Equitable chromatic threshold of direct products of complete graphs

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Abstract

A proper vertex coloring of a graph is equitable if the sizes of color classes differ by at most 1. The equitable chromatic threshold of a graph G, denoted by $\chi^*_=(G)$, is the minimum k such that G is equitably k'-colorable for all $k' \geq k$. Let $G \times H$ denote the direct product of graphs G and H. For $n \geq m \geq 2$ we prove that $\chi^*_=(K_m \times K_n)$ equals $\lceil \frac{mn}{m+1} \rceil$ if $n \equiv 2, \ldots, m \pmod{m+1}$, and equals $m \lceil \frac{n}{s^*} \rceil$ if $n \equiv 0, 1 \pmod{m+1}$, where s^* is the minimum positive integer such that $s^* \nmid n$ and $s^* \geq m+2$.

Keywords: Equitable coloring; Equitable chromatic threshold; Direct product; Complete graph.

1 Introduction

All graphs considered in this paper are finite, undirected and without loops or multiple edges. For a positive integer k, let $[k] = \{1, 2, \cdots, k\}$. A (proper) k-coloring of a graph G is a mapping $c : V(G) \to [k]$ such that $c(x) \neq c(y)$ whenever $xy \in E(G)$. We call the set $c^{-1}(i) = \{x \in V(G) : c(x) = i\}$ a color class for each $i \in [k]$. A graph is k-colorable if it has a k-coloring. The chromatic number of G, denoted by $\chi(G)$, is the smallest integer k for which G is k-colorable. An equitable k-coloring of G is a k-coloring for which any two color classes differ in size by at most 1, or equivalently, each color class is of size $\lfloor |V(G)|/k \rfloor$ or $\lceil |V(G)|/k \rceil$. The equitable chromatic number of G, denoted by $\chi_=(G)$, is the smallest integer k such that G is equitably k-colorable, and the equitable chromatic threshold of a graph G, denoted by $\chi^*_=(G)$, is the smallest integer k such that G is equitably k'-colorable for all $k' \geq k$. The concept of equitable colorability was first

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introduced by Meyer [13] and has received a lot of attention. We refer the reader to a survey given by Lih [9] for some related results.

The direct (or Kronecker) product of graphs G and H is the graph $G\times H$ with vertex set $V(G)\times V(H)$ and edge set $\{(x,y)(x',y') : xx' \in E(G), yy' \in E(H)\}$. This product is commutative and associative in a natural way (see [7] for a detailed description on product graphs).

Chen et al. [3], Furmańzyk [5] and Lin and Chang [10] investigated equitable colorability of direct products and obtained exact values and upper bounds on equitable chromatic numbers and thresholds for direct products of some highly-structured graphs. For example, if $m \leq n$ then $\chi_=(K_m\times K_n) = m$ [3], $\chi_=(K_{1,m}\times K_{1,n}) = m+1$ [5] and $\chi^*_=(K_{1,m}\times K_{1,n}) = m+1$ [10]. Chen et al. [3] gave the following conjecture.

Conjecture 1. *[3] $\chi^*_=(G\times H) \leq \max\{|V(G)|, |V(H)|\}$ for any graphs G and H.*

Notice that Conjecture 1 holds if it holds for complete graphs. At the end of [3], they leave it as an open problem to determine the exact value of $\chi^*_=(K_m \times K_n)$ for any m and n.

Conjecture 1 was verified by Lin and Chang [10]. In fact, they proved a slightly better upper bound on $\chi^*_=(K_m \times K_n)$.

Theorem 1. *[10] For positive integers $m \leq n$, we have $\chi^*_=(K_m \times K_n) \leq \left\lceil \frac{mn}{m+1} \right\rceil$.*

In the same paper, Lin and Chang also determined the exact value of $\chi^*_=(K_m \times K_n)$ for $m = 2, 3$, respectively. The case when $m = 1$ is trivial since $K_1 \times K_n$ is the empty graph O_n and hence $\chi^*_=(K_1 \times K_n) = 1$.

Theorem 2. *[10] If integer $n \geq 3$, then $\chi^*_=(K_2 \times K_n) = 2\left\lceil \frac{n}{s^*} \right\rceil$, where s^* is the minimum positive integer such that $s^* \nmid n$ and $2\left\lceil \frac{n}{s^*} \right\rceil \leq \left\lceil \frac{2n}{3} \right\rceil$.*

We note that the formula in the above theorem also holds when $n = 2$.

Theorem 3. *[10] If integer $n \geq 3$, then*

$$\chi^*_=(K_3 \times K_n) = \begin{cases} \left\lceil \frac{3n}{4} \right\rceil, & if\ n \equiv 2 \pmod 4; \\ 3\left\lceil \frac{n}{s^*} \right\rceil, & otherwise, \end{cases}$$

where s^* is the minimum positive integer such that $s^* \nmid n$ and $3\left\lceil \frac{n}{s^*} \right\rceil \leq \left\lceil \frac{3n}{4} \right\rceil$.

The main purpose of the current paper is to determine $\chi^*_=(K_m \times K_n)$ for any m and n. We may assume $n \geq m \geq 2$ by commutativity of the direct product and the trivial case mentioned above. Our main result is the following.

Theorem 4. *If integers $n \geq m \geq 2$, then*
$$\chi^*_{=}(K_m \times K_n) = \begin{cases} \left\lceil \frac{mn}{m+1} \right\rceil, & \text{if } n \equiv 2, \ldots, m-1 \pmod{m+1}; \\ m \left\lceil \frac{n}{s^*} \right\rceil, & \text{if } n \equiv 0, 1, m \pmod{m+1}, \end{cases}$$
where s^* is the minimum positive integer such that $s^* \nmid n$ and $m \left\lceil \frac{n}{s^*} \right\rceil \leq \left\lceil \frac{mn}{m+1} \right\rceil$.

Although Theorem 4 is a natural generalization of Theorems 2 and 3, the special skills in proving Theorems 2 and 3 do not apply in this generalization. Instead of proving Theorem 4 directly, we shall show that it has the following two equivalent forms.

The first form simplifies the case when $n \equiv m \pmod{m+1}$.

Theorem 5. *If integers $n \geq m \geq 2$, then*
$$\chi^*_{=}(K_m \times K_n) = \begin{cases} \left\lceil \frac{mn}{m+1} \right\rceil, & \text{if } n \equiv 2, \ldots, m \pmod{m+1}; \\ m \left\lceil \frac{n}{s^*} \right\rceil, & \text{if } n \equiv 0, 1 \pmod{m+1}, \end{cases}$$
where s^* is the minimum positive integer such that $s^* \nmid n$ and $m \left\lceil \frac{n}{s^*} \right\rceil \leq \left\lceil \frac{mn}{m+1} \right\rceil$.

Under the assumption that $n \equiv 0, 1 \pmod{m+1}$, we show that the definition of s^* can be simplified. To avoid confusion, we use notation $s^\star$ in the next equivalent form. (We shall show $s^* = s^\star$, see Lemma 2.)

Theorem 6. *If integers $n \geq m \geq 2$, then*
$$\chi^*_{=}(K_m \times K_n) = \begin{cases} \left\lceil \frac{mn}{m+1} \right\rceil, & \text{if } n \equiv 2, \ldots, m \pmod{m+1}; \\ m \left\lceil \frac{n}{s^\star} \right\rceil, & \text{if } n \equiv 0, 1 \pmod{m+1}, \end{cases}$$
where $s^\star$ is the minimum positive integer such that $s^\star \nmid n$ and $s^\star \geq m+2$.

In Section 2, we shall prove the equivalence of Theorems 4-6. The proof of Theorem 6 is given in Section 3. We end this section by remarking that many other properties of direct products of complete graphs have been studied, such as idomatic partitions [4, 8], cycle decomposition [1, 15], minimum cycle basis [6], $L(j,k)$-labelling [11], vertex vulnerability parameters [14], etc.

2 Equivalence of Theorems 4-6

Note that Theorems 4 and 5 give the same expression for $\chi^*_{=}(K_m \times K_n)$ when $n \not\equiv m \pmod{m+1}$. To establish the equivalence between the two theorems, we need to show that two different expressions for $\chi^*_{=}(K_m \times K_n)$ take the same value when $n \equiv m \pmod{m+1}$.

Lemma 1. *Let $n \geq m \geq 2$. If $n \equiv m \pmod{m+1}$, then $\left\lceil \frac{mn}{m+1} \right\rceil = m \left\lceil \frac{n}{s^*} \right\rceil$, where s^* is the minimum positive integer such that $s^* \nmid n$ and $m \left\lceil \frac{n}{s^*} \right\rceil \leq \left\lceil \frac{mn}{m+1} \right\rceil$.*

Proof. Since $n \equiv m \pmod{m+1}$, we have $n = (m+1)p + m$, where $p = \lfloor \frac{n}{m+1} \rfloor$. Since $\lceil \frac{n}{m+1} \rceil = \lfloor \frac{n}{m+1} \rfloor + 1 = p+1$ and $\lceil \frac{mn}{m+1} \rceil = \left\lceil \frac{m(m+1)p+m^2}{m+1} \right\rceil = mp + \left\lceil \frac{m^2}{m+1} \right\rceil = m(p+1)$, we see that $m\lceil \frac{n}{m+1} \rceil = \lceil \frac{mn}{m+1} \rceil$. Since $m+1 \nmid n$, the minimality of s^* implies $s^* \le m+1$. Combining this with the definition of s^*, we have

$$\left\lceil \frac{mn}{m+1} \right\rceil \ge m\left\lceil \frac{n}{s^*} \right\rceil \ge m\left\lceil \frac{n}{m+1} \right\rceil = \left\lceil \frac{mn}{m+1} \right\rceil. \tag{1}$$

Therefore, equalities hold throughout (1). The lemma follows. □

We are left to show the equivalence of Theorems 5 and 6.

Lemma 2. *Let $n \ge m \ge 2$. If $n \equiv 0, 1 \pmod{m+1}$, then for any positive $s \nmid n$, the following two conditions are equivalent.*
(1) $m\lceil \frac{n}{s} \rceil \le \lceil \frac{mn}{m+1} \rceil$.
(2) $s \ge m+2$.

Proof. Let $n = (m+1)p + r$, where $r = 0$ or $r = 1$, and hence $p \ge r$ by the assumption of the lemma. We have $\lceil \frac{mn}{m+1} \rceil = \lceil \frac{m(m+1)p+mr}{m+1} \rceil = mp$ when $r = 0$ and $\lceil \frac{mn}{m+1} \rceil = mp+1$ when $r = 1$. Either case implies $mp \le \lceil \frac{mn}{m+1} \rceil < m(p+1)$ since $m \ge 2$.

If $s \le m+1$ then $\lceil \frac{n}{s} \rceil > \frac{n}{s} \ge \frac{n}{m+1} \ge p$, where the strict inequality follows from the assumption that $s \nmid n$. Therefore, $m\lceil \frac{n}{s} \rceil \ge m(p+1) > \lceil \frac{mn}{m+1} \rceil$. This proves that (1) implies (2).

If $s \ge m+2$ then $\lceil \frac{n}{s} \rceil \le \lceil \frac{(m+1)p+r}{m+2} \rceil \le \lceil \frac{(m+1)p+p}{m+2} \rceil = p$. Hence, $m\lceil \frac{n}{s} \rceil \le mp \le \lceil \frac{mn}{m+1} \rceil$. This proves that (2) implies (1). □

From Lemma 2, we see that $s^* = s^\star$, where s^* and $s^\star$ are defined, under the same assumption that $n \ge m \ge 2$ and $n \equiv 0, 1 \pmod{m+1}$, in Theorem 5 and Theorem 6, respectively. The equivalence of the two theorems follows.

3 Proof of Theorem 6

By $K_{m(n)}$ we denote the complete m-partite graph with n vertices in each part. As noted in [10], $K_m \times K_n$ is a span subgraph of $K_{m(n)}$ and hence any equitable k-coloring of $K_{m(n)}$ yields an equitable k-coloring of $K_m \times K_n$. Therefore, studying the equitable colorability of $K_{m(n)}$ plays a key role in determining $\chi^*_=(K_m \times K_n)$. We mention that the equitable colorings of complete m-partite graphs are also studied in [2, 12, 16, 17]. Our discussion on the equitable colorability of complete m-partite graphs is similar to [16]. We start with the empty graph O_n.

Lemma 3. *The empty graph O_n is equitably k-colorable with color classes of size q or $q+1$ if and only if $\lceil \frac{n}{q+1} \rceil \le k \le \lfloor \frac{n}{q} \rfloor$.*

Proof. ($\Rightarrow$) Let c be an equitable k-coloring of O_n with color classes of size q or $q+1$. Then $kq \le n \le k(q+1)$, i.e., $\frac{n}{q+1} \le k \le \frac{n}{q}$. Since k is an integer, we have $\lceil \frac{n}{q+1} \rceil \le k \le \lfloor \frac{n}{q} \rfloor$.

($\Leftarrow$) Since $\lceil \frac{n}{q+1} \rceil \le k \le \lfloor \frac{n}{q} \rfloor$, we have $\frac{n}{q+1} \le k \le \frac{n}{q}$, i.e., $kq \le n \le k(q+1)$. Let $r = n-kq$. We have $0 \le r \le k$. Hence $n = kq+r = (k-r)q+r(q+1)$, where $k-r$ and r are nonnegative. We can partition $V(O_n)$ into $(k-r)$ subsets of size q and r subsets of size $q+1$. □

Lemma 4. *$K_{m(n)}$ is equitably k-colorable with color classes of size q or $q+1$ if and only if $m\lceil \frac{n}{q+1} \rceil \le k \le m\lfloor \frac{n}{q} \rfloor$.*

Proof. Suppose $X_1, \ldots, X_m$ are the parts of $K_{m(n)}$ with $|X_1| = \cdots = |X_m| = n$.

($\Leftarrow$) Let $k_i = \lfloor \frac{k+i-1}{m} \rfloor$ for $i \in [m]$. Then $\sum_{i=1}^{m} k_i = k$ and $\lfloor \frac{k}{m} \rfloor \le k_i \le \lceil \frac{k}{m} \rceil$ for each $i \in [m]$. Since $m\lceil \frac{n}{q+1} \rceil \le k \le m\lfloor \frac{n}{q} \rfloor$, we see that $\lceil \frac{n}{q+1} \rceil \le \frac{k}{m} \le \lfloor \frac{n}{q} \rfloor$, and hence $\lceil \frac{n}{q+1} \rceil \le \lfloor \frac{k}{m} \rfloor \le \lceil \frac{k}{m} \rceil \le \lfloor \frac{n}{q} \rfloor$. Therefore, $\lceil \frac{n}{q+1} \rceil \le k_i \le \lfloor \frac{n}{q} \rfloor$. By Lemma 3, we can partition X_i into k_i independent sets of size q or $q+1$ for $i \in [m]$. Hence, $K_{m(n)}$ is equitably k-colorable with color classes of size q or $q+1$ since $\sum_{i=1}^{m} k_i = k$.

($\Rightarrow$) Let c be an equitable k-coloring with color classes of size q or $q+1$. Since different parts of $K_{m(n)}$ should be colored with different colors, we see that each X_i is equitably colored with $|c(X_i)|$ colors and hence $\sum_{i=1}^{m} |c(X_i)| = k$. By Lemma 3, $\lceil \frac{n}{q+1} \rceil \le |c(X_i)| \le \lfloor \frac{n}{q} \rfloor$ for $i \in [m]$. By adding these m inequalities, we have $m\lceil \frac{n}{q+1} \rceil \le \sum_{i=1}^{m} |c(X_i)| = k \le m\lfloor \frac{n}{q} \rfloor$. □

Lemma 5. $m\lfloor \frac{n}{m+1} \rfloor \le \lfloor \frac{mn}{m+1} \rfloor$ *with equality if and only if* $n \equiv 0, 1 \pmod{m+1}$.

Proof. Let $n = (m+1)p + r$ with $0 \le r \le m$. Since $m\lfloor \frac{n}{m+1} \rfloor = mp$ and $\lfloor \frac{mn}{m+1} \rfloor = \lfloor \frac{m(m+1)p+mr}{m+1} \rfloor = mp + \lfloor \frac{mr}{m+1} \rfloor$, we see that $m\lfloor \frac{n}{m+1} \rfloor \le \lfloor \frac{mn}{m+1} \rfloor$, where the equality holds if and only if $\lfloor \frac{mr}{m+1} \rfloor = 0$. The lemma holds since $\lfloor \frac{mr}{m+1} \rfloor = 0$ if and only if $r = 0, 1$. □

As noted earlier, any equitable k-coloring of $K_{m(n)}$ naturally leads to an equitable k-coloring of $K_m \times K_n$. The next lemma indicates that the converse is also true when k is small.

Lemma 6. *If $K_m \times K_n$ is equitably k-colorable for some $k < \lceil \frac{mn}{m+1} \rceil$, then $K_{m(n)}$ is also equitably k-colorable.*

Proof. Let c be an equitable k-coloring of $K_m \times K_n$ for some $k < \lceil \frac{mn}{m+1} \rceil$. Then the size of each color class is at least $\lfloor \frac{mn}{k} \rfloor$. Now $\frac{mn}{k} \geq \frac{mn}{\lceil mn/(m+1) \rceil - 1} \geq \frac{mn}{mn/(m+1)} = m+1$. It follows that $\lfloor \frac{mn}{k} \rfloor \geq m+1$. We claim that each color class is contained in $\{x\} \times V(K_n)$ for some $x \in V(K_m)$.

We show the claim by the way of contradiction. If there exists some color class S containing two vertices (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$, then $y_1 = y_2$ since otherwise (x_1, y_1) is adjacent with (x_2, y_2) by the definition of direct products, contradicting the fact that S is a color class. Set $y = y_1 = y_2$. Let $(x, y') \in V(K_m) \times V(K_n - y)$. Note $y' \neq y$. If $x \neq x_1$ then (x, y') and (x_1, y) are adjacent, otherwise $x \neq x_2$ since $x_1 \neq x_2$ and hence (x, y') and (x_2, y) are adjacent. This proves that any vertex in $V(K_m) \times V(K_n - y)$ has a neighbor in S, hence $S \subset V(K_m) \times \{y\}$ since S is independent. Therefore, $|S| \leq m$, a contradiction. This proves the claim.

For the equitable coloring c of $K_m \times K_n$, by the claim, each color class is also independent in the spanning supergraph $K_{m(n)}$. Hence, $K_{m(n)}$ is also equitably k-colorable. □

Lemma 7. *Let $n \geq m \geq 2$. If c is an equitable $\lfloor \frac{mn}{m+1} \rfloor$-coloring of $K_{m(n)}$, then each color class is of size $m+1$ or $m+2$.*

Proof. Let $p = \lfloor \frac{mn}{m+1} \rfloor$. Since each color class is of size $\lfloor \frac{mn}{p} \rfloor$ or $\lceil \frac{mn}{p} \rceil$, it suffices to show that $m+1 \leq \frac{mn}{p} \leq m+2$.

It is clear that $\frac{mn}{p} \geq \frac{mn}{mn/(m+1)} = m+1$. To show $\frac{mn}{p} \leq m+2$, we consider the following three cases.

Case 1. $n = m$. We have $p = \lfloor \frac{m^2}{m+1} \rfloor = m-1$ and hence $\frac{mn}{p} = \frac{m^2}{m-1} = m+1+\frac{1}{m-1} \leq m+2$.

Case 2. $n = m+1$. We have $p = m$ and hence $\frac{mn}{p} = n \leq m+2$.

Case 3. $n \geq m+2$. Since $p = \lfloor \frac{mn}{m+1} \rfloor$, we have $mn = (m+1)p + r$ with $0 \leq r < m+1$ and hence $p = \frac{mn-r}{m+1} \geq \frac{mn-m}{m+1}$. Therefore, $\frac{mn}{p} \leq \frac{mn}{(mn-m)/(m+1)} = \frac{n(m+1)}{n-1} = m+1+\frac{m+1}{n-1} \leq m+2$. □

Now we show Theorem 6 when $n \equiv 2, \ldots, m \pmod{m+1}$.

Corollary 1. *Let $n \geq m \geq 2$. If $n \equiv 2, \ldots, m \pmod{m+1}$, then $\chi^*_=(K_m \times K_n) = \lceil \frac{mn}{m+1} \rceil$.*

Proof. Since $\chi^*_=(K_m \times K_n) \leq \lceil \frac{mn}{m+1} \rceil$ by Theorem 1, it suffices to show that $K_m \times K_n$ is not equitably $(\lceil \frac{mn}{m+1} \rceil - 1)$-colorable. Since $m+1$ and m are coprime, and $m+1 \nmid n$, we see that $m+1 \nmid mn$ and hence $\lceil \frac{mn}{m+1} \rceil - 1 = \lfloor \frac{mn}{m+1} \rfloor$.

Suppose to the contrary that $K_m \times K_n$ is equitably $\lfloor \frac{mn}{m+1} \rfloor$-colorable. By Lemma 6, $K_{m(n)}$ is also equitably $\lfloor \frac{mn}{m+1} \rfloor$-colorable. Let c be such a coloring of $K_{m(n)}$. By Lemma 7, each color class of c is of size $m+1$ or $m+2$. It follows that $\lfloor \frac{mn}{m+1} \rfloor \le m \lfloor \frac{n}{m+1} \rfloor$ by Lemma 4.

On the other hand, since $n \equiv 2, \ldots, m \pmod{m+1}$, Lemma 5 implies that $m \lfloor \frac{n}{m+1} \rfloor < \lfloor \frac{mn}{m+1} \rfloor$. This is a contradiction. $\square$

Lemma 8. *Let $n \ge m \ge 2$. If $n \equiv 0, 1 \pmod{m+1}$, then $\lceil \frac{n}{s^*} \rceil \le \lfloor \frac{n}{s^*-1} \rfloor$, where s^* is the minimum positive integer such that $s^* \nmid n$ and $s^* \ge m+2$.*

Proof. By the assumption that $s^* \ge m+2$, we consider the following two cases.

Case 1. $s^* > m+2$. Since $s^* - 1 \ge m+2$, the minimality of s^* implies $s^* - 1 \mid n$. Hence, $\lceil \frac{n}{s^*} \rceil \le \lceil \frac{n}{s^*-1} \rceil = \lfloor \frac{n}{s^*-1} \rfloor$.

Case 2. $s^* = m+2$. Let $n = (m+1)p + r$, where $r = 0$ or $r = 1$, and hence $p \ge r$ by the assumption of the lemma. Hence, $\lceil \frac{n}{s^*} \rceil = \lceil \frac{(m+1)p+r}{m+2} \rceil \le \lceil \frac{(m+1)p+p}{m+2} \rceil = p = \lfloor \frac{n}{m+1} \rfloor = \lfloor \frac{n}{s^*-1} \rfloor$. $\square$

The following lemma is from [10]. We give a different proof for convenience.

Lemma 9. *If m, s and n are positive integers with $m \ge 2$ and $s \nmid n$, then $K_{m(n)}$ is not equitably $(m \lceil \frac{n}{s} \rceil - i)$-colorable for $1 \le i < m$.*

Proof. Suppose to the contrary that $K_{m(n)}$ is equitably $(m \lceil \frac{n}{s} \rceil - i)$-colorable for some $i \in [m-1]$. By Lemma 4, there exists $q > 0$ such that

$$m \left\lceil \frac{n}{q+1} \right\rceil \le m \left\lceil \frac{n}{s} \right\rceil - i \le m \left\lfloor \frac{n}{q} \right\rfloor. \tag{2}$$

We shall show either of the following two cases will yield a contradiction.

Case 1. $q \ge s$. Since $s \nmid n$ and $i < m$, we have $m \lfloor \frac{n}{q} \rfloor \le m \lfloor \frac{n}{s} \rfloor = m(\lceil \frac{n}{s} \rceil - 1) < m \lceil \frac{n}{s} \rceil - i$. It contradicts the right inequality of (2).

Case 2. $q < s$. Since $i \ge 1$, we have $m \lceil \frac{n}{q+1} \rceil \ge m \lceil \frac{n}{s} \rceil > m \lceil \frac{n}{s} \rceil - i$, a contradiction to the left inequality of (2). $\square$

We are ready to prove Theorem 6 for the remaining case $n \equiv 0, 1 \pmod{m+1}$. For integers $a \le b$, by $[a, b]$ we denote the interval of all integers between a and b, including both.

Corollary 2. *Let $n \ge m \ge 2$. If $n \equiv 0, 1 \pmod{m+1}$, then $\chi^*_=(K_m \times K_n) = m \lceil \frac{n}{s^*} \rceil$, where s^* is the minimum positive integer such that $s^* \nmid n$ and $s^* \ge m+2$.*

Proof. By Lemma 2, the definition of s^* implies $m\left\lceil \frac{n}{s^*} \right\rceil \le \left\lceil \frac{mn}{m+1} \right\rceil$. Hence, $m\left\lceil \frac{n}{s^*} \right\rceil - 1 < \left\lceil \frac{mn}{m+1} \right\rceil$. Since $K_{m(n)}$ is not equitably $(m\left\lceil \frac{n}{s^*} \right\rceil - 1)$-colorable by Lemma 9, Lemma 6 implies that $K_m \times K_n$ is also not equitably $(m\left\lceil \frac{n}{s^*} \right\rceil - 1)$-colorable. We are left to show that $K_m \times K_n$ is equitably k-colorable for all $k \ge m\left\lceil \frac{n}{s^*} \right\rceil$.

Claim 1. *$K_{m(n)}$ is equitably k-colorable for $k \in \left[m\left\lceil \frac{n}{s^*} \right\rceil, m\left\lfloor \frac{n}{m+1} \right\rfloor\right]$.*

Since $\left\lceil \frac{n}{s^*} \right\rceil \le \left\lfloor \frac{n}{s^*-1} \right\rfloor$ by Lemma 8, we see that $K_{m(n)}$ is equitably k-colorable for $k \in \left[m\left\lceil \frac{n}{s^*} \right\rceil, m\left\lfloor \frac{n}{s^*-1} \right\rfloor\right]$ by Lemma 4. If $s^* = m+2$ we are done. Now we assume $s^* = m+2+l$ for some $l > 0$. Let $I_0 = \left[m\left\lceil \frac{n}{s^*} \right\rceil, m\left\lfloor \frac{n}{s^*-1} \right\rfloor\right]$ and we shall find additional l integer intervals $I_1, \ldots, I_l$ such that

$$\bigcup_{i=0}^{l} I_i = \left[m\left\lceil \frac{n}{s^*} \right\rceil, m\left\lfloor \frac{n}{m+1} \right\rfloor\right]. \tag{3}$$

and $K_{m(n)}$ is equitably k-colorable for $k \in I_i$ and $1 \le i \le l$.

Let $1 \le i \le l$. Then $s^* - i \ge m+2$ and hence the minimality of s^* implies $s^* - i \mid n$. Consequently,

$$\left\lceil \frac{n}{s^* - i} \right\rceil = \left\lfloor \frac{n}{s^* - i} \right\rfloor \le \left\lfloor \frac{n}{s^* - i - 1} \right\rfloor \text{ for } 1 \le i \le l. \tag{4}$$

Define $I_i = \left[m\left\lceil \frac{n}{s^*-i} \right\rceil, m\left\lfloor \frac{n}{s^*-i-1} \right\rfloor\right]$ for $1 \le i \le l$. By Lemma 4, we see that $K_{m(n)}$ is equitably k-colorable for $k \in I_i$ and $1 \le i \le l$.

Finally, by the equality of (4), we see that $m\left\lceil \frac{n}{s^*-i} \right\rceil = m\left\lfloor \frac{n}{s^*-i} \right\rfloor$, that is the left endpoint of interval I_i coincides with the right endpoint of previous one I_{i-1} for each $1 \le i \le l$. Therefore, equation (3) holds and hence Claim 1 follows.

Note $m\left\lfloor \frac{n}{m+1} \right\rfloor = \left\lfloor \frac{mn}{m+1} \right\rfloor$ by Lemma 5. Since $K_m \times K_n$ is a span subgraph of $K_{m(n)}$, by Claim 1 we see that $K_m \times K_n$ is equitably k-colorable for $k \in \left[m\left\lceil \frac{n}{s^*} \right\rceil, \left\lfloor \frac{mn}{m+1} \right\rfloor\right]$. If $k > \left\lfloor \frac{mn}{m+1} \right\rfloor$ then $k \ge \left\lceil \frac{mn}{m+1} \right\rceil$ and hence $K_m \times K_n$ is equitably k-colorable by Theorem 1. Hence, $K_m \times K_n$ is equitably k-colorable for all $k \ge m\left\lceil \frac{n}{s^*} \right\rceil$. This proves the corollary. □

The proof of Theorem 6 is complete by Corollaries 1 and 2.

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