Total edge irregularity strength of a categorical product of two paths*

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Abstract

An edge irregular total k-labeling of a graph $G = (V, E)$ is a labeling $f : V \cup E \to \{1, 2, \ldots, k\}$ such that the total edge-weights $wt(xy) = f(x) + f(xy) + f(y)$ are different for all pairs of distinct edges. The minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of G.

In this paper, we determine the exact value of the total edge irregularity strength of the categorical product of two paths P_n and P_m. Our result adds further support to a recent conjecture of Ivančo and Jendrol̆.

Keywords : irregularity strength, edge irregular total labeling, total edge irregularity strength, categorical product.

1 Introduction and Definitions

We consider finite undirected graphs $G = (V, E)$ without loops and multiple edges with vertex-set $V(G)$ and edge-set $E(G)$, where $|V(G)| = p$ and

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$|E(G)| = q$. The degree of a vertex x is the number of edges that have x as an endpoint, and the set of neighbors of x is denoted by $N(x)$.

By a *labeling* we mean any mapping that carries a set of graph elements to a set of numbers (usually positive integers), called *labels*. If the domain is the vertex-set or the edge-set, the labelings are called respectively *vertex labelings* or *edge labelings*. If the domain is $V \cup E$ then we call the labeling *total labeling*. Thus, for an edge k-labeling $\sigma : E(G) \to \{1, 2, \ldots, k\}$ the associated vertex-weight of a vertex $x \in V(G)$ is

$$w_\sigma(x) = \sum_{y \in N(x)} \sigma(xy)$$

and for a total k-labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ the associated edge-weight is

$$wt_\varphi(xy) = \varphi(x) + \varphi(xy) + \varphi(y).$$

Chartrand *et al.* in [6] introduced edge k-labeling of a graph G such that $w(x) \neq w(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k.

The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

This parameter has attracted much attention [1], [2], [4], [7], [8], [9], [11]. Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see a survey article by Lehel [14].

Motivated by these papers and by a book of Wallis [19], Bača *et al.* in [3] started to investigate the total edge irregularity strength of a graph, an invariant analogous to the irregularity strength for total labelings.

A total k-labeling φ is defined to be an *edge irregular total labeling* of a graph G if for every two different edges xy and $x'y'$ of G one has $wt_\varphi(xy) \neq wt_\varphi(x'y')$. The minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of G, $tes(G)$.

Let φ be an edge irregular total k-labeling of $G = (V, E)$. Since $3 \leq wt_\varphi(xy) = \varphi(x) + \varphi(xy) + \varphi(y) \leq 3k$ for every edge $xy \in E(G)$, we have $|E(G)| \leq 3k - 2$ which implies $tes(G) \geq \left\lceil \frac{|E(G)|+2}{3} \right\rceil$.

If $x \in V(G)$ is a fixed vertex of maximum degree $\Delta(G)$, then there is a range of $2k-1$ possible weights $\varphi(x)+2 \leq wt_\varphi(xy) \leq \varphi(x)+2k$ for the $\Delta(G)$ edges $xy \in E(G)$ incident with x which implies $tes(G) \geq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

So, we have that

$$tes(G) \geq \max\left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}. \tag{1}$$

The authors of [3] determined the exact value of the total edge irregularity strength for certain families of graphs, namely paths, cycles, stars, wheels and friendship graphs. They posed the problem to determine the total edge irregularity strength of trees. Recently Ivančo and Jendroľ [10] proved that for any tree T the total edge irregularity strength is equal to its lower bound, i.e.

$$tes(T) = \max\left\{ \left\lceil \frac{|E(T)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(T)+1}{2} \right\rceil \right\}.$$

Moreover, they posed the following conjecture.

Conjecture 1 [10] *Let $G = (V, E)$ be an arbitrary graph different from K_5 and maximum degree $\Delta(G)$. Then*

$$tes(G) = \max\left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}.$$

Note that for K_5 the maximum of the lower bounds is 4 while $tes(K_5) = 5$, (see Theorem 7 in [3]). Conjecture 1 has been verified for complete graphs and complete bipartite graphs by Jendroľ, Miškuf and Soták in [12] and [13], and for the Cartesian product of two paths by Miškuf and Jendroľ in [15]. Brandt *et al.* in [5] proved Conjecture 1 for large dense graphs, i.e. for graphs G with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$.

Motivated by the papers [7], [15] and [18] we investigate the total edge irregularity strength of the categorical product of two paths $P_n \times P_m$. This paper adds further support to Conjecture 1 by demonstrating that the categorical product $P_n \times P_m$ has total edge irregularity strength equal to $\left\lceil \frac{|E(P_n \times P_m)|+2}{3} \right\rceil$.

2 Main Result

For integers a and b let $[a, b]$ be an interval of integers x, $a \leq x \leq b$. In this section we deal with a categorical product of two graphs. The categorical product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, where two vertices (u, u') and (v, v') are adjacent if and only if u, v are adjacent in G and u', v' are adjacent in H (see e.g. [16] or [17]). If we consider graph G as the path P_n with $V(P_n) = \{x_i : i \in [1, n]\}$, $E(P_n) = \{x_i x_{i+1} : i \in [1, n-1]\}$ and graph H as the path P_m with $V(P_m) = \{y_j : j \in [1, m]\}$, $E(P_m) = \{y_j y_{j+1} : j \in [1, m-1]\}$ then $V(P_n \times P_m) = \{(x_i, y_j) : i \in [1, n],\ j \in [1, m]\}$ is the vertex set of $P_n \times P_m$ and $E(P_n \times P_m) = \{(x_i, y_j)(x_k, y_l) : i, k \in [1, n],\ j, l \in [1, m],\ |i - k| = 1,\ |j - l| = 1\}$ is the edge set of $P_n \times P_m$. So, $P_n \times P_m$ is the graph of order nm and size $2(n-1)(m-1)$. As the maximum degree $\Delta(P_n \times P_m) = 4$ then from (1) it follows that $tes(P_n \times P_m) \geq \left\lceil \frac{2(n-1)(m-1)+2}{3} \right\rceil$. The main goal of this paper is to prove equality.

Theorem 1 *Let $m, n \geq 2$ be positive integers and $P_n \times P_m$ be the categorical product of two paths P_n and P_m. Then*

$$tes(P_n \times P_m) = \left\lceil \frac{2(n-1)(m-1)+2}{3} \right\rceil.$$

Proof. Let $m, n \geq 2$ be positive integers and let $k = \left\lceil \frac{2(n-1)(m-1)+2}{3} \right\rceil$. We split the edge set of $P_n \times P_m$ into mutually disjoint subsets A_i and B_i, where
$A_i = \{(x_i, y_j)(x_{i+1}, y_{j+1}) : j \geq 1 \text{ odd}\} \cup \{(x_i, y_{j+1})(x_{i+1}, y_j) : j \geq 2 \text{ even}\}$ for $i \in [1, n-1]$ and
$B_i = \{(x_i, y_{j+1})(x_{i+1}, y_j) : j \geq 1 \text{ odd}\} \cup \{(x_i, y_j)(x_{i+1}, y_{j+1}) : j \geq 2 \text{ even}\}$ for $i \in [1, n-1]$.

Clearly, $|A_i| = |B_i| = m - 1$ and $\bigcup_{i=1}^{n-1} \{A_i \cup B_i\} = E(P_n \times P_m)$.

Because the graphs $P_m \times P_n$ and $P_n \times P_m$ are isomorphic, it is sufficient to prove the statement for $m \leq n$. Let us distinguish two cases.

Case 1. m and n have the same parity, $2 \leq m \leq n$

It is easy to see that for $m = n = 2$ the total edge irregularity strength is 2. Now, for $m \geq 2$ and $n \geq 3$ we construct the function φ as follows:

$$\varphi((x_i,y_j)) = \begin{cases} \frac{j}{2}, & \text{if } i=1 \text{ and } j \text{ is even} \\ \left(\lceil \frac{i}{2} \rceil - 1\right) m + \frac{j+1}{2}, & \text{if } 1 \le i \le \lfloor \frac{n}{2} \rfloor \text{ and } j \text{ is odd} \\ \lceil \frac{m}{2} \rceil + \left(\lfloor \frac{i}{2} \rfloor - 1\right) m + \frac{j}{2}, & \text{if } 2 \le i \le \lfloor \frac{n}{2} \rfloor \text{ and } j \text{ is even} \\ k - \lfloor \frac{n-i}{2} \rfloor m - \frac{m-j}{2}, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \le i \le n, \text{ and} \\ & j \text{ has the same parity as } m \\ k - \lfloor \frac{n-i-1}{2} \rfloor m - m + \lceil \frac{j}{2} \rceil, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \le i \le n-1, \text{ and} \\ & j \text{ has different parity as } m \\ k - \frac{m-j-1}{2}, & \text{if } i = n, \text{ and} \\ & j \text{ has different parity as } m. \end{cases}$$

Observe that under the vertex labeling φ the weights of the edges

(i) from the set B_1 admit the consecutive integers from 2 to m,

(ii) from the set A_i and A_{i+1} receive the consecutive integers from $\lceil \frac{m}{2} \rceil + m(i-1)+2$ to $\lceil \frac{m}{2} \rceil + mi$ for every m and $1 \le i < \lfloor \frac{n}{2} \rfloor - 1$ odd,

(iii) from the set B_i and B_{i+1} receive the consecutive integers from $\lceil \frac{m}{2} \rceil + m(i-1)+2$ to $\lceil \frac{m}{2} \rceil + mi$ for every m and $2 \le i < \lfloor \frac{n}{2} \rfloor - 1$ even,

(iv) from the set $A_{\lfloor \frac{n}{2} \rfloor - 1}$ for $\lfloor \frac{n}{2} \rfloor$ even (respectively, $B_{\lfloor \frac{n}{2} \rfloor - 1}$ for $\lfloor \frac{n}{2} \rfloor$ odd) admit the consecutive integers from $\lceil \frac{m}{2} \rceil + m\left(\lfloor \frac{n}{2} \rfloor - 2\right) + 2$ to $\lceil \frac{m}{2} \rceil + m\left(\lfloor \frac{n}{2} \rfloor - 1\right)$,

(v) from the set $A_{\lfloor \frac{n}{2} \rfloor}$ for $\lfloor \frac{n}{2} \rfloor$ odd (respectively, $B_{\lfloor \frac{n}{2} \rfloor}$ for $\lfloor \frac{n}{2} \rfloor$ even) admit the consecutive integers from $k+2-\frac{m}{2}$ to $k+\frac{m}{2}$ if m is even and from $k+2-m$ to k if m is odd,

(vi) from the set $B_{\lfloor \frac{n}{2} \rfloor}$ for $\lfloor \frac{n}{2} \rfloor$ odd (respectively, $A_{\lfloor \frac{n}{2} \rfloor}$ for $\lfloor \frac{n}{2} \rfloor$ even) admit the consecutive integers from $k+2-\frac{m}{2}$ to $k+\frac{m}{2}$ if m is even and from $k+2-m$ to k if m is odd,

(vii) from the set $A_{\lfloor \frac{n}{2} \rfloor + 1}$ for $\lfloor \frac{n}{2} \rfloor$ even (respectively, $B_{\lfloor \frac{n}{2} \rfloor + 1}$ for $\lfloor \frac{n}{2} \rfloor$ odd) admit the consecutive integers from $2k - m(n - \lfloor \frac{n}{2} \rfloor) + \lfloor \frac{m}{2} \rfloor + 2$ to $2k - m(n-1-\lfloor \frac{n}{2} \rfloor) + \lfloor \frac{m}{2} \rfloor$,

(viii) from the set A_i and A_{i+1} receive the consecutive integers from $2k + m(i-n) + \lfloor \frac{m}{2} \rfloor + 2$ to $2k + m(i+1-n) + \lfloor \frac{m}{2} \rfloor$ for every m and $\lfloor \frac{n}{2} \rfloor < i < n-1$ even,

(ix) from the set B_i and B_{i+1} receive the consecutive integers from $2k + m(i-n) + \lfloor \frac{m}{2} \rfloor + 2$ to $2k + m(i+1-n) + \lfloor \frac{m}{2} \rfloor$ for every m and $\lfloor \frac{n}{2} \rfloor < i < n-1$ odd,

(x) from the last set A_{n-1} for n odd (respectively, B_{n-1} for n even) admit the consecutive integers from $2k - m + 2$ to $2k$.

To complete the labeling to a total one we label the edges of the graph $P_n \times P_m$ according to which of the ten families they belong to:

(i) We label the edges from the set B_1 by the label 1 to obtain the first $m - 1$ edge-weights $[3, m + 1]$.

(ii) The edges from the set A_i (respectively, A_{i+1}) we label by the label $2 - \lceil \frac{m}{2} \rceil + i(m-2)$ (respectively, by the label $m - \lceil \frac{m}{2} \rceil + 1 + i(m-2)$ for $1 \le i < \lfloor \frac{n}{2} \rfloor - 1$ odd. So, we create the edge-weights from the integer interval $[2i(m-1)-m+4, 2i(m-1)+2]$ (respectively, $[2i(m-1)+3, 2i(m-1)+m+1]$.

(iii) The edges from the set B_i (respectively, B_{i+1}) we label by the label $2 - \lceil \frac{m}{2} \rceil + i(m-2)$ (respectively, by the label $m - \lceil \frac{m}{2} \rceil + 1 + i(m-2)$ for $2 \le i < \lfloor \frac{n}{2} \rfloor - 1$ even. We create the edge-weights from the integer interval $[2i(m-1)-m+4, 2i(m-1)+2]$ (respectively, $[2i(m-1)+3, 2i(m-1)+m+1]$).

(iv) We label the edges from the set $A_{\lfloor \frac{n}{2} \rfloor - 1}$ for $\lfloor \frac{n}{2} \rfloor$ even (respectively, $B_{\lfloor \frac{n}{2} \rfloor - 1}$ for $\lfloor \frac{n}{2} \rfloor$ odd) by the label $\lfloor \frac{n}{2} \rfloor (m-2) - m + 4 - \lceil \frac{m}{2} \rceil$ to obtain the edge-weights from the interval $[(2\lfloor \frac{n}{2} \rfloor - 3)(m-1)+3, (2\lfloor \frac{n}{2} \rfloor - 2)(m-1)+2]$.

(v) We label the edges from the set $A_{\lfloor \frac{n}{2} \rfloor}$ for $\lfloor \frac{n}{2} \rfloor$ odd (respectively, $B_{\lfloor \frac{n}{2} \rfloor}$ for $\lfloor \frac{n}{2} \rfloor$ even) by the label $2\lfloor \frac{n}{2} \rfloor (m-1) - k + 3 - \frac{3m}{2}$ if m is even and by the label $(2\lfloor \frac{n}{2} \rfloor - 1)(m-1) - k + 2$ if m is odd to create the edge-weights from the interval $[(2\lfloor \frac{n}{2} \rfloor - 2)(m-1) + 3, (2\lfloor \frac{n}{2} \rfloor - 1)(m-1) + 2]$.

(vi) The edges from the set $B_{\lfloor \frac{n}{2} \rfloor}$ for $\lfloor \frac{n}{2} \rfloor$ odd (respectively, $A_{\lfloor \frac{n}{2} \rfloor}$ for $\lfloor \frac{n}{2} \rfloor$ even) we label by the label $(2\lfloor \frac{n}{2} \rfloor - 2)(m-1) - k + \frac{3m}{2}$ if m is even and by the label $(2\lfloor \frac{n}{2} \rfloor - 2)(m-1) + 2m - k$ if m is odd. We obtain the edge-weights from the interval $[(2\lfloor \frac{n}{2} \rfloor - 1)(m-1) + 3, 2\lfloor \frac{n}{2} \rfloor (m-1) + 2]$.

(vii) We label the edges from the set $A_{\lfloor \frac{n}{2} \rfloor + 1}$ for $\lfloor \frac{n}{2} \rfloor$ even (respectively, $B_{\lfloor \frac{n}{2} \rfloor + 1}$ for $\lfloor \frac{n}{2} \rfloor$ odd) by the label $\lfloor \frac{n}{2} \rfloor (m-2) + mn + 1 - 2k - \lfloor \frac{m}{2} \rfloor$ to create the edge-weights from the interval $[2\lfloor \frac{n}{2} \rfloor (m-1)+3, (2\lfloor \frac{n}{2} \rfloor + 1)(m-1)+2]$.

(viii) For $\lfloor \frac{n}{2} \rfloor < i < n - 1$ even, we label the edges from the set A_i (respectively, A_{i+1}) by the label $m(n-1) + i(m-2) + 2 - 2k - \lfloor \frac{m}{2} \rfloor$ (respectively, by the label $mn + i(m-2) + 1 - 2k - \lfloor \frac{m}{2} \rfloor$) to create the edge-weights from the interval $[2i(m-1) - m + 4, 2i(m-1) + 2]$ (respectively, $[2i(m-1) + 3, 2i(m-1) + m + 1]$).

(ix) With the same technique, for $\lfloor \frac{n}{2} \rfloor < i < n - 1$ odd, we label the edges from the set B_i (respectively, B_{i+1}) by the label $m(n-1) + i(m-2) +$

$2-2k-\lfloor\frac{m}{2}\rfloor$ (respectively, by the label $mn+i(m-2)+1-2k-\lfloor\frac{m}{2}\rfloor$) to obtain the edge-weights from the interval $[2i(m-1)-m+4, 2i(m-1)+2]$ (respectively, $[2i(m-1)+3, 2i(m-1)+m+1]$).

(x) We label the edges from the set A_{n-1} for n odd (respectively, from the set B_{n-1} for n even) by the label $2(n-1)(m-1)+2-2k$ to obtain the last $m-1$ edge-weights, i.e. the edge-weights from the interval $[2(n-1)(m-1)-m+4, |E(P_n \times P_m)|+2]$.

Now, it is not hard to see that all vertex and edge labels are at most k and the edge-weights of the edges from the sets A_i and B_i, $i=1,2,\ldots,n-1$, are pairwise distinct and create the integer interval $[3, |E(P_n \times P_m)|+2]$. Thus, the resulting total labeling is desired edge irregular k-labeling.

Case 2. m and n have a different parity, $2 \le m < n$

For $2 \le m < n$, define the function ψ in the following way:

$$\psi((x_i,y_j)) = \begin{cases} \frac{j}{2}, & \text{if } i=1 \text{ and } j \text{ is even} \\ \left(\lceil\frac{i}{2}\rceil-1\right)m+\frac{j+1}{2}, & \text{if } 1\le i\le\lfloor\frac{n}{2}\rfloor \text{ and } j \text{ is odd} \\ \lceil\frac{m}{2}\rceil+\left(\lfloor\frac{i}{2}\rfloor-1\right)m+\frac{j}{2}, & \text{if } 2\le i\le\lfloor\frac{n}{2}\rfloor \text{ and } j \text{ is even} \\ k-\lfloor\frac{n-i-1}{2}\rfloor m-m+\lceil\frac{j}{2}\rceil, & \text{if } \lfloor\frac{n}{2}\rfloor+1\le i\le n-1, \text{ and} \\ & j \text{ has the same parity as } m \\ k-\frac{m-j}{2}, & \text{if } i=n, \text{ and} \\ & j \text{ has the same parity as } m \\ k-\lfloor\frac{n-i}{2}\rfloor m-\frac{m-j-1}{2}, & \text{if } \lfloor\frac{n}{2}\rfloor+1\le i\le n, \text{ and} \\ & j \text{ has different parity as } m. \end{cases}$$

It is a matter for routine checking to see that under the vertex labeling ψ the edges from the sets A_i and B_i, for $i=1,2,\ldots,\lfloor\frac{n}{2}\rfloor-1$, receive the weights like in *(i)*, *(ii)*, *(iii)* and *(iv)* from *Case 1.* For another sets A_i and B_i, for $i=\lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{2}\rfloor+1,\ldots,n-1$, we have.

(v) The edges from the set $A_{\lfloor\frac{n}{2}\rfloor}$ for $\lfloor\frac{n}{2}\rfloor$ odd (respectively, $B_{\lfloor\frac{n}{2}\rfloor}$ for $\lfloor\frac{n}{2}\rfloor$ even) admit the consecutive integers from $k+2-m$ to k if m is even and from $k+2-\frac{m-1}{2}$ to $k+\frac{m+1}{2}$ if m is odd.

(vi) The edges from the set $B_{\lfloor\frac{n}{2}\rfloor}$ for $\lfloor\frac{n}{2}\rfloor$ odd (respectively, $A_{\lfloor\frac{n}{2}\rfloor}$ for $\lfloor\frac{n}{2}\rfloor$ even) admit the consecutive integers from $k+2-m$ to k if m is even and from $k+2-\frac{m-1}{2}$ to $k+\frac{m+1}{2}$ if m is odd.

(vii) The edges from the set $A_{\lfloor \frac{n}{2} \rfloor+1}$ for $\lfloor \frac{n}{2} \rfloor$ even (respectively, $B_{\lfloor \frac{n}{2} \rfloor+1}$ for $\lfloor \frac{n}{2} \rfloor$ odd) admit the consecutive integers from $2k - m(n - \lfloor \frac{n}{2} \rfloor) + \lceil \frac{m}{2} \rceil + 2$ to $2k - m(n - 1 - \lfloor \frac{n}{2} \rfloor) + \lceil \frac{m}{2} \rceil$.

(viii) The edges from the set A_i and A_{i+1} receive the consecutive integers from $2k + m(i - n) + \lceil \frac{m}{2} \rceil + 2$ to $2k + m(i + 1 - n) + \lceil \frac{m}{2} \rceil$ for every m and $\lfloor \frac{n}{2} \rfloor < i < n - 1$ even.

(ix) The edges from the set B_i and B_{i+1} receive the consecutive integers from $2k + m(i - n) + \lceil \frac{m}{2} \rceil + 2$ to $2k + m(i + 1 - n) + \lceil \frac{m}{2} \rceil$ for every m and $\lfloor \frac{n}{2} \rfloor < i < n - 1$ odd.

(x) The edges from the set A_{n-1} for n odd (respectively, B_{n-1} for n even) admit the consecutive integers from $2k - m + 2$ to $2k$.

Now, we complete the edge labels and create a total labeling of the graph $P_n \times P_m$. The edges from the sets A_i and B_i, for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, we label by the same manner like in *(i)*, *(ii)*, *(iii)* and *(iv)* from *Case 1* and we obtain their edge-weights from the interval $[3, (2\lfloor \frac{n}{2} \rfloor - 2)(m - 1) + 2]$. Another edges from the sets A_i and B_i, for $i = \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, \ldots, n - 1$, we label as follows.

(v) Each edge from the set $A_{\lfloor \frac{n}{2} \rfloor}$ for $\lfloor \frac{n}{2} \rfloor$ odd (respectively, $B_{\lfloor \frac{n}{2} \rfloor}$ for $\lfloor \frac{n}{2} \rfloor$ even) we label by the label $(2\lfloor \frac{n}{2} \rfloor - 1)(m-1)+2-k$ if m is even and by the label $(2\lfloor \frac{n}{2} \rfloor - 2)(m-1)+1-k+\frac{m-1}{2}$ if m is odd. Thus, we obtain the edge-weights from the interval $[(2\lfloor \frac{n}{2} \rfloor - 2)(m - 1) + 3, (2\lfloor \frac{n}{2} \rfloor - 1)(m - 1) + 2]$.

(vi) The edges from the set $B_{\lfloor \frac{n}{2} \rfloor}$ for $\lfloor \frac{n}{2} \rfloor$ odd (respectively, $A_{\lfloor \frac{n}{2} \rfloor}$ for $\lfloor \frac{n}{2} \rfloor$ even) we label by $2\lfloor \frac{n}{2} \rfloor(m - 1) + 2 - k$ if m is even and by $(2\lfloor \frac{n}{2} \rfloor - 1)(m - 1) + 1 - k + \frac{m-1}{2}$ if m is odd. We receive the edge-weights from the interval $[(2\lfloor \frac{n}{2} \rfloor - 1)(m - 1) + 3, 2\lfloor \frac{n}{2} \rfloor(m - 1) + 2]$.

(vii) We label the edges from the set $A_{\lfloor \frac{n}{2} \rfloor+1}$ for $\lfloor \frac{n}{2} \rfloor$ even (respectively, $B_{\lfloor \frac{n}{2} \rfloor+1}$ for $\lfloor \frac{n}{2} \rfloor$ odd) by $\lfloor \frac{n}{2} \rfloor(m - 2) + mn + 1 - 2k - \lceil \frac{m}{2} \rceil$ to create the edge-weights from the interval $[2\lfloor \frac{n}{2} \rfloor(m - 1) + 3, (2\lfloor \frac{n}{2} \rfloor + 1)(m - 1) + 2]$.

(viii) For $\lfloor \frac{n}{2} \rfloor < i < n - 1$ even, we label the edges from the set A_i (respectively, A_{i+1}) by the label $m(n - 1) + i(m - 2) + 2 - 2k - \lceil \frac{m}{2} \rceil$ (respectively, by the label $mn + i(m - 2) + 1 - 2k - \lceil \frac{m}{2} \rceil$) to create the edge-weights from the interval $[2i(m - 1) - m + 4, 2i(m - 1) + 2]$ (respectively, $[2i(m - 1) + 3, 2i(m - 1) + m + 1]$).

(ix) For $\lfloor \frac{n}{2} \rfloor < i < n - 1$ odd, we label the edges from the set B_i (respectively, B_{i+1}) by the label $m(n - 1) + i(m - 2) + 2 - 2k - \lceil \frac{m}{2} \rceil$ (respectively, by the label $mn + i(m - 2) + 1 - 2k - \lceil \frac{m}{2} \rceil$) to obtain the edge-weights from the interval $[2i(m - 1) - m + 4, 2i(m - 1) + 2]$ (respectively,

$[2i(m-1)+3, 2i(m-1)+m+1]$).

(x) The last $m-1$ edges from the set A_{n-1} for n odd (respectively, from the set B_{n-1} for n even) we label by the same manner like in previous case and for edge-weights we obtain the same interval.

As in *Case 1* it is straightforward to see that the resulting total labeling has the required properties. This concludes the proof. □

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