left\lfloor\frac{f-1+k}{f}\right\rfloor = \left\lfloor\frac{f+k}{f}\right\rfloor = \left\lfloor\frac{f-1+k+1}{f}\right\rfloor$.
Therefore, the inductive hypothesis holds for $(k+1)$ when $k \neq af$.

Subcase 2: When k is a multiple of f, denoted by $k=af$. Suppose $k+1=af+1$. According to Lemma 2.1, if there exists $i_0 \in \{1,2,\cdots,n\}$ with $x_{i_0} \neq ft+(f-1)$, then $\left\lfloor\frac{x_{i_0}'}{f}\right\rfloor \geq \left\lfloor\frac{x_{i_0}}{f}\right\rfloor \geq a+1$. Otherwise, according to Lemma 2.1, for every $i_0 \in \{1,2,\cdots,n\}$ with $x_{i_0}=ft+(f-1)$ and for $(f-1)n+af+1$ objects, there exists $i_0 \in \{1,2,\cdots,n\}$ with $x_{i_0}' \geq x_{i_0}+1$, then $\left\lfloor\frac{x_{i_0}'}{f}\right\rfloor \geq \left\lfloor\frac{x_{i_0}}{f}\right\rfloor + 1$. In addition, for each $i \in \{1,2,\cdots,n\}-\{i_0\}$, we have $x_i'=x_i$.
Therefore, according to the principle of mathematical induction, $\sum_{i=1}^{n}\left\lfloor\frac{x_i'}{f}\right\rfloor \geq \sum_{i=1}^{n}\left\lfloor\frac{x_i}{f}\right\rfloor + 1 \geq a+1$. □

Theorem 2.3. *The pebbling number of* $D_{n,C_{2m}}$ *is* $[f(C_{2m})-1](n-2)+f(P_{2m+1})$.

Proof. For convenience, $[f(C_{2m})-1](n-2)+f(P_{2m+1})$ is denoted by A, respectively. Note u as the center vertex of all the cycles in $D_{n,C_{2m}}$. Let C^i be the cycle with the target vertex $u_{i,m}$ and let C^i/u be the cycle C^i without the center point u. Without lose of generality, we may assume that $u_{1,m}$ is the target vertex. First, suppose that there are $(A-1)$ pebbles on the vertices of $D_{n,C_{2m}}$, according to the distribution is given below: $p(u_{n,m}) = f(P_{2m+1})-1$, $p(u_{i,m}) = f(C_{2m})-1$ $(i = 2,3,\cdots n-1)$, $p(u) = p(u_{i,j}) = 0$ $(i = 2,3,\cdots n, j \neq m)$, $p(u_{1,m}) = 0$. In this case, there is no pebble can reach $u_{1,m}$. Thus $f(D_{n,C_{2m}}) \geq A$.

Next, we consider the distribution with A pebbles on the vertices of $D_{n,C_{2m}}$. The graph $D_{n,C_{2m}}$ has three kinds of target vertices, i.e.,(1) the center vertex u. (2) $u_{i,j}$, where $j \neq m$. (3) $u_{i,m}$, where $d(u_{i,m},u) = m$. The proof of (1) and (2) are easy to be checked, so we consider (3) in two cases. For convenience, the cycle C^1 is divided into two part $P_a = \langle u_{1,1}, u_{1,2}, \cdots, u_{1,m}\rangle$ and $P_b = \langle u_{1,2m-1}, u_{1,2m-2}, \cdots, u_{1,m}\rangle$, respectively.

Case 1: To prove the case when P_a and P_b are occupied by pebbles. If there exist $j_1, j_2, \cdots, j_k \in \{1,2,\cdots,m-1\}$ with $p(u_{1,j_t}) \neq 0$, where $t \in \{1,2,\cdots,k\}$ and there exist $i_1, i_2, \cdots, i_r \in \{m+1, m+2, \cdots, 2m-1\}$ with $p(u_{1,i_t}) \neq 0$, where $t \in \{1,2,\cdots,r\}$, then one pebble can be moved to $u_{1,m}$ when $\sum_{t=1}^{k} p(u_{1,j_t}) > 2^{m-1}-1$ or $\sum_{t=1}^{r} p(u_{1,i_t}) > 2^{m-1}-1$, since $d(u_{1,1}, u_{1,m}) = d(u_{1,2m-1}, u_{1,m}) = m-1$ and $f(P_a) = f(P_b) = 2^{m-1}$. Otherwise, both $\sum_{t=1}^{k} p(u_{1,j_t})$ and $\sum_{t=1}^{r} p(u_{1,i_t})$ are less then 2^{m-1}. There will be at least $A-2(2^{m-1}-1) = A-2^m+2$ pebbles are required to be put on $(n-1)$ cycles. Hence $f(C_{2m})-1$ pebbles can be moved to according to Lemma 2.2.

Case 2: To prove the case when P_a and P_b are not occupied by pebbles. For each $j \in \{1,2,\cdots,m,m+1,\cdots,2m-1\}$ with $p(u_{1,j}) = 0$, then A pebbles are demanded to be put on the rest $(n-1)$ cycles. According to Lemma 2.2, $f(C_{2m})$ pebbles can be moved to u, then one pebble can be moved to $u_{1,m}$. Thus, $f(D_{n,C_{2m}}) \leq A$.
Therefore, $f(D_{n,C_{2m}}) = [f(C_{2m})-1](n-2)+f(P_{2m+1})$. □

3. The pebbling number of $D_{n,C_{2m+1}}$

In order to calculate the number of the graph $D_{n,C_{2m+1}}$, we first give the following corollary, which relies on Theorem 1.1 and 1.2 given in Section 1.

Corollary 3.1. *A path $\tilde{P}_{m+2} = \langle u, u_{1,1}, u_{1,2} \cdots, u_{1,m+1}\rangle$ is equivalent to a cycle C_{m+1}, if and only if for the path P_{m+2}, both u and $u_{1,m+1}$ are target vertices. Then*
(i) $f(\tilde{P}_{m+2}) = f(C_{m+1}) = 2^{\frac{m+1}{2}}$ where is odd.
(ii) $f(\tilde{P}_{m+2}) = f(C_{m+1}) = \frac{2^{\frac{m}{2}+2}-(-1)^{\frac{m}{2}}}{3}$ where is even.

Theorem 3.2. *Let $f_{2^m}(C_{2m+1})$ be the 2^m-pebbling number of the graph C_{2m+1}. The pebbling number of $D_{n,C_{2m+1}}$ is $[f(C_{2m+1}) - 1](n-2) + [f_{2^m}(C_{2m+1}) - 1] + f(C_{m+1})$.*

Proof. For convenience, $[f(C_{2m+1})-1](n-2)+[f_{2^m}(C_{2m+1})-1]+f(C_{m+1})$ is denoted by B. First, we consider the following distribution such that we cannot move one pebble to the target vertex by a sequence of pebbling move, when the total number of pebbles is $(B-1)$.

Case i: For odd m, $p(u_{i,m}) = p(u_{i,m+1}) = \frac{f(C_{2m+1})-1}{2}$ $(i = 2, 3, \cdots, n-1)$,$p(u_{1,\frac{m+1}{2}}) = f(C_{m+1}) - 1$,let $f_{2^m}(C_{2m+1}) - 1$ pebbles be on the cycle C^n, then $2^m - 1$ pebbles can be moved to u. Thus no pebble can be moved to the vertex $u_{1,m}$
Case ii: For even m, $p(u_{i,m}) = p(u_{i,m+1}) = \frac{f(C_{2m+1})-1}{2}$ $(i = 2, 3, \cdots, n-1)$, $p(u_{1,\frac{m}{2}}) = p(u_{1,\frac{m}{2}+1}) = \frac{f(C_{m+1})-1}{2}$, let $f_{2^m}(C_{2m+1}) - 1$ pebbles be on the cycle C^n, then $2^m - 1$ pebbles can be moved to u. Thus no pebble can be moved to the vertex $u_{1,m}$, i.e. $f(D_{n,C_{2m+1}}) \geq B$.

Next, we consider the distribution with B pebbles on the vertices of $D_{n,C_{2m+1}}$. If the target vertex is u or $u_{i,j}$ ($j \neq m, m+1$ where m is odd, or $j \neq m$ where m is even), then the proof is easy to check by the previous theorems and lemmas. Therefore we consider the case when the target vertex is $u_{i,m}$ by discussing the range of $p(C^1/u)$. Without loss of generality, we assume that the target vertex is $u_{1,m}$. If $p(u) \geq 2^m$, as $d(u, u_{1,m}) = m$, then at least one pebble can reach $u_{1,m}$.

Case 1: If $p(C^1/u) \geq f(C_{2m+1})$, then at least one pebble can reach $u_{1,m}$.

Case 2: If $\frac{2^m+6-(-1)^m}{3} \leq p(C^1/u) \leq f(C_{2m+1})$, then the remaining number of pebbles on the vertices of the graph $D_{n,C_{2m+1}}$ without cycle C^1 will be at least $B - (f(C_{2m+1}) - 1)$. Those pebbles are demanded to be put into $(n-1)$ cycles of $D_{n,C_{2m+1}}$. According to the Pigeonhole principle and Theorem 1.2 (ii), we can put at least $2^m - 2$ pebbles on u. Then, $p(C^1) \geq \frac{2^m+6-(-1)^m}{3} + 2^m - 2 = \frac{2^{m+2}-(-1)^m}{3} = f(C_{2m+1})$. As in Case 1, we have done.

Case 3: If $f(C_{m+1}) \leq p(C^1/u) \leq \frac{2^m+6-(-1)^m}{3} - 1$, then the remaining number of pebbles on the vertices of the graph $D_{n,C_{2m+1}}$ without cycle C^1 will

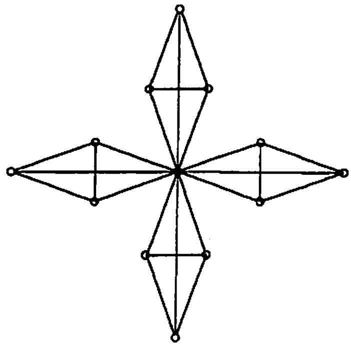

FIGURE 2. The Graph D_{n,K_4}.

be at least $B - \left(\frac{2^m+6-(-1)^m}{3} - 1\right)$. Those pebbles are demanded to be put into $(n-1)$ cycles of $D_{n,C_{2m+1}}$. As $B - \left(\frac{2^m+6-(-1)^m}{3} - 1\right) > (f(C_{2m+1}) - 1)(n-2) + f_{2^m-1}(C_{2m+1})$, at least $2^m - 1$ pebbles can reach u. Based on the pigeonhole principle and Theorem 1.2 (ii), if $p(u_{1,1}, \cdots, u_{1,m-1}) \geq 1$, then $p(u, u_{1,1}, \cdots, u_{1,m}) \geq 2^m$. Hence there will be at least one pebble on $u_{1,m}$. Otherwise, according to Corollary 3.1 more than one pebbles can be moved to u or $u_{1,m}$, since $p(u_{1,m+1}, u_{1,m+2}, \cdots, u_{1,2m}) \geq f(\tilde{P}_{m+2}) = f(C_{m+1})$.

Case 4: If $0 \leq p(C^1/u) \leq f(C_{m+1}) - 1$, then the other vertices have at least $B - (f(C_{m+1}) - 1) = (f(C_{2m+1}) - 1)(n-2) + f_{2^m}(C_{2m+1})$ pebbles. Thus at least one pebble can be moved to the target vertex.

Therefore, $f(D_{n,C_{2m+1}}) \leq B$.

Above all $f(D_{n,C_{2m+1}}) = [f(C_{2m+1}) - 1](n-2) + [f_{2^m}(C_{2m+1}) - 1] + f(C_{m+1})$. □

4. FURTHER PROBLEMS

This paper calculate the pebbling number of the graph D_{n,C_m}, and it is easy to verify for complete graph K_n, the pebbling number of the graph D_{n,K_n} is $[f(K_n) - 1](n-2) + [f_2(K_n) - 1] + (n-2) + 1$. D_{n,K_n} is shown in Figure 2. Since $diam(K_n) = 1$, it is likely to conjecture the pebbling number of a class of graph $D_{n,G}$ is $[f(G) - 1](n-2) + [f_t(G) - 1] + \mathcal{O}(1)$, where $t = f(P_d)$.

There are many types of graphs such as complete graph, product graph, and hypercube. It would be interesting to study whether a clique block graphs combine together through a common vertex have similar conclusions.

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