Spanning trees whose stems have a few leaves

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Abstract

For a tree T, $Leaf(T)$ denotes the set of leaves of T, and $T - Leaf(T)$ is called the stem of T. For a graph G and a positive integer m, $\sigma_m(G)$ denotes the minimum degree sum of m independent vertices of G. We prove the following theorem. Let G be a connected graph and $k \geq 2$ be an integer. If $\sigma_3(G) \geq |G| - 2k + 1$, then G has a spanning tree whose stem has at most k leaves.

Keywords: spanning tree, tree with k-ended stem, stem of a tree

1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $|G|$ for the order of G, that is, $|G| := |V(G)|$. Let $N_G(v)$ and $d_G(v)$ denote the set of neighbours of v and the degree of v in G, respectively. For a subgraph H of G and a vertex $v \in V(G)$, we define $N_H(v) := N_G(v) \cap V(H)$ and $d_H(v) := |N_H(v)|$. For a subset X of $V(G)$, the subgraph induced by X is denoted by $\langle X \rangle_G$. If there is no confusion, we often identify a subgraph H of a graph G with its vertex set $V(H)$.

*Supported by Chinese Academy of Science Fellowship for Young International Scientists. Grant No.2012Y1JA0004

For an integer $k \geq 2$, the invariant $\sigma_k(G)$ of a graph G is defined to be the minimum degree sum of k independent vertices of G, that is,

$$\sigma_k(G) := \min\{\sum_{x \in Y} d_G(x) \ : \ Y \subset V(G), \ |Y| = k, \ Y \text{ is independent}\}.$$

Recently, in [3], Kano, Yan and the first author considered a new concept of a spanning tree. We focused on the properties of a tree which can be obtained by removing all the leaves of a spanning tree. We call a vertex of a tree T, which has degree one, *leaf* of T. For convenience, we call an end-vertex, which has degree one, of a graph also a leaf. Let S be a graph. The set of leaves of S is denoted by $Leaf(S)$. The subgraph $S - Leaf(S)$ of S is called the *stem* of S and denoted by *Stem*(S). Note that a caterpillar is nothing but a tree whose stem is a path. Especially, we focused on a k-tree (a tree whose maximum degree is at most k) as a property of the stem of a spanning tree in a graph. We proved the following theorem, and also showed the best possibility of the lower bound of the degree sum condition.

Theorem 1 (Kano, Tsugaki and Yan [3]) *Let $k \geq 2$ be an integer. If a connected graph G satisfies $\sigma_{k+1}(G) \geq n - k - 1$, then G has a spanning tree T such that Stem(T) is a k-tree.*

In [2], Bondy gave a sufficient condition for a graph to have a caterpillar.

Theorem 2 (Bondy [2]) *Let G be a 2-connected graph. If $\sigma_3(G) \geq |G| + 2$, then G has a dominating cycle, in particular, G has a spanning caterpillar.*

But, the lower bound of $\sigma_3(G)$ of Theorem 2 is not best possible for a graph to have a spanning caterpillar. In fact, by Theorem 1, $|G| - 3$ is best possible.

In this paper, we deal with a k-ended tree as a property of the stem of a spanning tree in a graph. A tree which has at most k leaves is called a *k-ended tree.* A stem which is a k-ended tree is called a *k-ended stem*, and so a tree whose stem has at most k leaves is called a *tree with k-ended stem.*

In [4], a sufficient condition for a graph to have a spanning k-ended tree was introduced by Las Vergnas.

Theorem 3 **(Las Vergnas [4])** *Let $k \geq 2$ be an integer, and let G be a connected graph. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning k-ended tree.*

We shall prove a similar result for a graph to have a spanning tree with k-ended stem.

Theorem 4 *Let $k \geq 2$ be an integer. If a connected graph G satisfies $\sigma_3(G) \geq |G| - 2k + 1$, then G has a spanning tree T with k-ended stem.*

In fact, the condition of Theorem 4 is sharp. Let $k \geq 2$ be an integer. Let G be a graph of order $2k+3$ with $k+1$ leaves such that the stem of G is $K_{1,k+1}$, and for each $x \in Leaf(G)$, x connects to only one leaf of $K_{1,k+1}$(see Fig. 1). Then the spanning tree of G is itself whose stem is $K_{1,k+1}$. So G has no spanning tree with k-ended stem. Note that $\sigma_3(G) = 3 = |G| - 2k$, therefore the condition of Theorem 4 is sharp.

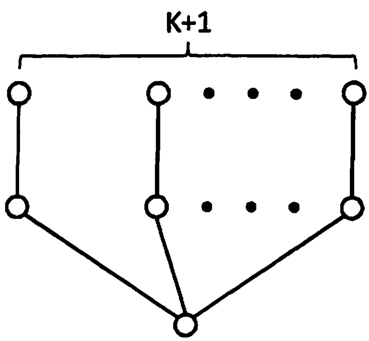

Figure 1: A graph G which has no spanning tree with k-ended stem.

We can also deal with spanning trees whose stems have some other properties. Many results on spanning k-ended trees and some other spanning trees can be found in a recent book [1] and a survey [5].

2 Proof of Theorem 4

We define a system to be a set of subgraphs of G whose stems are paths and cycles (including K_1 and K_2) and whose elements are pairwise vertexdisjoint. For convenience, we let $Stem(S) = S$ if $S \cong K_2$. We often view the system as a subgraph. Let $\mathcal{S}$ be a system in a graph. We define a function $f : \mathcal{S} \to \{1,2\}$ as follows. For each $S \in \mathcal{S}$, define $f(S) = 2$ if $Stem(S)$ is a path of order at least 3, and $f(S) = 1$ otherwise (i.e. $Stem(S)$ is a vertex, an edge or a cycle). We denote $\mathcal{P}_{\mathcal{S}} := \{S \in \mathcal{S} : f(S) = 2\}$ and $\mathcal{C}_{\mathcal{S}} := \{S \in \mathcal{S} : f(S) = 1\}$. Let $\mathcal{C}^1_{\mathcal{S}} := \{S \in \mathcal{C}_{\mathcal{S}} : Stem(S) \cong K_1\}$, $\mathcal{C}^2_{\mathcal{S}} := \{S \in \mathcal{C}_{\mathcal{S}} : Stem(S) \cong K_2\}$ and $\mathcal{C}^3_{\mathcal{S}} := \{S \in \mathcal{C}_{\mathcal{S}} : Stem(S)$ is a cycle$\}$. We define $V(\mathcal{S}) := \bigcup_{S \in \mathcal{S}} V(S)$ and $f(\mathcal{S}) := \sum_{S \in \mathcal{S}} f(S)$. We call $\mathcal{S}$ a *k-ended stem system* if $f(\mathcal{S}) \leq k$, and a spanning k-ended stem system if $f(\mathcal{S}) \leq k$ and $V(\mathcal{S}) = V(G)$.

For $S \in \mathcal{S}$, we sometimes give an orientation to $Stem(S)$. For each $P \in \mathcal{P}_{\mathcal{S}} \cup \mathcal{C}^2_{\mathcal{S}}$, let a_P and b_P be an initial and terminal vertices of $Stem(P)$, respectively, and let $End(P)$ be a set of vertices of $Leaf(P)$ which are adjacent to a_P or b_P in P. For $S \in \mathcal{S}$ and $x \in V(Stem(S))$, we denote the successor and the predecessor of x on $Stem(S)$ by $x^{+(S)}$ and $x^{-(S)}$, respectively (if exists). If there is no danger of confusion, we abbreviate $x^{+(S)}$ and $x^{-(S)}$ by x^+ and x^-, respectively. For $S \in \mathcal{S}$ and $X \subsetneq V(Stem(S))$, we define $X^- := \{x^- : x \in X\}$.

Theorem 4 is an immediate consequence of the following theorem.

Theorem 5 *Let $k \geq 2$ be an integer. If a connected graph G satisfies $\sigma_3(G) \geq |G| - 2k + 1$, then G has a spanning k-ended stem system.*

We now prove Theorem 5.

Proof of Theorem 5. Let G be a graph that satisfies the condition in Theorem 5. Suppose that G has no spanning k-ended stem system. Let $\mathcal{S}$ be a k-ended stem system in G. Choose $\mathcal{S}$ so that

(S1) $|V(\mathcal{S})|$ is as large as possible,

(S2) $\sum_{P \in \mathcal{P}_{\mathcal{S}}} |Stem(P)|$ is as large as possible subject to (S1),

(S3) $|\{S \in \mathcal{S} : S \cong K_1\}|$ is as small as possible subject to (S2) and,

(S4) $|\{S \in \mathcal{S} : S \cong K_2\}|$ is as small as possible subject to (S3).

We give an orientation to $Stem(S)$ for each $S \in \mathcal{S}$. Since G has no spanning k-ended stem system, there exists $w \in V(G) - V(\mathcal{S})$.

Claim 1 *Let $\mathcal{S}'$ be an l-ended stem system in G such that $V(\mathcal{S}) \subseteq V(\mathcal{S}')$. Then $l \geq k$ holds. Especially, $\sum_{S \in \mathcal{S}} f(S) = k$ holds.*

Proof. If $\sum_{S \in \mathcal{S}} f(S) < k$, then we can add w to $\mathcal{S}$ to get a larger k-ended stem system than $\mathcal{S}$, which contradicts the choice (S1). $\square$

Claim 2 $\mathcal{P}_{\mathcal{S}} \neq \emptyset$.

Proof. Suppose that $\mathcal{P}_{\mathcal{S}} = \emptyset$. Since $k \geq 2$, $|\mathcal{C}_{\mathcal{S}}| \geq 2$. Let $C_1, C_2 \in \mathcal{C}_{\mathcal{S}}$ with $C_1 \neq C_2$. Since G is connected, there exists a path (or an edge) P which connects a vertex of C_1 and a vertex of C_2. Since we can choose C_1 and C_2 arbitrarily, we may assume that $|V(P) \cap V(C_1)| = |V(P) \cap V(C_2)| = 1$ and $V(P) \cap (V(\mathcal{C}_{\mathcal{S}}) - V(C_1) - V(C_2)) = \emptyset$. By connecting C_1 and C_2 by P, we obtain a subgraph C_3 of G with $Stem(C_3)$ is a path. Now we show that $Stem(C_3)$ is a path of order at least 3. Let $\mathcal{S}_1 := (\mathcal{S} - \{C_1, C_2\}) \cup \{C_3\}$. Then $\mathcal{S}_1$ is a $(k - f(C_1) - f(C_2) + f(C_3))$-ended stem system such that $V(\mathcal{S}) \subseteq V(\mathcal{S}_1)$. By Claim 1, we obtain $2 = f(C_1) + f(C_2) \leq f(C_3) \leq 2$, that is $f(C_3) = 2$. This implies that $\mathcal{S}_1$ is a k-ended stem system and $Stem(C_3)$ is a path of order at least 3. By the choice (S1), $|V(\mathcal{S}_1)| = |V(\mathcal{S})|$. But, then $\sum_{P \in \mathcal{P}_{\mathcal{S}_1}} |Stem(P)| > \sum_{P \in \mathcal{P}_{\mathcal{S}}} |Stem(P)|$, which contradicts the choice (S2). $\square$

By Claim 2, there exists $P \in \mathcal{P}_{\mathcal{S}}$. Let $a \in N_P(a_P) \cap End(P)$ and $b \in N_P(b_P) \cap End(P)$. We choose $\mathcal{S}$, $P \in \mathcal{P}_{\mathcal{S}}$, and $a, b \in V(P)$ so that

(S5) $d_P(a) + d_P(b)$ is as small as possible subject to (S4).

Claim 3 (i) $(N_G(a) \cup N_G(b)) \cap (V(G) - V(\mathcal{S})) = \emptyset$.

(ii) $(N_G(a) \cup N_G(b)) \cap V(\mathcal{C}_{\mathcal{S}}) = \emptyset$.

(iii) $(N_G(a) \cup N_G(b)) \cap \{a_{P'}, a_{P'}^+, (a_{P'}^+)^+, b_{P'}\} = \emptyset$ *for any* $P' \in \mathcal{P}_{\mathcal{S}} - \{P\}$.

(iv) $(N_G(a) \cup N_G(b)) \cap Leaf(P') = \emptyset$ *for any* $P' \in \mathcal{P}_{\mathcal{S}}$.

(v) $b_P \notin N_G(a)$ *and* $a_P \notin N_G(b)$.

Proof. (i) Suppose not. By the symmetry of a and b, we may assume that there exists $r \in N_G(a) \cap (V(G) - V(\mathcal{S}))$. Then $\mathcal{S} + ra$ is a larger k-ended stem system than $\mathcal{S}$, which contradicts the choice (S1).

(ii) Suppose not. By the symmetry of a and b, we may assume that there exists $S \in \mathcal{C}_{\mathcal{S}}$ such that $N_G(a) \cap V(S) \neq \emptyset$. Let $r \in N_G(a) \cap V(S)$. Let $P_1 := P \cup ar \cup S$ if $S \in \mathcal{C}^1_{\mathcal{S}} \cup \mathcal{C}^2_{\mathcal{S}}$; otherwise if $r \in Stem(S)$ let $P_1 := P \cup ar \cup (S - rr^{-(S)})$, if $r \in Leaf(S)$ let $P_1 := P \cup ar \cup (S - xx^{-(S)})$ where $x \in N_S(r) \cap Stem(S)$. Let $\mathcal{S}_1 := (\mathcal{S} - \{P, S\}) \cup \{P_1\}$. Then $f(P_1) = f(P) + f(S) - 1$, and hence we can see that $\mathcal{S}_1$ is a $(k-1)$-ended stem system such that $V(\mathcal{S}_1) = V(\mathcal{S})$, which contradicts Claim 1.

(iii) Suppose that $a_{P'} \in N_G(a)$. Let $P_1 := P \cup aa_{P'} \cup P'$, and let $\mathcal{S}_1 := (\mathcal{S} - \{P, P'\}) \cup \{P_1\}$. Then $f(P_1) = f(P) + f(P') - 2$, and hence we can see that $\mathcal{S}_1$ is a $(k-2)$-ended stem system such that $V(\mathcal{S}_1) = V(\mathcal{S})$, which contradicts Claim 1. In the same way, we can get $b_{P'} \notin N_G(a)$ and $a_{P'}, b_{P'} \notin N_G(b)$.

Suppose that $a^+_{P'} \in N_G(a)$. Let P'_l and P'_r be two components of $P' - a_{P'}a^+_{P'}$ such that $a_{P'} \in V(P'_l)$ and $b_{P'} \in V(P'_r)$. Let $P_1 := P \cup aa^+_{P'} \cup P'_r$, and let $\mathcal{S}_1 := (\mathcal{S} - \{P, P'\}) \cup \{P_1, P'_l\}$. Then $f(P_1) + f(P'_l) = f(P) + f(P') - 1$, and hence we can see that $\mathcal{S}_1$ is a $(k-1)$-ended stem system such that $V(\mathcal{S}_1) = V(\mathcal{S})$, which contradicts Claim 1. Therefore, $a^+_{P'} \notin N_G(a)$. In the same way, we can get $a^+_{P'} \notin N_G(b)$.

By the same argument as above, we can also see that $(a^+_{P'})^+ \notin N_G(a)$ and $(a^+_{P'})^+ \notin N_G(b)$.

(iv) First, suppose that there exists $a' \in Leaf(P') - End(P')$ such that $a' \in N_G(a)$. Let $b' \in N_{P'}(a')$. If $P' \neq P$ let $P_1 := P \cup aa'$, $P_2 := P' - a'$ and let $\mathcal{S}_1 := (\mathcal{S} - \{P, P'\}) \cup \{P_1, P_2\}$. If $P' = P$ let $P_1 := (P - a'b') \cup aa'$, $\mathcal{S}_1 := (\mathcal{S} - \{P\}) \cup \{P_1\}$. Then $\mathcal{S}_1$ is a k-ended stem system such that $|V(\mathcal{S}_1)| = |V(\mathcal{S})|$ and $\sum_{Q \in \mathcal{P}_{\mathcal{S}_1}} |Stem(Q)| > \sum_{Q \in \mathcal{P}_{\mathcal{S}}} |Stem(Q)|$, which contradicts the choice (S2). Therefore $N_G(a) \cap (Leaf(P') - End(P')) = \emptyset$ for any $P' \in \mathcal{P}_{\mathcal{S}}$. Similarly, we can obtain $N_G(b) \cap (Leaf(P') - End(P')) = \emptyset$ for any $P' \in \mathcal{P}_{\mathcal{S}}$.

Next, suppose that there exists $a' \in End(P')$ such that $a' \in N_G(a)$. If $P' \neq P$, then we can obtain a contradiction as in the proof of statement (iii). Hence $P' = P$. If $a' \in N_P(a_P) \cap End(P)$, then, by letting $P_1 := (P - a'a_P) \cup aa'$ and $\mathcal{S}_1 := (\mathcal{S} - \{P\}) \cup \{P_1\}$, we can obtain a contradiction as in before case. Hence $a' \in N_P(b_P) \cap End(P)$. Let $C_1 := P \cup aa'$ and let $\mathcal{S}_1 := (\mathcal{S} - \{P\}) \cup \{C_1\}$. Then $f(C_1) = f(P) - 1$, and hence we can see that $\mathcal{S}_1$ is a $(k-1)$-ended stem system such that $V(\mathcal{S}_1) = V(\mathcal{S})$, which contradicts Claim 1. Therefore $N_G(a) \cap End(P') = \emptyset$ for any $P' \in \mathcal{P}_{\mathcal{S}}$.

Similarly, we can obtain $N_G(b) \cap End(P') = \emptyset$ for any $P' \in \mathcal{P}_{\mathcal{S}}$.

(v) Statement (v) can be shown by the same way as statement (iv). □

Claim 4 **(i)** *$N_G(w) \cap V(Stem(S)) = \emptyset$ for any $S \in \mathcal{S}$.*

(ii) *$N_G(w) \cap Leaf(S) = \emptyset$ for any $S \in \mathcal{C}^1_{\mathcal{S}}$.*

(iii) *$N_G(w) \cap End(P) = \emptyset$ for any $P \in \mathcal{P}_{\mathcal{S}}$.*

(iv) *$|N_G(w) \cap Leaf(S)| \leq |Leaf(S)| - 1$ for any $S \in \mathcal{C}^2_{\mathcal{S}}$ with $S \not\cong K_2$.*

Proof. Statements (i)–(iii) hold since otherwise we can obtain a larger k-ended stem system than $\mathcal{S}$, which contradicts the choice (S1).

(iv) Suppose that there exists $S \in \mathcal{C}^2_{\mathcal{S}}$ with $S \not\cong K_2$ such that $Leaf(S) \subseteq N_G(w)$. Let $a' \in N_S(a_S) \cap End(S)$, and $b' \in N_S(b_S) \cap End(S)$. Let $C_1 := S \cup wa' \cup wb'$, and let $\mathcal{S}_1 := (\mathcal{S} - \{S\}) \cup \{C_1\}$. Then $f(C_1) = f(S)$, and hence we can see that $\mathcal{S}_1$ is a k-ended stem system such that $|V(\mathcal{S}_1)| = |V(\mathcal{S})| + 1$, which contradicts the choice (S1). □

Claim 5 *$\left(N_G(a) \cap V(Stem(P'))\right)^- \cap N_G(b) = \emptyset$ for any $P' \in \mathcal{P}_{\mathcal{S}}$.*

Proof. Suppose that there exists $x \in \left(N_G(a) \cap V(Stem(P'))\right)^- \cap N_G(b)$.

First, suppose that $P' = P$. Let $C_1 := (P - xx^+) \cup ax^+ \cup bx$, and let $\mathcal{S}_1 := (\mathcal{S} - \{P\}) \cup \{C_1\}$. Then $f(C_1) = f(P) - 1$, and hence we can see that $\mathcal{S}_1$ is a $(k-1)$-ended stem system such that $V(\mathcal{S}_1) = V(\mathcal{S})$, which contradicts Claim 1.

Next, suppose that $P' \in \mathcal{P}_{\mathcal{S}} - \{P\}$. Let $P_1 := P \cup (P' - xx^+) \cup ax^+ \cup bx$, and let $\mathcal{S}_1 := (\mathcal{S} - \{P, P'\}) \cup \{P_1\}$. Then $f(P_1) = f(P) + f(P') - 2$, and hence we can see that $\mathcal{S}_1$ is a $(k-2)$-ended stem system such that $V(\mathcal{S}_1) = V(\mathcal{S})$, which contradicts Claim 1. □

By Claims 3 (iv) and 4 (iii), we can obtain the following claim.

Claim 6 *$\{a, b, w\}$ is an independent set.*

Claim 7 *$\{S \in \mathcal{S} : S \cong K_1\} = \emptyset$.*

Proof. Suppose that there exists $S' \in \mathcal{S}$ such that $S' \cong K_1$. Let $\{s\} := V(S')$. By Claim 1, $N_G(s) \cap V(Stem(S)) = \emptyset$ for any $S \in \mathcal{S}$. By the choice (S1) or (S3), $N_G(s) \cap Leaf(S) = \emptyset$ for any $S \in \mathcal{S}$. By the choice (S1), $N_G(s) \cap (V(G) - V(\mathcal{S})) = \emptyset$. These imply that $N_G(s) = \emptyset$. Since G is connected, this is a contradiction. □

Claim 8 (i) $d_S(a) + d_S(b) + d_S(w) \leq |S| - 2f(S) - 1$ *for any* $S \in \mathcal{S} - \{P\}$ *with* $S \not\cong K_2$.

(ii) $d_P(a) + d_P(b) + d_P(w) \leq |P| - 2f(P) + 2$.

Proof. (i) Let $S \in \mathcal{S} - \{P\}$ with $S \not\cong K_2$. By Claim 7, $S \not\cong K_1$, and hence $|S| \geq 3$.

First, suppose that $S \in \mathcal{C}_\mathcal{S}$. If $S \in \mathcal{C}^1_\mathcal{S}$, then, by Claims 3 (ii) and 4 (i), (ii), we obtain $d_S(a)+d_S(b)+d_S(w) = 0 \leq |S|-3 = |S|-2f(S)-1$. If $S \in \mathcal{C}^2_\mathcal{S}$, then, by Claims 3 (ii) and 4 (i), (iv), we obtain $d_S(a)+d_S(b)+d_S(w) = d_S(w) \leq |Leaf(S)|-1 = (|S|-|Stem(S)|)-1 = |S|-3 = |S|-2f(S)-1$. If $S \in \mathcal{C}^3_\mathcal{S}$, then, by Claims 3 (ii) and 4 (i), $d_S(a)+d_S(b)+d_S(w) = d_S(w) \leq |Leaf(S)| \leq |S| - 3 = |S| - 2f(S) - 1$. Therefore, in any cases, we obtain $d_S(a) + d_S(b) + d_S(w) \leq |S| - 2f(S) - 1$.

Next, suppose that $S \in \mathcal{P}_\mathcal{S}$. By Claim 5, $\left(N_G(a) \cap V(Stem(S))\right)^- \cap N_G(b) = \emptyset$. By Claim 3 (iii), $\left(N_G(a) \cap V(Stem(S))\right)^- \cup \left(N_G(b) \cap V(Stem(S))\right) \subseteq V(Stem(S)) - \{a_S, a_S^+, b_S\}$. These imply that $d_{Stem(S)}(a) + d_{Stem(S)}(b) \leq |Stem(S)| - 3$. Therefore, it follows from Claim 4 (i) that

$$d_{Stem(S)}(a) + d_{Stem(S)}(b) + d_{Stem(S)}(w) \leq |Stem(S)| - 3. \qquad (1)$$

By Claims 3 (iv) and 4 (iii), $d_{\langle Leaf(S)\rangle_G}(a) + d_{\langle Leaf(S)\rangle_G}(b) + d_{\langle Leaf(S)\rangle_G}(w) = d_{\langle Leaf(S)\rangle_G}(w) \leq |Leaf(S)| - |End(S)|$. Since $|End(S)| \geq 2$, we obtain

$$d_{\langle Leaf(S)\rangle_G}(a) + d_{\langle Leaf(S)\rangle_G}(b) + d_{\langle Leaf(S)\rangle_G}(w) \leq |Leaf(S)| - 2. \qquad (2)$$

By the inequalities (1) and (2), we obtain $d_S(a)+d_S(b)+d_S(w) \leq |S|-5 = |S| - 2f(S) - 1$.

(ii) Suppose that $d_P(a) + d_P(b) + d_P(w) \geq |P| - 2f(P) + 3$. By Claim 5, $\left((N_G(a) - \{a_P\}) \cap V(Stem(P))\right)^- \cap N_G(b) = \emptyset$. Hence, by Claim 4 (i), we obtain

$$d_{Stem(P)}(a) + d_{Stem(P)}(b) + d_{Stem(P)}(w) \leq |Stem(P)| + 1. \qquad (3)$$

By Claim 3 (iv), $(N_G(a) \cup N_G(b)) \cap Leaf(P) = \emptyset$. By Claim 4 (iii), $N_G(w) \cap Leaf(P) \subseteq Leaf(P) - End(P)$. Hence, we obtain

$$d_{\langle Leaf(S)\rangle_G}(a) + d_{\langle Leaf(S)\rangle_G}(b) + d_{\langle Leaf(S)\rangle_G}(w) \leq |Leaf(P)| - 2. \quad (4)$$

By the inequalities (3) and (4), we obtain $|P| - 2f(P) + 3 \leq d_P(a) + d_P(b) + d_P(w) \leq |P| - 1 = |P| - 2f(P) + 3$. Hence equalities hold above inequalities. By the equality (4), $End(P) = \{a, b\}$. We consider two cases.

Case 1. $|N_G(a) \cap N_G(b) \cap V(Stem(P))| = 1$

In this case, $V(Stem(P)) \subseteq N_G(a) \cup N_G(b)$ by the equality (3). Let $\{c\} := N_G(a) \cap N_G(b) \cap V(Stem(P))$. Let P_l and P_r be two components of $Stem(P) - c$ such that $a_P \in V(P_l)$ and $b_P \in V(P_r)$. Since $((N_G(a) - \{a_P\}) \cap V(Stem(P)))^- \cap N_G(b) = \emptyset$ and $V(Stem(P)) \subseteq N_G(a) \cup N_G(b)$, $v \in (N_G(a) - \{a_P\}) \cap V(Stem(P))$ implies $v^- \in N_G(a)$. This implies that $V(P_l) \cup \{c\} \subseteq N_G(a) \cap V(Stem(P))$. Similarly, $V(P_r) \cup \{c\} \subseteq N_G(b) \cap V(Stem(P))$. Since $((N_G(a) - \{a_P\}) \cap V(Stem(P)))^- \cap N_G(b) = \emptyset$, these imply that $V(P_l) \cup \{c\} = N_G(a) \cap V(Stem(P))$ and $V(P_r) \cup \{c\} = N_G(b) \cap V(Stem(P))$.

Let $a_1 \in V(P_l)$ and $P_1 := (P - a_1 a_1^+) \cup a a_1^+$. If $(Leaf(P) - \{a\}) \cap N_P(a_1) \neq \emptyset$, then $V(P) = V(P_1)$ and $V(Stem(P)) \cup \{a\} \subseteq V(Stem(P_1))$, which contradicts the choice (S2). Hence $(Leaf(P) - \{a\}) \cap N_P(a_1) = \emptyset$. Since $d_P(a) + d_P(b) = |Stem(P)| + 1$, it follows from the choice (S5) that $d_{P_1}(a_1) + d_{P_1}(b) \geq |Stem(P)| + 1$. Since $V(P_r) \cup \{c\} = N_G(b) \cap V(Stem(P))$, this implies that $V(P_l) \cup \{c\} = N_G(a_1) \cap V(Stem(P))$, especially $a_1 \in N_G(c)$. These imply that $V(P_l) \cup \{a\} \subseteq N_G(c)$ and $Leaf(P) \cap N_P(P_l) = \{a\}$. Similarly, $V(P_r) \cup \{b\} \subseteq N_G(c)$ and $Leaf(P) \cap N_P(P_r) = \{b\}$. Therefore, there exists a star S_1 such that $V(S_1) = V(P)$. Let $\mathcal{S}_1 := (\mathcal{S} - \{P\}) \cup \{S_1\}$. Then $f(S_1) = f(P) - 1$, and hence we can see that $\mathcal{S}_1$ is a $(k-1)$-ended stem system such that $V(\mathcal{S}_1) = V(\mathcal{S})$, which contradicts Claim 1.

Case 2. $|N_G(a) \cap N_G(b) \cap V(Stem(P)| \geq 2$

In this case, there exist $x_0 \in N_P(a)$ and $y_0 \in N_P(b)$ such that y_0 and x_0 are distinct, and are arranged in this order along $Stem(P)$. Choose such x_0 and y_0 so that distance between x_0 and y_0 along $Stem(P)$ is as small as possible. Since $((N_G(a) - \{a_P\}) \cap V(Stem(P)))^- \cap N_G(b) = \emptyset$, it follows from the equality (3) that $y_0^+ = x_0^-$. Suppose that $Leaf(P) \cap N_P(y_0^+) \neq \emptyset$. Let $P_1 := (P - y_0^+ x_0) \cup a x_0$. Then $V(P_1) = V(P)$ and $V(Stem(P)) \cup \{a\} \subseteq V(Stem(P_1))$, which contradicts the choice (S2). Hence $Leaf(P) \cap N_P(y_0^+) = \emptyset$. Let $C_1 := (P - y_0^+ x_0) \cup a x_0 \cup b y_0$ and $\mathcal{S}_1 := (\mathcal{S} - \{P\}) \cup \{C_1\}$. Then $Stem(C_1)$ is a cycle, and hence $\mathcal{S}_1$ is a $(k-1)$-ended stem system such that $V(\mathcal{S}_1) = V(\mathcal{S})$, which contradicts Claim 1. □

In the case $k = 2$, the theorem is proved in [3], therefore, we may assume that $k \geq 3$. Then $\mathcal{S} - \{P\} \neq \emptyset$.

Claim 9 *$S \cong K_2$ holds for any $S \in \mathcal{S} - \{P\}$.*

Proof. Suppose that there exists $S_0 \in \mathcal{S} - \{P\}$ such that $S_0 \ncong K_2$. By Claims 3 (ii) and 4 (i), $d_S(a) + d_S(b) + d_S(w) = 0 = |S| - 2f(S)$ for any $S \in \mathcal{S}$ with $S \cong K_2$. Therefore, by Claims 1, 3 (i), 6, 7 and 8, we obtain

$$\begin{aligned}
\sigma_3(G) &\leq d_G(a) + d_G(b) + d_G(w) \\
&\leq \sum_{S \in \mathcal{S}} \big(d_S(a) + d_S(b) + d_S(w)\big) + d_{\langle V(G) - V(\mathcal{S})\rangle_G}(w) \\
&\leq \sum_{S \in \mathcal{S} - \{P, S_0\}} \big(|S| - 2f(S)\big) + \big(|P| - 2f(P) + 2\big) \\
&\quad + \big(|S_0| - 2f(S_0) - 1\big) + \big(|V(G) - V(\mathcal{S})| - 1\big) \\
&\leq |G| - 2\sum_{S \in \mathcal{S}} f(S) \\
&= |G| - 2k.
\end{aligned}$$

This contradicts the assumption of Theorem 5. □

Let $S \in \mathcal{S} - \{P\}$. By Claim 9, $S \cong K_2$. Let $V(S) := \{s, s'\}$. By Claims 1 and 9, $N_G(S) \cap V(S') = \emptyset$ for any $S' \in \mathcal{S} - \{P, S\}$. By the choice (S1), $N_G(S) \cap \big(V(G) - V(\mathcal{S})\big) = \emptyset$. Suppose that there exists $t \in Leaf(P)$ such that $st \in E(G)$. Note that $t \notin End(P)$ by Claim 1. Let $\mathcal{S}_1 := (\mathcal{S} - \{S, P\}) \cup \{s'st, P - t\}$. Then $\mathcal{S}_1$ is a k-ended stem system such that $V(\mathcal{S}_1) = V(\mathcal{S})$, $\sum_{U \in \mathcal{P}_{\mathcal{S}_1}} |Stem(U)| = \sum_{U \in \mathcal{P}_{\mathcal{S}}} |Stem(U)|$, $|\{U \in \mathcal{S}_1 : U \cong K_1\}| = |\{U \in \mathcal{S} : U \cong K_1\}|$ and $|\{U \in \mathcal{S}_1 : U \cong K_2\}| < |\{U \in \mathcal{S} : U \cong K_2\}|$, which contradicts the choice (S4). Therefore, by the symmetry of s and s', we may assume that $N_G(s') \cap V(Stem(P)) \neq \emptyset$ because G is connected. Let

$r \in N_G(s') \cap V(Stem(P))$. If $N_G(s) \cap V(Stem(P)) \neq \emptyset$, we can obtain a $(k-1)$-ended stem system $\mathcal{S}'$ such that $V(\mathcal{S}') = V(\mathcal{S})$, which contradicts Claim 1. This implies that $N_G(s) = \{s'\}$, and hence $d_G(s) = 1$.

Case 1. $|Stem(P)| = 3$ and $|End(P)| = 2$

Note that $|G| \geq |\{w\}| + 2\sum_{S \in \mathcal{S}} f(S) + |P| - 4 \geq 2 + 2k$.

If $r = a_P$ or $r = b_P$, then $\mathcal{S}_1 := \mathcal{S} + s'r$ is a $(k-1)$-ended stem system such that $V(\mathcal{S}_1) = V(\mathcal{S})$, which contradicts Claim 1. Hence $r = a_P^+$. Let $\mathcal{S}_2 := (\mathcal{S} - a_P r) + s'r$. Then $\mathcal{S}_2$ is a k-ended stem system such that $V(\mathcal{S}_1) = V(\mathcal{S})$. In this system $\mathcal{S}_2$, a plays the same role of s in the system $\mathcal{S}$. By the symmetry of a and b, we can see that b also plays the same role of s. Note that $\{a, b, s\}$ is an independent set by Claims 3 (ii) and (iv). Then $|G| - 2k + 1 \leq \sigma_3(G) \leq d_G(a) + d_G(b) + d_G(s) = 3 \leq |G| - 2k + 1$. Hence equalities hold above inequalities. These equalities imply that $V(G) - V(\mathcal{S}) = \{w\}$ and $Leaf(P) = \{a, b\}$. Then $N_G(w) = \emptyset$ by Claims 4 (i) and (iii). Since G is connected, this is a contradiction.

Case 2. $|V(Stem(P))| \geq 4$ or $|End(P)| \geq 3$

We shall prove that $d_P(a) + d_P(w) \leq |P| - 2f(P)$. If $|End(P)| \geq 3$, then, by Claims 3 (iv), (v), 4 (i) and (iii), $d_P(a) + d_P(w) \leq |P| - |End(P)| - |\{b_P\}| \leq |P| - 4 = |P| - 2f(P)$. Suppose that $|Stem(P)| \geq 4$. Then, if necessary, by changing the orientation of P, we may assume that $r^+ \neq b_P$. Suppose that $r^+ \in N_G(a)$. Let $\mathcal{S}_1 := (\mathcal{S} - rr^+) + s'r + ar^+$. Then $\mathcal{S}_1$ is a $(k-1)$-ended stem system such that $V(\mathcal{S}_1) = V(\mathcal{S})$, which contradicts Claim 1. Therefore, $r^+ \notin N_G(a)$. Then, it follows from Claims 3 (iv), (v), 4 (i) and (iii) that $d_P(a) + d_P(w) \leq |P| - |End(P)| - |\{b_P, r^+\}| \leq |P| - 4 = |P| - 2f(P)$.

By Claims 3 (ii) and 4 (i), $d_Q(a) + d_Q(w) = 0 = |Q| - 2 = |Q| - 2f(Q)$ for any $Q \in \mathcal{S} - \{P\}$.

By Claim 3 (i), $d_{\langle V(G) - V(\mathcal{S})\rangle_G}(a) + d_{\langle V(G) - V(\mathcal{S})\rangle_G}(w) \leq |V(G) - V(\mathcal{S})| - 1$.

Note that $\{a, s, w\}$ is an independent set.

Therefore,

$$
\begin{aligned}
\sigma_3(G) &\le d_G(a) + d_G(w) + d_G(s) \\
&\le \sum_{S\in\mathcal{S}} \left(d_S(a) + d_S(w)\right) + \left(|V(G) - V(\mathcal{S})| - 1\right) + 1 \\
&\le \sum_{S\in\mathcal{S}} (|S| - 2f(S)) + |V(G) - V(\mathcal{S})| \\
&= |G| - 2\sum_{S\in\mathcal{S}} f(S) \\
&= |G| - 2k.
\end{aligned}
$$

This contradicts the assumption of Theorem 5. Therefore Theorem 5 holds and so does Theorem 4. □

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