On the lexicographical ordering by spectral moments of bicyclic graphs

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Abstract. A graph G of order n is called a bicyclic graph if G is connected and the number of edges of G is $n+1$. In this paper, we study the lexicographic ordering of bicyclic graphs by spectral moments. For each of the three basic types of bicyclic graphs on a fixed number of vertices maximal and minimal graphs in the mentioned order are determined.

Keywords: bicyclic graph, spectral moment, lexicographical order.

AMS: 05C50

1 Introduction

In this paper, we will only consider simple graphs. We will generally follow [1] for undefined notation and terminology. The path and cycle with n vertices are denoted by P_n and C_n, respectively. The minimum length of a cycle (contained) in a graph G is the girth $g(G)$ of G. Suppose $H \subseteq G$ and let $N_G(H)$ be the number of subgraphs of G, which isomorphic to H.

Let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ be the eigenvalues in non-increasing order of a graph G. The number $\sum_{i=1}^{n} \lambda_i^k(G)$ $(k = 0, 1, \ldots, n-1)$ is called the kth spectral moment of G, denoted by $S_k(G)$. Let $S(G) = (S_0(G), S_1(G), \ldots, S_{n-1}(G))$ be the sequence of spectral moments of G. For two graphs G_1,

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G_2 we shall write $G_1 =_S G_2$ if $S_i(G_1) = S_i(G_2)$ for $i = 0, 1, \ldots, n-1$. Similarly, we have $G_1 \prec_S G_2$ (G_1 comes before G_2 in an S-order) if for some $k(k = 1, 2, \ldots, n-1)$ we have $S_i(G_1) = S_i(G_2)(i = 0, \ldots, k-1)$ and $S_k(G_1) < S_k(G_2)$. We shall also write $G_1 \preceq_S G_2$ if $G_1 \prec_S G_2$ or $G_1 =_S G_2$.

D. Cvetković et al [3] present a catalogue of the 236(connected) bicyclic graphs on eight vertices. Up to now, few results on the S- order of graphs are obtained. D. Cvetković et al [4] obtained the following results.

Theorem 1 *([4]) In an S-order of trees on n vertices, the first graph is the path P_n and the last graph is the star $K_{1,n-1}$.*

A graph G of order n is called a unicyclic graph if G is connected and the number of edges of G is n. Let $\mathcal{U}(n)$ be the set of all unicyclic graphs on n vertices. The set of unicyclic graphs on $e+f$ vertices which contain a cycle C_e will be denoted by U_{ef}. Let E_{ef} be the graph obtained by the coalescence of a cycle C_e with a path P_{f+1} at one of its end vertices. Let F_{ef} be the graph obtained by the coalescence of a cycle C_e and a star $K_{1,f}$ at its central vertex.

Theorem 2 *([4]) In an S-order of U_{ef}, the first graph is E_{ef} and the last graph is F_{ef}.*

Theorem 3 *In an S-order of graphs in $\mathcal{U}(n)$, the first graph is C_n and the last graph is $F_{3,n-3}$.*

Proof. Suppose that $G \in \mathcal{U}(n)$ and $G \neq C_n$. Since $N_{C_n}(P_3) = n$ and $N_G(P_3) \geq n+1$, $S_4(C_n) < S_4(G)$. Hence $C_n \prec_S G$. If $S_3(G) = 6$, then $G \in U_{3,n-3}$. By Theorem 2, $F_{3,n-3}$ is the last graph in $\mathcal{U}(n)$. □

Bicyclic graphs are connected in which the number of edges equals the number of vertices plus one. Let $\mathcal{B}(n)$ be the set of all bicyclic graphs on n vertices. In this paper, we study the S- order of bicyclic graphs. We will determine the first and the last graphs in the S-order in the class $\mathcal{B}(n)$.

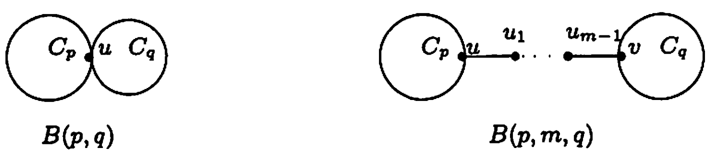


Fig. 2.1: $B(p,q)$ and $B(p,m,q)$.

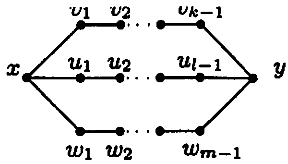


Fig. 2.2: $B(P_{k+1}, P_{l+1}, P_{m+1})$.

2 Three classes of bicyclic graphs and some basic lemmas

A graph G of order n is called a bicyclic graph if G is connected and the number of edges of G is $n+1$.

It is easy to see from the definition that G is a bicyclic graph if and only if G can be obtained from a tree T (with the same order) by adding two new edges to T.

A pendant vertex of a graph is a vertex of degree 1.

Let G be a bicyclic graph. The base of G, denoted by $\widehat{G}$, is the (unique) minimal bicyclic subgraph of G.

It is easy to see that $\widehat{G}$ is the unique subgraph of G containing no pendant vertices, while G can be obtained from $\widehat{G}$ by attaching trees to some vertices of $\widehat{G}$. It is well known that there are the following three types of bicyclic graphs containing no pendant vertices [5]:

Let $B(p,q)$ be the bicyclic graph obtained from two vertex-disjoint cycles C_p and C_q by identifying vertices u of C_p and v of C_q(see Fig 2.1).

Let $B(p,m,q)$ be the bicyclic graph obtained from two vertex-disjoint cycles C_p and C_q by joining vertices u of C_p and v of C_q by a new path $uu_1u_2\dots u_{m-1}v$ with length $m(m \geq 1)$(see Fig 2.1).

Let $B(P_{k+1}, P_{l+1}, P_{m+1})(1 \leq m \leq \min\{k,l\})$ be the bicyclic graph obtained from three pairwise disjoint paths from a vertex x to a vertex y. These three paths are $xv_1v_2\dots v_{k-1}y$ with length k, $xu_1u_2\dots u_{l-1}y$ with length l and $xw_1w_2\dots w_{m-1}y$ with length m(see Fig 2.2).

Now we can define the following three classes of bicyclic graphs of order n:

$$\mathcal{B}_1(n) = \{G \in \mathcal{B}(n) | \widehat{G} = B(p,q) \text{ for some } p \geq 3 \text{ and } q \geq 3\},$$

$$\mathcal{B}_2(n) = \{G \in \mathcal{B}(n) | \widehat{G} = B(p,m,q) \text{ for some } p \geq 3, q \geq 3 \text{ and } m \geq 1\},$$

$\mathcal{B}_3(n) = \{G \in \mathcal{B}(n) | \widehat{G} = B(P_{k+1}, P_{l+1}, P_{m+1})$ for some $1 \leq m \leq \min\{k,l\}\}$.

The union of graphs $G_1, \dots, G_k$, written $G_1 \cup \dots \cup G_k$, is the graph with vertex set $\cup_{i=1}^{k} V(G_i)$ and edge set $\cup_{i=1}^{k} E(G_i)$. Further, we write

$G_1 \uplus \ldots \uplus G_k$ to denote $G_1 \cup \ldots \cup G_k$ with constrains that $V(G_i) \cap V(G_j) = \emptyset, 1 \leq i \neq j \leq k$. It is easy to see that

$\mathcal{B}(n) = \mathcal{B}_1(n) \uplus \mathcal{B}_2(n) \uplus \mathcal{B}_3(n)$. Furthermore, $\mathcal{B}_i(n), i = 1, 2, 3$, consists of two types of graphs: one type, denoted by $\mathcal{B}_i^+(n)$, are those graphs whose bases are spanning subgraphs; the other type, denoted by $\mathcal{B}_i^{++}(n)$, are those graphs whose bases are not spanning subgraphs.

Now we quote some basic lemmas which will be used in the proofs of our main results.

Lemma 1 *([2]) The kth spectral moment of G is equal the number of closed walks of length k.*

Lemma 2 *([2]) For every graph, we have $S_0 = n, S_1 = l, S_2 = 2m, S_3 = 6t, S_4 = 2m+4p+8q$, where n, l, m, t, p, q denote the number of vertices, the number of loops, the number of edges, the number of triangles, the number of pairwise adjacent edges and the number of quadrangles of G, respectively.*

Lemma 3 *([6]) Suppose that N is a positive integer. The number of partitions of N divided into r ordered parts with repetitions is $\binom{N-1}{r-1}$.*

Let G_0 be a minimal bicyclic graph and $|V(G_0)| = l (4 \leq l \leq n)$. Suppose that u, v are two vertices of G_0 with $d(u) = \delta(G_0)$ and $d(v) = \Delta(G_0)$. Let G_0^* be the graph obtained from G_0 by attaching a new path $uu_1 \ldots u_{n-l}$ at u. Suppose that v is the central vertex of star $K_{1,n-l}$. Let G_0^{**} be the graph obtained from G_0 by attaching $K_{1,n-l}$ at v.

Lemma 4 *Suppose $G \in \mathcal{B}(n)$. If $\widehat{G} = G_0$, then $G_0^* \preceq_S G \preceq_S G_0^{**}$.*

Proof. Set $|V(G_0)| = n - m$ and $G_0' = G_0$. Let G_i be obtained from G_{i-1} by joining u_{i-1}(such that $d_{G_{i-1}}(u_{i-1}) = \delta(G_{i-1})$) to an isolated vertex w_{i-1}, and G_i' be obtained from G_{i-1}' by joining v_{i-1}(such that $d_{G_{i-1}'}(v_{i-1}) = \Delta(G_{i-1}')$) to an isolated vertex w_{i-1}, $i = 1, \ldots, m$. Then $N_{G_i}(P_3) = N_{G_{i-1}}(P_3) + \delta(G_{i-1})$ and $N_{G_i'}(P_3) = N_{G_{i-1}'}(P_3) + \Delta(G_{i-1})$, $i = 1, \ldots, m$. By Lemma 2, $S_4(G_i) < S_4(G_i')$, $i = 1, \ldots, m$. Thus $S_4(G_m) \leq S_4(G) \leq S_4(G_m')$. Hence $G_m \preceq_S G \preceq_S G_m'$. By definitions of G_0^* and G_0^{**}, $G_0^* = G_m$ and $G_0^{**} = G_m'$. Hence, Lemma 4 is true. □

Lemma 5 *Suppose that $G_i \in \mathcal{B}_i^+(n)$ and $G_i' \in \mathcal{B}_i^{++}(n)$, $i = 1, 2, 3$. If $\min\{g(G_i), g(G_i')\} \geq 5$, then $G_i \prec_S G_i'$, $i = 1, 2, 3$.*

Proof. Since $S_j(G_i) = S_j(G_i'), j \in \{0, 1, 2, 3\}$, it suffices to show that $S_4(G_i') > S_4(G_i), i = 1, 2, 3$. By Lemma 4, $\widehat{G_i'}^* \preceq_S G_i'$ and $S_4(G_i') \geq S_4(\widehat{G_i'}^*), i = 1, 2, 3$. Since $N_{G_1}(P_3) = n+5$ and $N_{\widehat{G_1'}^*}(P_3) = n+6$, $S_4(G_1) =$

$6n+22$ and $S_4(\widehat{G_1'}^*) = 6n+26$. Since $N_{G_i}(P_3) = n+4$ and $N_{\widehat{G_i'}^*}(P_3) = n+5$, $S_4(G_i) = 6n + 18$ and $S_4(\widehat{G_i'}^*) = 6n + 22, i = 2,3$. Hence $S_4(G_i) < S_4(\widehat{G_i'}^*) \le S_4(G_i)$, $i = 1,2,3$. Therefore, Lemma 5 is true. □

3 Main results

A graph H' which is obtained from a graph H by replacing some edges of H with independent paths between their vertices is called a subdivision of H. Let $TH = \{H' | H' \text{ is a subdivision of } H\}$. Define

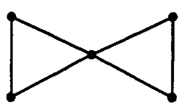
$$\mathcal{T}_k(G) = \{T | T \subset G, T \in TK_{1,3}, |E(T)| \le k\},$$
$$\mathcal{T}'_k(G) = \{T | T \subset G, T \in TK_{1,4}, |E(T)| \le k\}.$$

Define

$$X_i(G) = \{v | v \in V(G), G \text{ has four } P_{i+1} \text{ with } v \text{ as an end-vertex .}\}$$
$$Y_i(G) = \{v | v \in V(G), G \text{ has three } P_{i+1} \text{ with } v \text{ as an end-vertex .}\}$$
$$Z_i(G) = \{v | v \in V(G), G \text{ has two } P_{i+1} \text{ with } v \text{ as an end-vertex .}\}$$

If $G \in \mathcal{B}_1^+(n) \cup \mathcal{B}_2^+(n) \cup \mathcal{B}_3^+(n)$ and $i \le \lfloor \frac{g(G)}{2} \rfloor$, then

$$N_G(P_{i+1}) = \frac{4|X_i(G)| + 3|Y_i(G)| + 2|Z_i(G)|}{2} = \frac{2n + 2|X_i(G)| + |Y_i(G)|}{2} \tag{1}$$



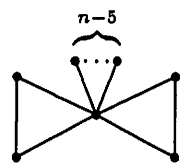


Fig. 3.1 $B(3,3)$ and $B^{**}(3,3)$

Theorem 4 *In an S-order of graphs in* $\mathcal{B}_1(n)$, $B(\lfloor \frac{n+1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil)$ *is the first graph and* $B^{**}(3,3)$ *is the last graph (see Fig. 3.1).*

Proof. By Lemma 5, It suffices to show that $B(\lfloor \frac{n+1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil)$ is the first graph of $\mathcal{B}_1^+(n)$.

Claim 1 Suppose $G_i \in \mathcal{B}_1^+(n), i = 1,2$. If $g(G_1) < g(G_2)$, then $G_2 \prec_S G_1$.

By Lemma 1, $S_k(G_i)$ are only related to the numbers of connected subgraphs (such that the numbers of edges of them are at most $\frac{k}{2}$) in G_i, $i = 1,2$. First suppose $k < g(G_1)$. Since tree subgraphs only generate even closed walks, $S_k(G_1) = S_k(G_2) = 0$ when k is odd. Furthermore, for each tree $T \subseteq G$ with $|E(T)| \leq \frac{k}{2}$, T can generate some closed walks of even length k. These tree subgraphs of G_j are: paths $P_{i+1} (i \leq \frac{k}{2})$, trees $T \in \mathcal{T}_{\frac{k}{2}}(G_j)$ and $T' \in \mathcal{T}'_{\frac{k}{2}}(G_j), j = 1,2$.

Since $i \leq \frac{k}{2}$, $|X_i(G_j)| = 4(i-1)+1, |Y_i(G_j)| = 0$ for $j \in \{1,2\}$. By (1),

$$N_{G_j}(P_{i+1}) = n + 4i - 3, j = 1,2. \tag{2}$$

Since $\frac{k}{2} \leq \lfloor \frac{g(G_1)}{2} \rfloor$,

$$\mathcal{T}_{\frac{k}{2}}(G_1) = \mathcal{T}_{\frac{k}{2}}(G_2), \mathcal{T}'_{\frac{k}{2}}(G_1) = \mathcal{T}'_{\frac{k}{2}}(G_2). \tag{3}$$

And for each $T \in \mathcal{T}_{\frac{k}{2}}(G_j)$, $T' \in \mathcal{T}'_{\frac{k}{2}}(G_j)$, by Lemma 3, we have

$$N_{G_j}(T) = 4\binom{|E(T)|-1}{2}, N_{G_j}(T') = \binom{|E(T')|-1}{3}, j = 1,2. \tag{4}$$

By (2),(3) and (4), when $k < g(G_1)$ and k is even, $S_k(G_1) = S_k(G_2)$.

Now suppose $k \geq g(G_1)$, and we consider two cases.

Case 1: $g(G_1)$ is odd.

Since C_k can generate $2k$ closed walks of length k, $S_{g(G_1)}(G_1) = 2g(G_1)$. Since $g(G_1) < g(G_2)$, $S_{g(G_1)}(G_2) = 0$. So $S_{g(G_1)}(G_1) > S_{g(G_1)}(G_2)$. Hence $G_2 \prec_S G_1$.

Case 2: $g(G_1)$ is even.

Since $N_{G_1}(P_{\frac{g(G_1)}{2}+1}) = \frac{2n+2[g(G_1)-1+2\times(\frac{g(G_1)}{2}-1)]}{2} = n + 2g(G_1) - 3$, when $i \leq \frac{g(G_1)}{2}$, we have

$$N_{G_j}(P_{i+1}) = n + 4i - 3, j = 1,2. \tag{5}$$

By (3),(4) and (5), $S_{g(G_1)}(G_1) = S_{g(G_1)}(G_2) + 2g(G_1)$. Thus $G_2 \prec_S G_1$. Hence Claim 1 is true.

If $G \in \mathcal{B}_1^+(n)$, then $g(G) \leq \frac{n+1}{2}$. $B(\lfloor \frac{n+1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil)$ is the only graph in $\mathcal{B}_1^+(n)$ with girth $\lfloor \frac{n+1}{2} \rfloor$. By Lemma 5 and Claim 1, $B(\lfloor \frac{n+1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil)$ is the first graph in an S-order of graphs in $\mathcal{B}_1(n)$.

For every $G \in \mathcal{B}_1(n)$, $S_3(G) \leq 12$. If $S_3(G) = 12$, then $\widehat{G} = B(3,3)$. By Lemma 4, $G \preceq_S B^{**}(3,3)$. So $B^{**}(3,3)$ is the last graph of $\mathcal{B}_1(n)$. Hence we complete the proof of Theorem 4. □

Define $\mathcal{H}_{m,l}(G) = \{H | H \subset G, H$ is obtained from P_{m+1} through attaching its two end vertices to an inner vertex (i.e. not its end vertex) of P_{i+1} and P_{j+1}, respectively, $|E(H)| \le l\}$. Define $\mathcal{G}^+_{2,m} = \{G | G \in \mathcal{B}^+_2(n), \widehat{G} = B(p,q,m)\}$ and $\mathcal{G}^+_{3,m} = \{G | G \in \mathcal{B}^+_3(n), \widehat{G} = B(P_{k+1}, P_{l+1}, P_{m+1})\}$.

Lemma 6 *Suppose $G_1, G_2 \in \mathcal{G}^+_{j,m}, j = 2,3$. If $g(G_1) < g(G_2)$, then $G_2 \prec_S G_1$.*

Proof. First suppose that $k < g(G_1)$ and k is even. Tree subgraphs of $G_j, j = 1,2$, which can generate closed walks of even length k are : paths $P_{i+1} (i \le \frac{k}{2})$, trees $T \in \mathcal{T}_{\frac{k}{2}}(G_j)$ and $H \in \mathcal{H}_{m,\frac{k}{2}}(G_j)$.

When $\frac{k}{2} \le \lfloor \frac{g(G_1)}{2} \rfloor$, we have

$$\mathcal{T}_{\frac{k}{2}}(G_1) = \mathcal{T}_{\frac{k}{2}}(G_2), \mathcal{H}_{m,\frac{k}{2}}(G_1) = \mathcal{H}_{m,\frac{k}{2}}(G_2). \tag{6}$$

For each $T \in \mathcal{T}_{\frac{k}{2}}(G_j)$, by Lemma 3, if $|E(T)| \le m+2$, then

$$N_{G_j}(T) = 2\binom{|E(T)|-1}{2}, j = 1,2. \tag{7}$$

And if $|E(T)| > m+2$, then

$$N_{G_j}(T) = 2\binom{|E(T)|-1}{2} + 2\binom{|E(T)|-m-1}{2}, j = 1,2. \tag{8}$$

For each $H \in \mathcal{H}_{m,\frac{k}{2}}(G_j)$,

$$N_{G_j}(H) = \binom{|E(H)|-m-1}{3}, j = 1,2. \tag{9}$$

By (7) and (8), $N_{G_j}(T)$ is only related to m and the number of edge of T; by (9), $N_{G_j}(H)$ is only related to m and the number of edge of H.

Claim 2 Suppose $G \in \mathcal{G}^+_{j,m}, j = 2,3$. If $i \le \lfloor \frac{g(G)}{2} \rfloor$, then

$$N_G(P_{i+1}) = \begin{cases} n+3i-2, & i-1 \le m \\ n+4i-m-3. & i-1 > m \end{cases} \tag{10}$$

By (1), it suffices to calculate $|X_i(G)|$ and $|Y_i(G)|$ for $G \in \mathcal{G}^+_{j,m}, j = 2,3$.

First suppose $G \in \mathcal{G}^+_{2,m}$. Let $G = B(p,q,m)$.

If $m+1 \ge 2i$, then $|X_i(G)| = 0$, $|Y_i(G)| = 4(i-1) + 2i$.

If $i \le m+1 < 2i$, then $|X_i(G)| = 2i - (m+1)$, $|Y_i(G)| = 4(i-1) + 2(m+1) - 2i$.

If $m+1 < i$, then $|X_i(G)| = 4[i-(m+1)] + m + 1, |Y_i(G)| = 4m$.

Now suppose $G \in \mathcal{G}_{3,m}^{+}$. Suppose $G = B(P_{k+1}, P_{l+1}, P_{m+1})$ and $1 \le m \le l \le k$. Suppose $i \le \lfloor \frac{m+l}{2} \rfloor$. In what follows, we consider two cases.

Case 1: $2(i-1) \le l-1$.

If $m+1 \ge 2i$, then $|X_i(G)| = 0$, $|Y_i(G)| = 6(i-1)+2$.
If $i \le m+1 < 2i$, then $|X_i(G)| = 2i-(m+1)$, $|Y_i(G)| = 2m+2i-2$.
If $m+1 < i$, then $|X_i(G)| = 4i-3(m+1), |Y_i(G)| = 4m$.

Case 2: $l-1 < 2(i-1)$.

First suppose $2(i-1) \le k-1$. Let $a_1 = 2i-1-l$, $a_2 = 2i-m-1$.
If $m+1 \ge 2i$, then $|X_i(G)| = a_1, |Y_i(G)| = 2(i-1-a_1)+4i-2$.
If $i \le m+1 < 2i$, then $|X_i(G)| = a_1+a_2, |Y_i(G)| = 2(i-1-a_1)+(m+1-a_2)+2(i-1)$.
If $m+1 < i$, then $|X_i(G)| = (m+1)+4[i-(m+1)]+a_1, |Y_i(G)| = 2(m-a_1)$.

Now suppose $2(i-1) > k-1$. Let $a_3 = 2i-1-k$.
If $m \ge 2i$, then $|X_i(G)| = a_1+a_3, |Y_i(G)| = 2(i-1-a_1)+2(i-1-a_3)+2i$.
If $i \le m+1 < 2i$, then $|X_i(G)| = a_1+a_2+a_3, |Y_i(G)| = 2(i-1-a_1)+(m+1-a_2)+2(i-1-a_3)$.
If $m+1 < i$, then $|X_i(G)| = (m+1)+4[(i-(m+1)]+a_1+a_3, |Y_i(G)| = 2(m-a_1)+2(m-a_3)$.

Using (1), we thus obtain the values of $N_G(P_{i+1})$ for every case. Hence Claim 2 is true.

Since $g(G_1) < g(G_2)$, when $k < g(G_1)$, by (6)-(10), we have $S_k(G_1) = S_k(G_2)$ for $G_i \in \mathcal{G}_{j,m}^{+}, i = 1, 2, j \in \{2, 3\}$.

Now suppose $k \ge g(G_1)$. By (6)-(10), $S_{g(G_1)}(G_1) = S_{g(G_2)}(G_2) + 2g(G_1)$. Hence Lemma 6 is proved. □

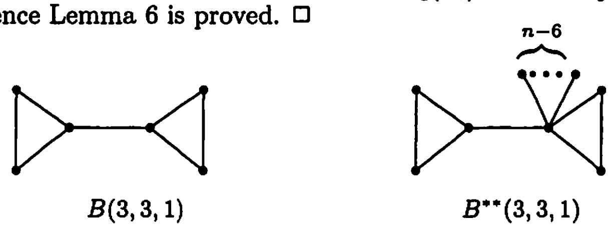


Fig. 3.2 $B(3,3,1)$ and $B^{**}(3,3,1)$

Theorem 5 *In an S-order of graphs in $\mathcal{G}_{2,m}^{+}$, $B(\lfloor \frac{n-m+1}{2} \rfloor, \lceil \frac{n-m+1}{2} \rceil, m)$ is the first graph. And $B^{**}(3,3,1)$ (see Fig.3.2) is the last graph in an S-order of graphs in $\mathcal{B}_2(n)$.*

Proof. Suppose $G = B(p,q,m) \in \mathcal{G}_{2,m}^{+}$. Then $g(G) \le \lfloor \frac{n-m+1}{2} \rfloor$. $B(\lfloor \frac{n-m+1}{2} \rfloor, \lceil \frac{n-m+1}{2} \rceil, m)$ is the only graph with girth $\lfloor \frac{n-m+1}{2} \rfloor$ in $\mathcal{G}_{2,m}^{+}$. By Lemma 6,

$$B(\lfloor \frac{n-m+1}{2} \rfloor, \lceil \frac{n-m+1}{2} \rceil, m) \preceq_S G.$$

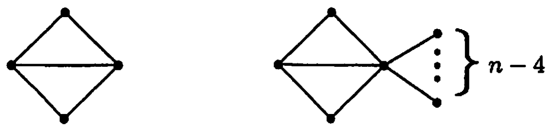
Claim 3 Suppose that $B(3,3,m_i)$ are two minimal bicyclic graphs, $i = 1,2$. If $1 \le m_1 < m_2$, then $B^{**}(3,3,m_2) \prec_S B^{**}(3,3,m_1) \preceq_S B^{**}(3,3,1)$.

By Lemma 2, $S_i(B^{**}(3,3,m_1)) = S_i(B^{**}(3,3,m_2))$ for $i \in \{0,1,2,3\}$. The number of pairs of adjacent edges of $B^{**}(3,3,m_i)$ are $\binom{n+8-m_i}{2} + 6 - m_i, i = 1,2$. When $m_1 < m_2$, we have

$$\binom{n+8-m_1}{2} + 6 - m_1 > \binom{n+8-m_2}{2} + 6 - m_2.$$

Thus $S_4(B^{**}(3,3,m_1)) \ge S_4(B^{**}(3,3,m_2))$. Since $m_i \ge 1$, $B^{**}(3,3,m_i) \preceq_S B^{**}(3,3,1)$. Hence we complete the proof of Claim 3.

For $G \in \mathcal{B}_2^{++}(n)$, $S_3(G) = 12$ only when $\widehat{G} = B(3,3,m_i)$. By Lemma 4 and Claim 3, $B^{**}(3,3,1)$ is the last graph in an S-order of graphs in $\mathcal{B}_2(n)$. Hence we complete the proof of Theorem 5. □



$B(P_3, P_3, P_2)$ $B^{**}(P_3, P_3, P_2)$

Fig. 3.3 $B(P_3,P_3,P_2)$ and $B^{**}(P_3,P_3,P_2)$

Theorem 6 *In an S-order of graphs in $\mathcal{B}(n)$, the first graph is $B(P_{\lceil\frac{n-2}{3}\rceil+2}, P_{n-\lceil\frac{n-2}{3}\rceil-\lfloor\frac{n-2}{3}\rfloor}, P_{\lfloor\frac{n-2}{3}\rfloor+2})$ and the last graph is $B^{**}(P_3,P_3,P_2)$ (Fig. 3.3).*

Proof. Let $G = B(P_{k+1}, P_{l+1}, P_{m+1}) \in \mathcal{B}_3^{+}(n)$, $1 \le m \le l \le k$. Since $g(G) \le \lfloor\frac{n+m+1}{2}\rfloor$, by Lemma 6, $B(P_{\lfloor\frac{n+m+1}{2}\rfloor-m+1}, P_{\lceil\frac{n+m+1}{2}\rceil-m+1}, P_{m+1})$ is the first graph in an S-order of graphs in $\mathcal{G}_{3,m}^{+}$. Since $m \le \lfloor\frac{n-2}{3}\rfloor + 1$, by (7)-(10), $G_0 = B(P_{\lceil\frac{n-2}{3}\rceil+2}, P_{n-\lceil\frac{n-2}{3}\rceil-\lfloor\frac{n-2}{3}\rfloor}, P_{\lfloor\frac{n-2}{3}\rfloor+2})$ is the first graph in an S-order of graphs in $\mathcal{B}_3^{+}(n)$.

As in the proof of Lemma 5, the first graph must be in $\mathcal{B}_2(n)$ or $\mathcal{B}_3(n)$. If $G \in \mathcal{B}_2(n)$, then $g(G) \le \lfloor\frac{n}{2}\rfloor$. Since $\lfloor\frac{n}{2}\rfloor < n - \lceil\frac{n-2}{3}\rceil$, $g(G) < g(G_0)$, where $g(G_0) = n - \lceil\frac{n-2}{3}\rceil$.

Now we compare $S_k(G_0)$ with $S_k(G)$ such that $k < g(G)$ and k is even. Let $m_0 = \lfloor\frac{n-2}{3}\rfloor + 1$. Then $\mathcal{H}_{m_0,\frac{k}{2}}(G_0) = \emptyset$. Since $k < \min\{g(G_0), g(G)\}$, $\mathcal{T}_{\frac{k}{2}}(G_0) = \mathcal{T}_{\frac{k}{2}}(G)$. For every $T \in \mathcal{T}_{\frac{k}{2}}(G_0)$, by (7) and (8), $N_G(T) \ge N_{G_0}(T)$. When $i \le \lfloor\frac{g(G)}{2}\rfloor < m_0$, by (10), we have $N_G(P_{i+1}) \ge N_{G_0}(P_{i+1})$. Then, $S_k(G_0) \le S_k(G)$. Since $g(G) < g(G_0)$, $S_{g(G)}(G_0) < S_{g(G)}(G)$. Hence G_0 is the first graph in an S- order of bicyclic graphs.

For each $G \in \mathcal{B}_3^{++}(n)$, $S_3(G) = 12$ only when $\widehat{G} = B(P_3,P_3,P_2)$. By Lemma 4, $B^{**}(P_3,P_3,P_2)$ is the last graph of $\mathcal{B}_3(n)$. Then $B_1^{**}(3,3)$,

$B_2^{**}(3,3,1)$, $B_3^{**}(P_3,P_3,P_2)$ are the last graph of $\mathcal{B}_1(n)$, $\mathcal{B}_2(n)$, $\mathcal{B}_3(n)$, respectively. The number of pairs of adjacent edges of them are $\binom{n-1}{2}+4$, $\binom{n-1}{2}+5$, $\binom{n-3}{2}+7$, respectively. If $n \geq 4$, then $\binom{n-1}{2}+5 > \binom{n-3}{2}+7$. By Lemma 2, $B_3^{**}(P_3,P_3,P_2)$ is the last graph in an S- order of bicyclic graphs. Hence Theorem 6 is true. □

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