G-designs without blocking sets *

NOTE

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Abstract

For an undirected graph G and a natural number n, a G-design of order n is an edge partition of the complete graph K_n with n vertices into subgraphs $G_1, G_2, \ldots$, each isomorphic to G. A set $T \subset V(K_n)$ is called a blocking set if it meets the vertex set $V(G_i)$ of each G_i in the decomposition, but contains none of them. In a previous paper [*J. Combin. Designs* 4 (1996), 135–142] the first and third authors proved that if G is a cycle, then there exists a G-design without blocking sets. Here we extend this theorem for all graphs G, moreover we prove that for every G and every integer $k \geq 2$ there exists a non-k-colorable G-design.

1 Introduction

Given a simple undirected graph G without isolated vertices, a G-*design* of order n is a pair $(X, \mathcal{B})$, where X is the vertex set of the complete graph K_n on n vertices, and $\mathcal{B}$ is an edge partition of K_n into subgraphs isomorphic to G. For example, a Steiner triple system on n points is a K_3-design of order n.

Much attention has been focused on G-designs, and also on G-designs with additional properties. The existence problem (see the surveys [1, 5] and references therein) has been determined for many graphs G.

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A related issue is the existence problem concerning G-designs that satisfy additional properties, such as those having or not having a *2-coloring*. If $(X, \mathcal{B})$ is a G-design, a subset T of X is called a *blocking set* of $(X, \mathcal{B})$ if, for each $B \in \mathcal{B}$, $V(B) \cap T \neq \emptyset$ and $V(B) \cap (X \setminus T) \neq \emptyset$ where $V(B)$ is the vertex set of the graph having edge set B. If T is a blocking set, the partition $\{T, X \setminus T\}$ is called a *2-coloring*.

The terminology can be extended for any natural number $k \geq 3$ in the standard way as follows. A *k-coloring* of $(X, \mathcal{B})$ is a partition of X into k classes such that no class contains $V(B)$ for any $B \in \mathcal{B}$. A G-design admitting a k-coloring is said to be *k-colorable*, and otherwise it is non-k-colorable.

Numerous papers (see the survey [2] and the research articles [3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19]) have been written on determining the spectrum for 2-colorable t-designs, 2-colorable projective planes, 2-colorable symmetric designs, 2-colorable block designs, 2-colorable balanced and almost balanced path designs and 2-colorable G-designs when G has fewer than 5 edges.

The negative side (i.e., the existence of G-designs *without* a 2-coloring) was proved by the first and third authors in [15] for the case where G is a cycle. As it was noted in [16], the proof technique presented in [15] can be applied also to show that the chromatic number of an m-cycle design can be arbitrarily large. In this paper we extend that result to the following very general one.

Theorem 1 *For every graph G and every integer $k \geq 2$, there exists a non-k-colorable G-design.*

In the proof, the following further terminology will be needed.

- A hypergraph $\mathcal{H}$ is a pair $(X, \mathcal{E})$ where X is a set called *vertex set* and $\mathcal{E}$ is a set system over X whose members $e \in \mathcal{E}$ are called *hyperedges*. In the proof, only *r-uniform* hypergraphs will be considered for some integer r; that is, $|e| = r$ will hold for all $e \in \mathcal{E}$.

- A *cycle of length* $\ell \geq 2$ in a hypergraph $\mathcal{H} = (X, \mathcal{E})$ is a sequence

$$x_1 e_1 x_2 e_2 \ldots x_\ell e_\ell x_{\ell+1}$$

  where it is $x_1 = x_{\ell+1}$, $x_i, x_{i+1} \in e_i$ for all $1 \leq i \leq \ell$ and $x_i \neq x_j$, $e_i \neq e_j$ for all $1 \leq i < j \leq \ell$. The *girth* of $\mathcal{H}$ is the smallest integer g such that $\mathcal{H}$ contains a cycle of length g. By the definition of cycles, $g \geq 2$ always is valid.

- The *chromatic number* of $\mathcal{H}$ is the smallest integer $m \geq 1$ for which there exists a vertex partition $X_1 \cup \cdots \cup X_m = X$ such that no X_i $(1 \leq i \leq m)$ contains any $e \in \mathcal{E}$.

2 The proof

(1) Given G, denote by r the number of (non-isolated) vertices in G. We temporarily forget about G itself, only consider r and the given integer k. It was proved by Lovász [14] that, for every r, there exist r-uniform hypergraphs of arbitrarily large girth and chromatic number. We need here girth at least three, i.e., where any two edges of the hypergraph share at most one vertex (also called *linear hypergraph* in the literature). Let $\mathcal{H}_{k,r} = (X_{k,r}, \mathcal{E}_{k,r})$ be any non-k-colorable r-uniform hypergraph of girth at least three, for the value k given in Theorem 1.

(2) For each $e \in \mathcal{E}_{k,r}$ we take a bijection $f_e : V(G) \to e$ from the vertex set of G onto the hyperedge e and put $f_e(xy) = \{f_e(x), f_e(y)\}$ for any edge $\{x,y\} \in E(G)$. Let then H be the graph with vertex set $V(H) = X_{k,r}$ and edge set

$$E(H) = \{f_e(xy) \mid \{x,y\} \in E(G),\ e \in \mathcal{E}_{k,r}\}$$

Since $\mathcal{H}_{k,r}$ has girth three or more, no two $f_{e'}, f_{e''}$ ($e', e'' \in \mathcal{E}_{k,r}$ and $e' \neq e''$) can map any edge of G onto the same edge of H. Hence, by construction, H admits an edge decomposition into subgraphs isomorphic to G in the natural way, along the hyperedges of $\mathcal{H}_{k,r}$.

(3) By a theorem of Wilson [20], there exists an H-design over a K_n, for some n. (In fact, the general theorem of [20] states that every sufficiently large n satisfying certain divisibility conditions admits an H-design of order n, but we do not need this strong assertion.) We denote such an H-design by $(X, \mathcal{B}_H)$, where $X = V(K_n)$. Since G decomposes H, each block $B \in \mathcal{B}_H$ can be decomposed into a certain number, $|E(H)|/|E(G)|$, of subgraphs isomorphic to G. Doing this for all $B \in \mathcal{B}_H$, we obtain a G-design $(X, \mathcal{B})$ over K_n. Since H may admit several G-decompositions, it is essential to take the one in which the blocks isomorphic to G are exactly the subgraphs corresponding to the hyperedges $e \in \mathcal{E}$ of $\mathcal{H}$.

(4) It remains to show that $(X, \mathcal{B})$ is not k-colorable. Consider an arbitrary vertex k-partition $\mathcal{P} = X_1 \cup \cdots \cup X_k = X$, and let $B \in \mathcal{B}_H$ be any block of the H-design. Then $\mathcal{P}$ induces a partition of $V(B)$ into at most k nonempty classes $B_i = X_i \cap V(B)$ ($1 \leq i \leq k$). Moreover, B is isomorphic to H, hence the isomorphism defines a vertex k-partition, say $\mathcal{P}'$, on $V(H) = X_{k,r}$. Since $\mathcal{H}_{k,r}$ is not k-colorable, some partition class entirely contains some hyperedge $e \in \mathcal{E}_{k,r}$, and this e contains the image of G under the mapping f_e. By assumption, $f_e(E(G))$ is then a block in $(X, \mathcal{B})$, entirely contained in some B_i. Thus, no $\mathcal{P}$ can be a k-coloring of $(X, \mathcal{B})$.

3 Concluding remarks

(1) The construction above is very far from being economical. It remains an open problem to determine — at least asymptotically — the smallest order $n = n(G, k)$ for which K_n admits a non-k-colorable G-design. Already some tight estimates for $k = 2$ (the case of blocking sets) would be of great interest. For some types of bipartite graphs (e.g., graceful ones), and also for some 3-colorable graphs, the method of [15] yields smaller constructions than the present one, but it is not clear whether it can be extended for all graphs G.

(2) Since Wilson's theorem provides a sequence of positive density for the orders of H-designs, we obtain infinitely many examples satisfying the conditions of Theorem 1 for every G and k. It is very likely, however, that the density of existing G-designs with the required properties is much larger than the one guaranteed by our method.

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