$[a, b]$-Factors With Given Edges In Graphs *

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Abstract

Let G be a graph, and let a and b be integers with $1 \leq a \leq b$. An $[a, b]$-factor of G is defined as a spanning subgraph F of G such that $a \leq d_F(x) \leq b$ for each $x \in V(G)$. In this paper, we obtain a sufficient condition for a graph to have $[a, b]$-factors including given edges that extends a well-known sufficient condition for the existence of a k-factor.

Keywords: graph, subgraph, minimum degree, $[a, b]$-factor.

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1 Introduction

The graphs considered in this paper will be finite undirected graphs without loops or multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G. Furthermore, we denote the minimum degree of G by $\delta(G)$. For a subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S and $G - S$ denote the subgraph obtained from G by deleting all the vertices of S together with the edges incident with the vertices of S. Let S and T be two disjoint vertex subsets of G, we use $e_G(S, T)$ to denote the number

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of edges joining S to T. Let a and b be nonnegative integers with $a \leq b$. An $[a,b]$-factor of G is defined as a spanning subgraph F of G such that $a \leq d_F(x) \leq b$ for each $x \in V(G)$. Note that if $a = b = k$, then an $[a,b]$-factor is a k-factor.

Many authors have investigated k-factors [3, 6, 7] and $[a,b]$-factors [5, 9, 10, 11]. The following result on k-factors is known.

Theorem 1 *(Egawa [1]). Let $k \geq 2$ be an integer, and let G be a graph of order n with kn even. If $\delta(G) > n + 2k - 2\sqrt{kn+1}$, then G has a k-factor.*

The following results on the existence of an $[a,b]$-factor including given edges are known.

Theorem 2 *(Matsuda [4]). Let a, b, m and t be integers such that $1 \leq a < b$ and $2 \leq \lceil \frac{b-m+1}{a} \rceil$. Suppose that G be a graph of order $n > \frac{(a+b)(t(a+b-1)-1)+2m}{b}$ and $\delta(G) \geq a$. Let H be any subgraph of G with $|E(H)| = m$. If*

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_t)| \geq \frac{an+2m}{a+b}$$

for every independent set $\{x_1, x_2, \cdots x_t\} \subseteq V(G)$, then G has an $[a,b]$-factor including H.

Theorem 3 *(Zhou [8]). Let G be a graph of order n, and let a, b and m be nonnegative integers with $1 \leq a < b$ and $b \geq m$. If bind$(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-2m+2}$ and $n \geq \frac{(b-1)(a+b-1)(a+b-2)+2bm}{b(b-1)}$, then for any subgraph H of G with m edges, G has an $[a,b]$-factor including H.*

In this paper, we obtain a new sufficient condition for graphs to have $[a,b]$-factors including given edges. Our result is an extension of Theorem 1. The main result will be given in the following section.

2 The Main Theorem and Its Proof

Now, we show the main theorem in this paper.

Theorem 4 *Let a, b and m be nonnegative integers with $1 \leq a < b$ and $m \leq b$, and let G be a graph of order $n \geq \frac{(a+b+\sqrt{2m})^2-1}{b}$. Let H be any subgraph of G with $|E(H)| = m$. If $\delta(G) > n + a + b - 2\sqrt{bn+1}$, then G has an $[a,b]$-factor F including H (i.e., $E(H) \subseteq E(F)$).*

We do not know whether the condition $\delta(G) > n+a+b-2\sqrt{bn+1}$ in Theorem 4 can be replaced by $\delta(G) \geq n+a+b-2\sqrt{bn+1}$.

In Theorem 4, if $m=0$, then we get the following corollary.

Corollary 1 *Let a and b be integers with $1 \leq a < b$, and let G be a graph of order $n \geq \frac{(a+b)^2-1}{b}$. If $\delta(G) > n+a+b-2\sqrt{bn+1}$, then G has an $[a,b]$-factor.*

In Theorem 4, if $m=1$, then we get the following corollary.

Corollary 2 *Let a and b be integers with $1 \leq a < b$, and let G be a graph of order $n \geq \frac{(a+b+\sqrt{2})^2-1}{b}$. If $\delta(G) > n+a+b-2\sqrt{bn+1}$, then for any $e \in E(G)$, G has an $[a,b]$-factor including e.*

In order to prove Theorem 4, we depend on the following theorem.

Theorem 5 *(Lam etc. [2]). Let $1 \leq a < b$ be integers, and let G be a graph and H a subgraph of G. Then G has an $[a,b]$-factor F such that $E(H) \subseteq E(F)$ if and only if*

$$b|S| + \sum_{x\in T} d_{G-S}(x) - a|T| \geq \sum_{x\in S} d_H(x) - e_H(S,T)$$

for all disjoint subsets S and T of $V(G)$.

Proof of Theorem 4. Suppose a graph G satisfies the condition of the theorem, but has no desired $[a,b]$-factor for some subgraph H with m edges. Then by Theorem 5, there exist two disjoint subsets S and T of $V(G)$ such that

$$b|S| + \sum_{x\in T}(d_{G-S}(x) + e_H(S,x) - a) - \sum_{x\in S} d_H(x) \leq -1. \tag{1}$$

We choose such subsets S and T so that $|T|$ is minimum.

At first, we prove the following claims.

Claim 1. $S \neq \emptyset$.

Proof. If $S = \emptyset$, then by (1) and $n \geq \frac{(a+b+\sqrt{2m})^2-1}{b}$, we obtain

$$\begin{aligned} -1 &\geq \sum_{x\in T}(d_G(x) - a) \geq \sum_{x\in T}(\delta(G) - a) \\ &> \sum_{x\in T}(n+a+b-2\sqrt{bn+1} - a) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{x\in T}(n+b-2\sqrt{bn+1})\\
&= \sum_{x\in T}\frac{bn+b^2-2b\sqrt{bn+1}}{b}\\
&= \sum_{x\in T}\frac{(\sqrt{bn+1}-b)^2-1}{b}\\
&\geq \sum_{x\in T}\frac{(a+\sqrt{2m})^2-1}{b}\\
&\geq 0.
\end{aligned}
$$

which is a contradiction.

Claim 2. $|T| \geq b-m+1$.

Proof. If $|T| \leq b-m$, then by (1), Claim 1 and $\sum_{x\in S} d_H(x) \leq m|S|$, we have

$$
\begin{aligned}
-1 &\geq b|S| + \sum_{x\in T}(d_{G-S}(x)+e_H(S,x)-a) - \sum_{x\in S} d_H(x)\\
&\geq m|S| + \sum_{x\in T}(|S|+d_{G-S}(x)+e_H(S,x)-a) - \sum_{x\in S} d_H(x)\\
&\geq m|S| + \sum_{x\in T}(|S|+d_{G-S}(x)-a) - \sum_{x\in S} d_H(x)\\
&\geq m|S| + \sum_{x\in T}(\delta(G)-a) - \sum_{x\in S} d_H(x)\\
&\geq \sum_{x\in T}(\delta(G)-a)\\
&> \sum_{x\in T}(n+a+b-2\sqrt{bn+1}-a)\\
&= \sum_{x\in T}(n+b-2\sqrt{bn+1})\\
&\geq 0.
\end{aligned}
$$

This is a contradiction.

Claim 3. $d_{G-S}(x) + e_H(S,x) \leq a-1$ for each $x \in T$.

Proof. If $d_{G-S}(x) + e_H(S,x) \geq a$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (1). This contradicts the choice of S and T.

According to Claim 2, obviously $T \neq \emptyset$. In the following, we define

$$h = \min\{d_{G-S}(x) + e_H(S,x) | x \in T\},$$

and choose $x_1 \in T$ such that $d_{G-S}(x_1) + e_H(S, x_1) = h$. Then by Claim 3, we obtain

$$0 \le h \le a - 1.$$

In view of the condition of Theorem 4, the following inequalities hold:

$$\begin{aligned} n + a + b - 2\sqrt{bn+1} &< \delta(G) \le d_G(x_1) \le d_{G-S}(x_1) + |S| \\ &\le h - e_H(S, x_1) + |S| \\ &\le h + |S|, \end{aligned}$$

which implies,

$$|S| > n + a + b - 2\sqrt{bn+1} - h. \qquad (2)$$

According to (1), (2), $n \ge \frac{(a+b+\sqrt{2m})^2-1}{b}$, $|S|+|T| \le n$ and $\sum_{x\in S} d_H(x) \le 2m$, we have

$$\begin{aligned} -1 &\ge b|S| + \sum_{x\in T}(d_{G-S}(x) + e_H(S,x) - a) - \sum_{x\in S} d_H(x) \\ &\ge b|S| + (h-a)|T| - 2m \\ &\ge b|S| - (a-h)(n-|S|) - 2m \\ &= (a+b-h)|S| - (a-h)n - 2m \\ &> (a+b-h)(n+a+b-2\sqrt{bn+1}-h) - (a-h)n - 2m \\ &= (a+b-h)n + (a+b-h)^2 - 2(a+b-h)\sqrt{bn+1} - (a-h)n - 2m \\ &= bn + 1 - 2(a+b-h)\sqrt{bn+1} + (a+b-h)^2 - 2m - 1 \\ &= (\sqrt{bn+1} - (a+b-h))^2 - 2m - 1 \\ &\ge (\sqrt{bn+1} - (a+b))^2 - 2m - 1 \\ &\ge (\sqrt{2m})^2 - 2m - 1 \\ &= -1, \end{aligned}$$

which is a contradiction.
Completing the proof of Theorem 4.

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