The Linear 2-Arboricity of Some Planar Graphs *

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Abstract

Let G be a planar graph with maximum degree $\Delta(G)$. The least integer k such that G can be partitioned into k edge disjoint forests, whose each component is a path of length at most 2, is called the linear 2-arboricity of G, which is denoted by $la_2(G)$. we give new upper bound of the linear 2-arboricity of some planar graphs.
Keywords: planar graph, linear arboricity, linear 2-arboricity.

1 Introduction

In this paper we consider finite simple graphs. Let G be a planar graph with maximum degree $\Delta(G)$ and minimum degree $\delta(G)$. The linear 2-arboricity of G is the least integer k such that G can be partitioned into k edge disjoint forests, whose component trees are paths of length at most 2. It is denoted by $la_2(G)$.

Lih, Tong and Wang [4] studied the linear 2-arboricity of a planar graph G with maximum degree Δ and girth g. For planar graph they gave the following upper bound of the linear 2-arboricity.

Theorem 1. [4] *If G is a planar graph, then $la_2(G) \leq \lceil (\Delta+1)/2 \rceil + 12$.*

Theorem 2. [4] *Let G be a planar graph with girth g. If $g \geq 4$, then $la_2(G) \leq \lceil (\Delta+1)/2 \rceil + 6$.*

For a planar graph without 5-cycles or without 6-cycles, Ma, Wu and Yu [5] got the following result.

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Theorem 3. [5] *If G is a planar graph without 5-cycles or without 6-cycles, then* $la_2(G) \leq \lceil(\Delta+1)/2\rceil + 6$.

For planar graphs without adjacent short cycles, Chen, Tan and Wu [3] got the following results.

Theorem 4. [3] *If G is a planar graph without adjacent 3-cycles, then* $la_2(G) \leq \lceil(\Delta)/2\rceil + 8$.

Theorem 5. [3] *If G is a planar graph without adjacent 4-cycles, then* $la_2(G) \leq \lceil(\Delta)/2\rceil + 10$.

Theorem 6. [3] *If G is a planar graph that any 3-cycle is not adjacent to a 4-cycle, then* $la_2(G) \leq \lceil(\Delta)/2\rceil + 6$.

In [7], the upper bound of $la_2(G)$ is improved to $la_2(G) \leq \lceil\Delta/2\rceil + 8$ if $\Delta = 0, 3(mod\ 4)$ and $la_2(G) \leq \lceil\Delta/2\rceil + 7$ if $\Delta = 1, 2(mod\ 4)$. In this note, we improve the upper bound of $la_2(G)$ in Theorem 2 $\sim$ Theorem 6.

2 The Main Results

To get our results, we need the following Lemmas.

Lemma 7. [4] *If a graph G can be edge-partitioned into m subgraphs* $G_1, G_2, \ldots, G_m$, *then* $la_2(G) \leq \sum_{i=1}^{m} la_2(G_i)$.

Lemma 8. [2] *For a forest T, we have* $la_2(T) \leq \lceil(\Delta(T)+1)/2\rceil$.

Lemma 9. [1] *For a graph G, we have* $la_2(G) \leq \Delta(G)$.

For $s \geq 2$, an even cycle $C = v_1v_2\ldots v_{2s}v_1$ is called a 2-alternating cycle if $d_G(v_1) = d_G(v_3) = \ldots = d_G(v_{2s-1}) = 2$.

Definition 10. A planar graph G is called a $(k,1)$-graph if for each of its nontrivial components H, one of the following holds:

(1) $\delta(H) = 1$;

(2) $\delta(H) \geq 2$ and there exists an edge $xy \in E(H)$ such that $d_G(x) + d_G(y) \leq k$ or there exists a 2-alternating cycle.

If G is a $(k,1)$-graph and each subgraph of G is also a $(k,1)$-graph, then G is called a $(k,1)$-hereditary graph.

Lemma 11. *Let G be a* $(9,1)$*-hereditary planar graph. Then G has an edge-partition into two forests* T_1, T_2 *and a subgraph H such that, for every* $v \in V(G)$:

(1) *If* $d_G(v) > 4$, *then* $d_{T_i}(v) \leq \lceil d_G(v)/2\rceil - 1$ $(i = 1, 2)$, *and* $d_H(v) \leq 4$.

(2) *If* $d_G(v) \leq 4$, *then* $d_{T_i}(v) \leq \min\{2, \lceil d_G(v)/2\rceil\}$ $(i = 1, 2)$.

Proof. We prove the result by induction on $|V(G)|+|E(G)|$. If $|V(G)|+|E(G)|\leq 5$, the result holds trivially. Now let $|V(G)|+|E(G)|\geq 6$. If $\Delta(G)\leq 4$, then let $H=G, T_1=T_2=\phi$, the result holds.

Suppose that $\Delta(G)\geq 5$, without loss of generality assume that G is connected. By induction, for any proper subgraph G' of G, G' has an edge-partition into two forests T_1', T_2' and a subgraph H', for every $v\in V(G')$, $d_{H'}(v)\leq 4$ and if $d_{G'}(v)>4$, then $d_{T_i'}(v)\leq\lceil d_{G'}(v)/2\rceil-1(i=1,2)$, if $d_{G'}(v)\leq 4$, then $d_{T_i'}(v)\leq\min\{2,\lceil d_{G'}(v)/2\rceil\}(i=1,2)$.

Case 1. $\delta(G)=1$. Let $uv\in E(G)$ with $d(u)=1$. Define $G'=G-uv$.

Subcase 1.1. $d_{H'}(v)\leq 3$. Let $H=H'+uv$, $T_i=T_i'$ ($i=1,2$).

If $d_G(v)>5$, then $d_{G'}(v)>4$, we get that $d_{T_i}(v)=d_{T_i'}(v)\leq\lceil d_{G'}(v)/2\rceil-1\leq\lceil d_G(v)/2\rceil-1$, $i=1,2$.

If $d_G(v)\leq 4$, then $d_{G'}(v)\leq 3$, we have $d_{T_i}(v)=d_{T_i'}(v)\leq\min\{2,\lceil d_{G'}(v)/2\rceil\}\leq\min\{2,\lceil d_G(v)/2\rceil\}$, $i=1,2$.

If $d_G(v)=5$, then $d_{G'}(v)=4$, we have $d_{T_i}(v)=d_{T_i'}(v)\leq min\{2,\lceil 4/2\rceil\}=2=\lceil 5/2\rceil-1$, $i=1,2$.

Subcase 1.2. $d_{H'}(v)$=4. We may suppose that $d_{T_1'}(v)\leq d_{T_2'}(v)$. Since $d_{G'}(v)=d_G(v)-1=d_{T_1'}(v)+d_{T_2'}(v)+4$, we have $d_{T_1'}(v)\leq(d_G(v)-5)/2$. Let $T_1=T_1'+uv$, $T_2=T_2'$ and $H=H'$. Note that $d_G(v)\geq 5$.

If $d_G(v)>5$, then $d_{T_1}(v)=1+d_{T_1'}(v)\leq 1+(d_G(v)-5)/2\leq\lceil d_G(v)/2\rceil-1$.

If $d_G(v)=5$, then $d_{T_1'}(v)=0$. $d_{T_1}(v)=1<\lceil d_G(v)/2\rceil-1$.

Since $d_G(u)=1$, $d_{T_1}(u)=1=\min\{2,\lceil d_G(u)/2\rceil\}$. Obviously, for all $x\in V(G)$, $x\neq\{u,v\}$, $d_{T_1}(x)=d_{T_1'}(x)$; and for all $x\in V(G)$, $d_{T_2}(x)=d_{T_2'}(x)$, $d_H(x)=d_{H'}(x)$.

Case 2. $\delta(G)\geq 2$. Based on the definition of (9,1)-hereditary graph, we consider two subcases.

Subcase 2.1. There is an edge $xy\in E(G)$ such that $d_G(x)+d_G(y)\leq 9$. Define $G'=G-xy$, and assume that $d_{H'}(x)\leq d_{H'}(y)$.

If $d_{H'}(y)\leq 3$, let $H=H'+xy$, $T_i=T_i', i=1,2$. Then with the same discussion as in $\delta(G)=1$, the lemma holds.

If $d_{H'}(y)=4$, then $1\leq d_{G'}(x)\leq 3$ and $d_{G'}(x)+d_{T_1'}(y)+d_{T_2'}(y)\leq 3$. We may assume that $d_{T_1'}(x)\leq d_{T_2'}(x)$. Based on the degree of $d_{G'}(x)$ we consider three cases.

(a) $d_{G'}(x)=3$, then $y\notin T_1'$, $y\notin T_2'$. Let $T_1=T_1'+xy$, $T_2=T_2'$, $H=H'$. Now $d_G(x)=4$, $d_{T_1}(x)\leq 2=\min\{2,\lceil d_G(x)/2\rceil\}$, $d_G(y)=5$, $d_{T_1}(y)=1<\lceil d_G(y)/2\rceil-1$.

(b) $d_{G'}(x)=2$, then $d_{T_i'}(x)\leq\min\{2,\lceil d_{G'}(x)/2\rceil\}=1$ $(i=1,2)$. Since $d_{T_1'}(y)+d_{T_2'}(y)\leq 1$, then $y\notin T_1'$ or $y\notin T_2'$. Suppose that $y\notin T_1'$ (the proof of the case $y\notin T_2'$ is similar). Let $T_1=T_1'+xy$, $T_2=T_2'$, $H=H'$. T_1 is a forest, $d_{T_1}(x)\leq 2=\min\{2,\lceil d_G(x)/2\rceil\}$, $d_G(y)\geq 5$, $d_{T_1}(y)=1<\lceil d_G(y)/2\rceil-1$.

(c) $d_{G'}(x) = 1$, then $x \notin T_1'$. Let $T_1 = T_1' + xy$, $T_2 = T_2'$, $H = H'$. Obviously, T_1 is a forest and $d_{T_1}(x) = 1 = \min\{2, \lceil d_G(x)/2 \rceil\}$, $d_{T_1'}(y) \leq 2$, $d_{T_1}(y)=d_{T_1'}(y) + 1$, and $5 \leq d_G(y) \leq 7$. If $d_{T_1'}(y) \leq 1$, then $d_{T_1}(y) \leq 2 \leq \lceil d_G(y)/2 \rceil - 1$. If $d_{T_1'}(y) = 2$, then $d_G(y) = 7$, $d_{T_1}(y) = 3 = \lceil 7/2 \rceil - 1$.

From the above discussion the lemma holds in case 2.1. In the following we assume that for all $xy \in E(G)$, $d_G(x) + d_G(y) > 9$.

Subcase 2.2. There is a 2-alternating cycle $C = v_1 v_2 \dots v_{2s} v_1$, $s \geq 2$, such that $d_G(v_1) = d_G(v_3) = \dots = d_G(v_{2s-1}) = 2$.

Define $G' = G - E(C)$. Let $T_1 = T_1' + \{v_1v_2, v_3v_4, \dots, v_{2s-1}v_{2s}\}$, $T_2 = T_2' + \{v_2v_3, v_4v_5, \dots, v_{2s}v_1\}$, and $H = H'$. Note that T_1 and T_2 are forests. For each $x \in V(C)$, $d_G(x) = d_{G'}(x) + 2$, and $d_{T_1}(v_j) = d_{T_2}(v_j) = 1 = \min\{2, \lceil d_G(v_j)/2 \rceil\}$, $j = 1, 3, \dots, 2s-1$. Since $d_G(v_j) > 7$, $d_{G'}(v_j) > 5, j = 2, 4, \dots, 2s$, we have $d_{T_i}(v_j) = d_{T_i'}(v_j) + 1 \leq \lceil d_{G'}(v_j)/2 \rceil - 1 + 1 = \lceil d_G(v_j)/2 \rceil - 1$, $i = 1, 2$.

Form the above discussion the result holds. □

Similarly, we can prove that.

Lemma 12. *Let G be a $(k,1)$-hereditary planar graph ($10 \leq k \leq 14$). Then G has an edge-partition into two forests T_1, T_2 and a subgraph H such that, for every $v \in V(G)$:*

(1) If $d_G(v) > k-5$, then $d_{T_i}(v) \leq \lceil d_G(v)/2 \rceil - \lceil k/2 \rceil + 4$ ($i = 1, 2$), and $d_H(v) \leq k-5$.

(2) If $d_G(v) \leq k-5$, then $d_{T_i}(v) \leq \min\{2, \lceil d_G(v)/2 \rceil\}$ ($i = 1, 2$).

By Lemma 11 and Lemma 12, we get the following corollary.

Corollary 13. *Let G be a $(k,1)$-hereditary planar graph ($9 \leq k \leq 14$) and $\Delta(G) > k-5$. Then G has an edge-partition into two forests T_1, T_2 and a subgraph H such that $\Delta(T_i) \leq \lceil \Delta(G)/2 \rceil - \lceil k/2 \rceil + 4$ ($i = 1, 2$) and $\Delta(H) \leq k-5$.*

Theorem 14. *Let G be a $(k,1)$-hereditary planar graph ($9 \leq k \leq 14$), then*

$$la_2(G) \leq \lceil \Delta(G)/2 \rceil + [k/2] + 1.$$

Proof. If $\Delta(G) \leq k-5$, the result hold trivially. Now suppose that $\Delta(G) \geq k-4$. By Corollary 13, and combining Lemma 7, 8, and 9, we get that:

$$\begin{aligned} la_2(G) &\leq la_2(T_1) + la_2(T_2) + la_2(H) \\ &\leq \lceil (\Delta(T_1)+1)/2 \rceil + \lceil (\Delta(T_2)+1)/2 \rceil + \Delta(H) \\ &\leq 2\lceil (\lceil \Delta(G)/2 \rceil - \lceil k/2 \rceil + 4 + 1)/2 \rceil + k - 5 \\ &\leq \lceil \Delta(G)/2 \rceil - \lceil k/2 \rceil + 6 + k - 5 \\ &= \lceil \Delta(G)/2 \rceil + [k/2] + 1. \end{aligned}$$

□

It is proved in [6], [5] and [3] that a planar graph without 3-cycles, or without 5-cycles or without 6-cycles, or a planar graph that any 3-cycle is not adjacent to a 4-cycle are (9,1)-hereditary graphs. So we have

Corollary 15. *Let G be a planar graph without 3-cycles, then* $la_2(G) \leq \lceil \Delta)/2 \rceil + 5$.

Corollary 16. *If G is a planar graph without 5-cycles or without 6-cycles, then* $la_2(G) \leq \lceil \Delta/2 \rceil + 5$.

Corollary 17. *If G is a planar graph that any 3-cylce is not adjacent to a 4-cycle, then* $la_2(G) \leq \lceil (\Delta)/2 \rceil + 5$.

It is proved in [3] that a planar graph without adjacent 3-cycles is a (11,1)-hereditary graph and a planar graph without adjacent 4-cycles is a (13,1)-hereditary graph. So we have

Corollary 18. *If G is a planar graph without adjacent 3-cycles, then* $la_2(G) \leq \lceil (\Delta)/2 \rceil + 6$.

Corollary 19. *If G is a planar graph without adjacent 4-cycles, then* $la_2(G) \leq \lceil (\Delta)/2 \rceil + 7$.

References

[1] J.C. Bermond, J.L. Fouquet, M. Habib, B. Péroche. On linear k-arboricity. Discrete Math., 52 (1984), 123-132

[2] B.L. Chen, H.L. Fu, K.C. Huang. Decomposing graphs into forests of paths with size less than three. Austral J Combin., 3 (1991), 55-73

[3] H.Y. Chen, X. Tan, J. Wu. The Linear 2-Arboricity of planar graphs without adjacent short cycles. Bull. Korean Math. Soc, 49 (2012), 145-154

[4] K.W. Lih, L.D. Tong, W.F. Wang. The Linear 2-Arboricity of planar Graphs. Graphs and Combinatorics, 19 (2003), 241-248

[5] Q. Ma, J.L. Wu, X. Yu. Planar graphs without 5-cycles or without 6-cycles. Discrete Math., 309 (2009), 2998-3005

[6] J.L. Wu. On the linear arboricity of planar graphs. Graphs Theory, 31(2) (1999), 129-134

[7] C.Q. Xu, L.S. An. An Upper Bound of the Linear 2-Arboricity of Planar Graph, submitted