

Introduction to Domination Polynomial of a Graph

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ABSTRACT

We introduce a domination polynomial of a graph G . The domination polynomial of a graph G of order n is the polynomial $D(G, x) = \sum_{i=\gamma(G)}^n d(G, i)x^i$, where $d(G, i)$ is the number of dominating sets of G of size i , and $\gamma(G)$ is the domination number of G . We obtain some properties of $D(G, x)$ and its coefficients. Also we compute this polynomial for some specific graphs.

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1 Introduction

Let $G = (V, E)$ be a simple graph. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is

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$N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* of G , if $N[S] = V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in S . The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G . A dominating set with cardinality $\gamma(G)$ is called a γ -set. For a detailed treatment of this parameter, the reader is referred to [5]. We denote the family of dominating sets of graph G with cardinality i by $\mathcal{D}(G, i)$.

The *corona* of two graphs G_1 and G_2 , as defined by Frucht and Harary in [4], is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 , where the i th vertex of G_1 is adjacent to every vertex in the i th copy of G_2 . The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G , where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added. The *join* of two graphs G_1 and G_2 , denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$.

A finite sequence of real numbers $(a_0, a_1, a_2, \dots, a_n)$ is said to be *unimodal* if there is some $k \in \{0, 1, \dots, n\}$, called the *mode* of sequence, such that

$$a_0 \leq \dots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \dots \geq a_n;$$

the mode is *unique* if $a_{k-1} < a_k > a_{k+1}$. A polynomial is called *unimodal* if the sequence of its coefficients is unimodal.

In the next section, we introduce the domination polynomial and obtain some of its properties. In Section 3, we study the coefficients of the domination polynomials. In the last section, we investigate the domination polynomial of the graph $G \circ K_1$, where $G \circ K_1$ is the corona of two graphs G and K_1 . Also we show that $D(G \circ K_1, x)$ is unimodal.

2 Introduction to domination polynomial

In this section, we state the definition of domination polynomial and some of its properties.

Definition 1. Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i) = |\mathcal{D}(G, i)|$. Then the domination polynomial $D(G, x)$ of G is defined as

$$D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i,$$

where $\gamma(G)$ is the domination number of G .

The path P_4 on 4 vertices, for example, has one dominating set of cardinality 4, four dominating sets of cardinalities 3 and 2; its domination polynomial is then $D(P_4, x) = x^4 + 4x^3 + 4x^2$. As another example, it is easy to see that, for every $n \in \mathbb{N}$, $D(K_n, x) = (1 + x)^n - 1$.

Theorem 1. If a graph G consists of m components G_1, \dots, G_m , then $D(G, x) = D(G_1, x) \cdots D(G_m, x)$.

Proof. It suffices to prove this theorem for $m = 2$. For $k \geq \gamma(G)$, a dominating set of k vertices in G arises by choosing a dominating set of j vertices in G_1 (for some $j \in \{\gamma(G_1), \gamma(G_1) + 1, \dots, |V(G_1)|\}$) and a dominating set of $k - j$ vertices in G_2 . The number of way of doing this over all $j = \gamma(G_1), \dots, |V(G_1)|$ is exactly the coefficient of x^k in $D(G_1, x)D(G_2, x)$. Hence both side of the above equation have the same coefficient, so they are identical polynomial. \square

As a consequence of Theorem 1, we have the following corollary for the empty graphs:

Corollary 1. Let \overline{K}_n be the empty graph with n vertices. Then $D(\overline{K}_n, x) = x^n$.

Proof. Since $D(\overline{K}_1, x) = x$, we have the result by Theorem 1. \square

Here, we provide a formula for the domination polynomial of the join of two graphs.

Theorem 2. *Let G_1 and G_2 be graphs of order n_1 and n_2 , respectively. Then*

$$D(G_1 \vee G_2, x) = \left((1+x)^{n_1} - 1 \right) \left((1+x)^{n_2} - 1 \right) + D(G_1, x) + D(G_2, x).$$

Proof. Let i be a natural number $1 \leq i \leq n_1 + n_2$. We want to determine $d(G_1 \vee G_2, i)$. If i_1 and i_2 are two natural numbers such that $i_1 + i_2 = i$, then clearly, for every $D_1 \subseteq V(G_1)$ and $D_2 \subseteq V(G_2)$, such that $|D_j| = i_j$, $j = 1, 2$, $D_1 \cup D_2$ is a dominating set of $G_1 \vee G_2$. Moreover, if $D \in \mathcal{D}(G_1, i)$, then D is a dominating set for $G_1 \vee G_2$ of size i . The same is true for every $D \in \mathcal{D}(G_2, i)$. Thus

$$D(G_1 \vee G_2, x) = \left((1+x)^{n_1} - 1 \right) \left((1+x)^{n_2} - 1 \right) + D(G_1, x) + D(G_2, x). \quad \square$$

As a corollary, we have the following formula for the domination polynomial of the complete bipartite graph $K_{m,n}$, the star $K_{1,n}$ and the wheel W_n .

Corollary 2.

- (i) $D(K_{m,n}, x) = ((1+x)^m - 1)((1+x)^n - 1) + x^m + x^n$.
- (ii) $D(K_{1,n}, x) = x^n + x(1+x)^n$.
- (iii) If $n \geq 4$, then $D(W_n, x) = x(1+x)^{n-1} + D(C_{n-1}, x)$.

Proof.

- (i) By applying Theorem 2 with $G_1 = \overline{K}_n$ and $G_2 = \overline{K}_m$, we have the result.
- (ii) It's suffices to apply Part (i) for $m = 1$.
- (iii) Since for every $n \geq 4$, $W_n = C_{n-1} \vee K_1$, we have the result by Theorem 2. \square

In Corollary 2(iii), we have a relationship between the domination polynomials of wheels and cycles. For the study of the domination polynomial of cycles, the reader is referred to [1].

3 Coefficients of domination polynomial

In this section, we obtain some properties of the coefficients of the domination polynomial of a graph.

The following theorem is an easy consequence of the definition of the domination polynomial.

Theorem 3. *Let G be a graph with $|V(G)| = n$. Then*

- (i) *If G is connected, then $d(G, n) = 1$ and $d(G, n - 1) = n$,*
- (ii) *$d(G, i) = 0$ if and only if $i < \gamma(G)$ or $i > n$.*
- (iii) *$D(G, x)$ has no constant term.*
- (iv) *$D(G, x)$ is a strictly increasing function in $[0, \infty)$.*
- (v) *Let G be a graph and H be any induced subgraph of G . Then $\deg(D(G, x)) \geq \deg(D(H, x))$.*
- (vi) *Zero is a root of $D(G, x)$, with multiplicity $\gamma(G)$.*

In the following theorem, we want to show that, from the domination polynomial of a graph G , we can obtain the number of isolated vertices, the number of K_2 -components and the number of vertices of degree one in G .

Theorem 4. *Let G be a graph of order n with t vertices of degree one and r isolated vertices. If $D(G, x) = \sum_{i=1}^n d(G, i)x^i$ is its domination polynomial, then the following hold:*

- (i) $r = n - d(G, n - 1)$.
- (ii) *If G has s K_2 -components, then $d(G, n - 2) = \binom{n}{2} - t + s - r(n - 1) + \binom{r}{2}$.*
- (iii) *If G has no isolated vertices and $D(G, -2) \neq 0$, then $t = \binom{n}{2} - d(G, n - 2)$.*

$$(iv) d(G, 1) = \left| \left\{ v \in V(G) \mid \deg(v) = n - 1 \right\} \right|.$$

Proof.

(i) Suppose that $A \subseteq V(G)$ is the set of all isolated vertices. Therefore by assumption, $|A| = r$. For any vertex $v \in V(G) \setminus A$, the set $V(G) \setminus \{v\}$ is a dominating set of G . Therefore $d(G, n - 1) = |V(G \setminus A)| = n - r$, and $r = n - d(G, n - 1)$.

(ii) Suppose that $D \subseteq V(G)$ is a set of cardinality $n - 2$ which is not a dominating set of G . We have three cases for D :

Case 1. $D = V(G) \setminus \{v, w\}$, where v is an isolated vertex and $v \in V(G) \setminus \{w\}$. Thus for every isolated vertex v , there are $n - 1$ vertices such that $V(G) \setminus \{v, w\}$ is not a dominating set. Therefore the total number of $(n - 2)$ -subsets of $V(G)$ of the form $V(G) \setminus \{v, w\}$ which is not dominating set (v or w is an isolated vertex) is $r(n - 1) - \binom{r}{2}$, since if v and w are isolated vertices, then we count $V(G) \setminus \{v, w\}$ for both v and w .

Case 2. $D = V(G) \setminus \{v, w\}$, for two adjacent vertices v and w with $\deg(v) = 1$. Since we have s K_2 -components, the number of such $\{v, w\}$ is $t - s$ and the proof is complete.

(iii) Since $D(G, -2) \neq 0$, by Theorem 1, G has no K_2 -component, and so by Part (ii), we obtain the result.

(iv) For every $v \in V(G)$, $\{v\}$ is a dominating set if and only if v is adjacent to all vertices. The proof is complete. \square

We recall that a subset M of $E(G)$ is called a *matching* in G if its elements are not loops and no two of them are adjacent in G ; the two ends of an edge in M are said to be *matched* under M . A matching M *saturates* a vertex v , and v is said to be *M -saturated* if some edges of M is incident with v ; otherwise v is *M -unsaturated*.

We need the following result to prove Theorem 6:

Theorem 5. (Hall [2], p.72) *Let G be a bipartite graph with bipartition (X, Y) . Then G contains a matching that saturates every vertex in X if and only if for all $S \subseteq X$, $|N(S)| \geq |S|$.*

Theorem 6. *Let G be a graph of order n . Then for every $0 \leq i < \frac{n}{2}$, we have $d(G, i) \leq d(G, i + 1)$.*

Proof. Consider a bipartite graph with two partite sets X and Y . The vertices of X are dominating sets of G of cardinality i , and the vertices of Y are all $(i + 1)$ -subsets of $V(G)$. Join a vertex A of X to a vertex B of Y , if $A \subseteq B$. Clearly, the degree of each vertex in X is $n - i$. Also for any $B \in Y$, the degree of B is at most $i + 1$. We claim that for any $S \subseteq X$, $|N(S)| \geq |S|$ and so by Hall's Marriage Theorem, the bipartite graph has a matching which saturate all vertices of X . By contradiction suppose that there exists $S \subseteq X$ such that $|N(S)| < |S|$. The number of edges incident with S is $|S|(n - i)$. Thus by pigeon hole principle, there exists a vertex $B \in Y$ with degree more than $n - i$. This implies that $i + 1 \geq n - i + 1$. Hence $i \geq \frac{n}{2}$, a contradiction. Thus for every $S \subseteq X$, $|N(S)| \geq |S|$ and the claim is proved. Since for every $A \in X$, and every $v \in V(G) \setminus A$, $A \cup \{v\}$ is a dominating set of cardinality $i + 1$, we conclude that $d(G, i + 1) \geq d(G, i)$ and the proof is complete. \square

Obviously the result in Theorem 6 is useful for the study of unimodality of domination polynomial. We state the following conjecture which is similar to the unimodal conjecture for chromatic polynomial (See [3], p.47):

Conjecture. *The domination polynomial of any graph is unimodal.*

4 Domination polynomial of $G \circ K_1$

Let G be any graph with vertex set $\{v_1, \dots, v_n\}$. Add n new vertices $\{u_1, \dots, u_n\}$ and join u_i to v_i for $1 \leq i \leq n$. By the definition of the corona

of two graphs, we shall denote this graph by $G \circ K_1$. We study $D(G \circ K_1, x)$ in this section. Also we show that $D(G \circ K_1, x)$ is unimodal.

We start with the following lemma:

Lemma 1. *For any graph G of order n , $\gamma(G \circ K_1) = n$.*

Proof. If D is a dominating set of G , then for every $1 \leq i \leq n$, $u_i \in D$ or $v_i \in D$. Therefore $|D| \geq n$. Since $\{u_1, \dots, u_n\}$ is a dominating set of $G \circ K_1$, we have $\gamma(G \circ K_1) = n$. \square

By Lemma 1, $d(G \circ K_1, m) = 0$ for $m < n$, so we shall compute $d(G \circ K_1, m)$ for $n \leq m \leq 2n$.

Theorem 7. *For any graph G of order n and $n \leq m \leq 2n$, we have $d(G \circ K_1, m) = \binom{n}{m-n} 2^{2n-m}$. Hence $D(G \circ K_1, x) = x^n(x+2)^n$.*

Proof. Suppose that D is a dominating set of $G \circ K_1$ of size m . There are $\binom{n}{m-n}$ possibilities to choose both vertices of an edge $\{u_i, v_i\}$ for D . Then there remain 2^{2n-m} possibilities to choose the other vertices by selecting for each pair $\{u_j, v_j\}$ exactly one of these vertices. Therefore

$$d(G \circ K_1, m) = \binom{n}{m-n} 2^{2n-m}. \quad \square$$

Here, we study the unimodality of the domination polynomial of $G_n \circ K_1$, where G_n denote a graph with n vertices. Let us denote $G \circ K_1$ simply by G^* . First we state and prove the following theorem for G_n^* .

Theorem 8. *For every $n \in \mathbb{N}$, $d(G_{3n+2}^*, 4n+2) = d(G_{3n+2}^*, 4n+3)$.*

Proof. By Theorem 7, $d(G_{3n+2}^*, 4n+2) = 2^{2n+2} \binom{3n+2}{n}$ and $d(G_{3n+2}^*, 4n+3) = 2^{2n+1} \binom{3n+2}{n+1}$. Since $2^{2n+2} \binom{3n+2}{n} = 2^{n+1} \binom{3n+2}{n+1}$, we have the result. \square

Theorem 9. (Unimodal Theorem for $G \circ K_1$) For every $n \in \mathbb{N}$,

- (i) $2^{3n} = d(G_{3n}^*, 3n) < d(G_{3n}^*, 3n + 1) < \dots < d(G_{3n}^*, 4n - 1) < d(G_{3n}^*, 4n) > d(G_{3n}^*, 4n + 1) > \dots > d(G_{3n}^*, 6n - 1) > d(G_{3n}^*, 6n) = 1$,
- (ii) $2^{3n+1} = d(G_{3n+1}^*, 3n + 1) < d(G_{3n+1}^*, 3n + 2) < \dots < d(G_{3n+1}^*, 4n) < d(G_{3n+1}^*, 4n + 1) > d(G_{3n+1}^*, 4n + 2) > \dots > d(G_{3n+1}^*, 6n + 1) > d(G_{3n+1}^*, 6n + 2) = 1$,
- (iii) $2^{3n+2} = d(G_{3n+2}^*, 3n + 2) < d(G_{3n+2}^*, 3n + 3) < \dots < d(G_{3n+2}^*, 4n + 2) = d(G_{3n+2}^*, 4n + 3) > d(G_{3n+2}^*, 4n + 4) > \dots > d(G_{3n+2}^*, 6n + 3) > d(G_{3n+2}^*, 6n + 4) = 1$.

Proof. Since the proof of all part are similar, we only prove the part (i):

- (i) We shall prove that $d(G_{3n}^*, i) < d(G_{3n}^*, i + 1)$ for $3n \leq i \leq 4n - 1$ and $d(G_{3n}^*, i) > d(G_{3n}^*, i + 1)$ for $4n \leq i \leq 6n - 1$. Suppose that $d(G_{3n}^*, i) < d(G_{3n}^*, i + 1)$. By Theorem 7 we have

$$2^{6n-i} \binom{3n}{i-3n} < 2^{6n-i-1} \binom{3n}{i-3n+1}$$

So we have $i < 4n - \frac{2}{3}$. On the other hand $i \geq 3n$. Together we have $3n \leq i \leq 4n - 1$. Similarly, we have $d(G_{3n}^*, i) > d(G_{3n}^*, i + 1)$ for $4n \leq i \leq 6n - 1$. \square

By Theorems 8 and 9, we observe that the mode for the family $\{D(G_{3n+2}^*, x)\}$ is not unique, but for the families $\{D(G_{3n}^*, x)\}$ and $\{D(G_{3n+1}^*, x)\}$, the mode is unique.

Remark. The unimodality of $D(G^*, x)$ (Theorem 9) also follows immediately from the fact that this polynomial has (except zero) only negative real roots. Hence, $D(G^*, x)$ is log-concave and consequently unimodal (see, for example, Wilf [7]).

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