Uniquely colorable graphs on surfaces

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Abstract

A k-chromatic graph G is *uniquely k-colorable* if G has only one k-coloring up to permutation of the colors. In this paper, we focus on uniquely k-colorable graphs on surfaces. Let F^2 be a closed surface except the sphere, and let $\chi(F^2)$ be the maximum number of the chromatic number of graphs which can be embedded on F^2. Then we shall prove that the number of uniquely k-colorable graphs on F^2 is finite if $k \geq 5$, and we characterize uniquely $\chi(F^2)$-colorable graphs on F^2. Moreover, we completely determine uniquely k-colorable graphs on the projective plane, where $k \geq 5$.

1 Introduction

In this paper, we only deal with finite undirected simple graphs. Moreover, let K_n be a complete graph with n vertices and let H_n be a graph obtained from K_n by deleting an edge of K_n. A *k-coloring* of a graph G is a map $c : V(G) \to \{1, 2, \dots, k\}$ such that for any edge $uv \in E(G)$, $c(u) \neq c(v)$. A graph G is *k-colorable* if there exists a k-coloring of G, and a *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k such that G is k-colorable. Moreover, a graph G with $\chi(G) = k$ is called a *k-chromatic* graph. A graph G is *uniquely k-colorable* if $k = \chi(G)$ and G has only one k-coloring up to permutation of the colors, where the coloring is called a *unique k-coloring*. In other words, any uniquely k-colorable graph G has only one partition of $V(G)$ into k-independent subsets. Trivially, a complete graph K_n is uniquely n-colorable. Moreover, we denote the set of uniquely k-colorable graphs by UC_k. For two distinct colors $i, j \in \{1, 2, \dots, k\}$ in a k-coloring c of a graph G, define $G_{i,j}$ to be the subgraph of G induced by $c^{-1}(i) \cup c^{-1}(j)$. For uniquely k-colorable graphs, Harary et al. proved the following theorem.

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Theorem 1 (Harary et al. [7]) *If $c : V(G) \to \{1, 2, \ldots, k\}$ is a unique k-coloring of $G \in UC_k$, then the graph $G_{i,j}$ is connected for all $i \neq j$ ($i, j \in \{1, 2, \ldots, k\}$).*

If a graph G is uniquely 1-colorable, then G has no edges. Hence, throughout this paper, we only consider $k \geq 2$ for any uniquely k-colorable graphs. Moreover, the following corollaries hold by Theorem 1. For a graph G, we define $\delta(G)$ to be the minimum degree of G.

Corollary 2 *If $G \in UC_k$ with $|V(G)| = n$, then G has at least $(k-1)n - \binom{k}{2}$ edges.*

Corollary 3 *If $G \in UC_k$, then $\delta(G) \geq k - 1$.*

Corollary 4 *If $G \in UC_k$, then G is $(k-1)$-connected.*

For other results and related topics, see [6].

In this paper, we focus on uniquely colorable graphs on surfaces. Then, following this, we regard a graph on a surface F^2 as a map on F^2. For uniquely colorable graphs on the sphere, there are several known results, see [3] and [6]. However, for those on any other surface, there are no results and research. Hence, in this paper, we consider uniquely colorable graphs on surfaces except the sphere. Let F^2 denote a closed surface and let $\chi(F^2)$ be the maximum number of the chromatic number of graphs which can be embedded on F^2. By known results [1, 2, 5, 10, 12], we have the following. In the equation, $\mathbb{K}$ and $\varepsilon(F^2)$ mean Klein bottle and Euler characteristic of F^2, respectively.

$$\chi(F^2) = \begin{cases} \lfloor (7 + \sqrt{49 - 24\varepsilon(F^2)})/2 \rfloor & (F^2 \neq \mathbb{K}) \\ 6 & (F^2 = \mathbb{K}) \end{cases} \tag{1}$$

Hence, we have $\chi(F^2) \geq 6$ for any surface F^2 except the sphere since $\chi(\mathbb{P}) = \chi(\mathbb{K}) = 6$ by the above equation, where $\mathbb{P}$ stands for the projective plane.

First, we focus on the number of uniquely k-colorable graphs. It is easy to see that there are infinitely many uniquely k-colorable graphs for any positive integer k. However, the number of uniquely k-colorable graphs on a closed surface F^2 is a (finite) constant that depends upon k and the Euler characteristic $\varepsilon(F^2)$ as follows.

Theorem 5 *The number of uniquely k-colorable graphs on a closed surface F^2 is finite if $k \geq 5$ except the case F^2 is the sphere.*

Following this, F^2 stands for a surface except the sphere. For an integer $k \geq 5$, we can also see the larger k becomes, the smaller the number of uniquely k-colorable graphs on F^2 is, where F^2 is fixed since the connectivity of corresponding graphs becomes large by Corollary 4. Then, we focus on uniquely $\chi(F^2)$-colorable graphs on F^2, and we completely determine them as follows.

Theorem 6 *Let G be a graph on a closed surface F^2 except the sphere. Then G is uniquely $\chi(F^2)$-colorable if and only if G is isomorphic to either $K_{\chi(F^2)}$ or $H_{\chi(F^2)+1}$.*

By the above result, any uniquely $\chi(F^2)$-colorable graph G on any closed surface F^2 except the sphere must have a complete graph $K_{\chi(F^2)}$ as its subgraph. Classically, Albertson and Hutchinson [1] and Dirac [5] proved that any $\chi(F^2)$-chromatic graph on F^2 except the Klein bottle contains $K_{\chi(F^2)}$ as its subgraph. However, for the Klein bottle $\mathbb{K}$, there exists a $\chi(\mathbb{K})$-chromatic graph which has no $K_{\chi(\mathbb{K})}$ as its subgraph [1]. In other words, by adding the property "uniquely" into $\chi(F^2)$-chromatic graphs on any surface F^2 except the sphere, it must be guaranteed that those graphs have $K_{\chi(F^2)}$ as their subgraphs. (Note that there exists a uniquely r-colorable graph containing no K_r for any integer $r \geq 3$ [7]. Moreover, for the sphere, it was proved that any uniquely 4-colorable planar graph is isomorphic to a *spherical* 3-*tree*, see [6].) Moreover, Chenette et al. [4] recently exhibit an explicit list of nine graphs such that a graph on the Klein bottle is 5-colorable if and only if it contains no subgraph isomorphic to a member of the list. Since a lot is known about embeddings of $K_{\chi(F^2)}$ and $H_{\chi(F^2)+1}$, we shall discuss it in Section 3.

Finally, we also focus on uniquely $(\chi(F^2)-1)$-colorable graphs on F^2. By Theorem 5 and $\chi(F^2) \geq 6$, the number of those graphs is finite. Hence we consider making a list of uniquely $(\chi(F^2)-1)$-colorable graphs on F^2. Then, this time, we complete the list of uniquely 5-colorable graphs on the projective plane as follows. The graph B_7 is a graph on the projective plane with 7 vertices shown in Figure 1, which is obtained from K_4 on the projective plane with the vertices x, y, z and w by joining three independent vertices a, b and c as shown in Figure 1. Moreover, it is easy to see that B_7 is uniquely 5-colorable.

Theorem 7 *Let G be a graph on the projective plane. Then G is a uniquely 5-colorable if and only if G is isomorphic to either K_5, H_6 or B_7.*

By Theorems 6 and 7, we have the following corollary.

Corollary 8 *Let $\mathcal{G}$ be a set of uniquely k-colorable graphs on the projective plane, where $k \geq 5$. Then we have $\mathcal{G} = \{K_5, H_6, K_6, B_7\}$.*

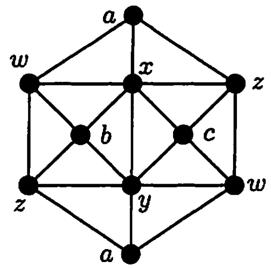

Figure 1: B_7

2 Proof of theorems

In this section, we prove Theorems 5, 6 and 7.

Proof of Theorem 5. Let G be a uniquely k-colorable graph on F^2, where $k \geq 5$ and $|V(G)| = n$. By Corollary 2, we have $|E(G)| \geq (k-1)n - \binom{k}{2}$. Moreover, by Euler's formula, we have $|E(G)| \leq 3n - 3\varepsilon(F^2)$. Hence, by combining the above two inequalities, we have

$$n \leq \frac{\binom{k}{2} - 3\varepsilon(F^2)}{k-4}.$$

Therefore, since the number of vertices is finite, the theorem holds. ∎

Proof of Theorem 6. It is easy to see that the "if" part follows. Hence we prove the "only if" part.

Let G be a uniquely $\chi(F^2)$-colorable graph on a closed surface F^2 except the sphere and the Klein bottle. By Corollary 2, we have $|E(G)| \geq (\chi(F^2) - 1)|V(G)| - \binom{\chi(F^2)}{2}$, and by Corollary 3, any vertex of G has degree at least $\chi(F^2) - 1$. Hence, if $|V(G)| = \chi(F^2)$, then G is isomophic to $K_{\chi(F^2)}$, that is, G is the complete graph with the maximum number of vertices which can be embedded on F^2. In this case, the number of edges which can be added to G on F^2 is at most $|V(G)| - 4$ (in this case, since the largest face size is $|V(G)| - 1$ and disregarding simplicity.) Otherwise, we can embed $K_{|V(G)|+1}$ on F^2 by [10], this is a contradiction.

Let us consider the number of vertices of G. Now, we have $|E(G)| \geq (\chi(F^2) - 1)|V(G)| - \binom{\chi(F^2)}{2}$ by Corollary 2. On the other hand, for any graph on F^2, if the number of vertices increases by one, then the upper bound of the number of edges increases by at most three by Euler's formula, that is, $|E(G)| \leq 3|V(G)| - 3\varepsilon$. Hence, two bounds on $|E(G)|$ give that $|V(G)| \leq \frac{\chi(F^2)^2 - \chi(F^2) - 3\varepsilon}{2\chi(F^2) - 8}$. Therefore, by an easy calculation using the previous inequality, we can see that any uniquely $\chi(F^2)$-colorable graph on F^2 has at most $\chi(F^2)+1$ vertices. Then, it is easy to check that G is isomorphic to either $K_{\chi(F^2)}$ or $H_{\chi(F^2)+1}$.

Finally, we consider the Klein bottle case. Let G be a uniquely $\chi(\mathbb{K})$-colorable graph on $\mathbb{K}$. By Corollary 2, the equation (1) and Euler's formula, we have $5|V(G)|-15 \le |E(G)| \le 3|V(G)|$. Hence, we have $|V(G)| \le 7$ by $|V(G)| \le \frac{15}{2}$. Therefore, we can also see that G is isomorphic to either $K_{\chi(\mathbb{K})}$ or $H_{\chi(\mathbb{K})+1}$. ■

Proof of Theorem 7. We first show the "if" part. Clearly, if G is isomorphic to K_5 or H_6, then G is uniquely 5-colorable. Moreover, by Figure 1, it is easy to see that B_7 is uniquely 5-colorable since K_4 is uniquely 4-colorable. Hence, the "if" part holds.

Next, we show the "only if" part. Let G be a uniquely 5-colorable graph on the projective plane. By Corollary 2, G has at least $4|V(G)|-10$ edges. On the other hand, since G is on the projective plane, we have $|E(G)| \le 3|V(G)|-3$ by Euler's formula. Hence, by combining the above inequalities, we have $|V(G)| \le 7$. Moreover, if $|V(G)| = 5$ (resp., 6), then it is easy to see that G is isomorphic to K_5 (resp., H_6). (In this case, if $|V(G)| = 5$ (resp., 6), then G has exactly 10 (resp., 14) edges. If G has $|V(G)| = 6$ and 15 edges, then G is isomorphic to K_6. However, this contradicts $G \in UC_5$.) Hence, we may suppose that $|V(G)| = 7$ and $|E(G)| = 18$.

Now, we re-define G to be a graph on the projective plane obtained from K_7 by removing three edges e_1, e_2, e_3 of K_7 (note that $|E(K_7)| = 21$). Let $e_1 = u_1v_1, e_2 = u_2v_2$ and $e_3 = u_3v_3$ be the three edges. Then let us prove that e_1, e_2 and e_3 form a triangle, that is, we shall show $v_1 = u_2, v_2 = u_3$ and $v_3 = u_1$ by symmetry. If the three edges have no common vertex, then it is easy to see that G is 4-colorable since u_i and v_i can be colored by the same color for each $i \in \{1,2,3\}$. Hence, we may suppose that the three edges have at least one common vertex.

We first suppose $v_1 = v_2 = v_3$ (the three edges form a $K_{1,3}$). In this case, the graph induced by the six vertices of K_7 except v_1 is isomorphic to K_6, but this contradicts $G \in UC_5$. Next suppose that $v_1 = u_2$ and $\{u_1, v_1, v_2\} \cap \{u_3, v_3\} = \emptyset$. Then we give 5-coloring c to G, where $c(u_3) = c(v_3) = 1, c(u_1) = 2, c(v_2) = 3$ and other two vertices except v_1 are colored by 4, 5. In this case, since there are two choices of $c(v_1)$, namely 2 or 3, this contradicts G is uniquely colorable. Hence, we may suppose that e_1, e_2 and e_3 share at least two vertices.

Without loss of the generality, we may suppose that $v_1 = u_2, v_2 = u_3$. In this case, we consider whether the graph G can be embedded on the projective plane. We now see that the degree sequence of G is $(6,6,6,5,5,4,4)$. Let v be a vertex of degree 6 and let v_1 be a vertex of degree 6 which is

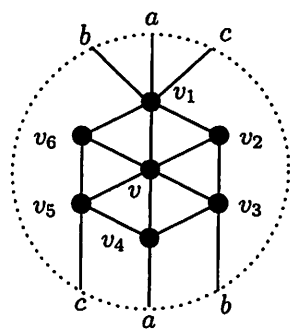

Figure 2: Surrounding of v

a neighbor of v. Then, we now have a configuration shown in Figure 2. (Note that G has only triangular faces on the projective plane by Euler's formula.)

In Figure 2, the third vertex of degree 6 is either v_3 or v_5 since any vertex of degree 6 must be adjacent to all other vertices. Hence, without loss of generality, we may suppose that $\deg(v_3) = 6$, and then, we have $\deg(v_5) = 5$ and $\deg(v_6) = 4$. In this case, it is easy to see that $\deg(v_2)$ cannot be five, and hence, G cannot be embedded on the projective plane satisfying the degree sequence $(6,6,6,5,5,4,4)$. Therefore, G is obtained from K_7 by removing three edges which form a triangle.

Then we now see that the degree sequence of G is $(6,6,6,6,4,4,4)$. Therefore, since G has only triangular faces on the projective plane by Euler's formula, it is not difficult to see that G is isomorphic to B_7 by being careful about any two vertices of degree 4 are not adjacent. ■

3 Embeddings of $H_{\chi(F^2)+1}$ on F^2

Embeddings of $H_{\chi(F^2)+1}$ on F^2 in Theorem 6 are now completely understood. By [12], $K_{\chi(F^2)}$ embeds on F^2. By Euler's formula, except when F^2 is the Klein Bottle $\mathbb{K}$, $H_{\chi(F^2)+1}$ embeds on F^2 only if $\chi(F^2) \equiv 1 \pmod 3$, and then when $\chi(F^2) = 3k+1$ $(k \geq 2)$, $\varepsilon(F^2) = \frac{4+3k-3k^2}{2}$. When $F^2 = \mathbb{K}$, it is not difficult to see that H_7 can be embedded on $\mathbb{K}$. When F^2 is a surface of Euler characteristic -1 (and $\chi(F^2)+1 = 8, k = 2$), Ringel [11] proved that H_8 does not embed on F^2. Moreover, Ringel [10] and Korzhik [8] proved that for all $k \geq 3$, $H_{\chi(F^2)+1} = H_{3k+2}$ does embed on F^2, a surface of Euler characteristic $\frac{4+3k-3k^2}{2}$ when $\chi(F^2) = 3k+1$. Thus we need to include $H_{\chi(F^2)+1}$ in the statement of Theorem 6 to cover these exceptional cases.

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