On a Class of Blockwise-Bursts in Array Codes

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Abstract. In this paper, we introduce the notion of blockwise-bursts in array codes equippped with m-metric [13] and obtain some bounds on the parameters of m-metric array codes for the detection and correction of blockwise-burst array errors.

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1. Introduction

In a classical coding setting, codes are subsets/subspaces of ambient space F_q^n and are investigated with respect to the Hamming metric [12]. In [13], m-metric or RT-metric array codes which are subsets/subspaces of linear space of all m by s matrices $\text{Mat}_{m\times s}(F_q)$ with entries from a finite field F_q endowed with a non-Hamming metric were introduced and some bounds on code parameters were obtained. Motivated by the occurrence of cluster errors in parallel channel communication systems, the author has already introduced the class of usual bursts [7], CT-bursts [8], cyclic bursts [11] in m-metric array codes. In this paper, we introduce another category of bursts viz. blockwise-bursts of order $p \times r$ and study the error detecting and error correcting capabilities of linear m-metric array codes with respect to these types of errors.

2. Definitions and Notations

Let F_q be a finite field of q elements. Let $\text{Mat}_{m\times s}(F_q)$ denote the linear space of all $m \times s$ matrices with entries from F_q. An m-metric array code is a subset of $\text{Mat}_{m\times s}(F_q)$ and a linear m-metric array code is an F_q–linear

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subspace of $\text{Mat}_{m\times s}(F_q)$. Note that the space $\text{Mat}_{m\times s}(F_q)$ is identifiable with the space F_q^{ms}. Every matrix in $\text{Mat}_{m\times s}(F_q)$ can be represented as a $1 \times ms$ vector by writing the first row of matrix followed by second row and so on. Similarly, every vector in F_q^{ms} can be represented as an $m \times s$ matrix in $\text{Mat}_{m\times s}(F_q)$ by separating the co-ordinates of the vector into m groups of s-coordinates.

The weight and metric defined by Rosenbloom and Tsfasman [13] on the space $\text{Mat}_{m\times s}(F_q)$ are as follows :

Let $X \in \text{Mat}_{m\times 1}(F_q)$ with

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{pmatrix},$$

then column weight (or weight) of X is given by

$$wt_c(X) = \begin{cases} m - \max\{\, i \mid x_k = 0 \quad \text{for any } k \leq i\} & \text{if } X \neq 0 \\ 0 & \text{if } X = 0. \end{cases}$$

This definition of wt_c can be extended to $m \times s$ matrices in the space $\text{Mat}_{m\times s}(F_q)$ as

$$wt_c(A) = \sum_{j=1}^{s} wt_c(A_j)$$

where $A = [A_1, A_2, \cdots, A_s] \in \text{Mat}_{m\times s}(F_q)$ and A_j denotes the j^{th} column of A. Then wt_c satisfies $0 \leq wt_c(A) \leq n(= ms)$ and determines a metric on $\text{Mat}_{m\times s}(F_q)$ if we set $d(A, A') = wt_c(A - A')\ \forall\ A, A' \in \text{Mat}_{m\times s}(F_q)$. We call this metric as column-metric. Note that for $m = 1$, it is just the usual Hamming metric.

There is an alternative equivalent way of defining the weight of an $m \times s$ matrix using the weight of its rows [4]:

Let $Y \in \text{Mat}_{1\times s}(F_q)$ with $Y = (y_1, y_2, \cdots, y_s)$. Define row weight (or weight) of Y as

$$wt_\rho(Y) = \begin{cases} \max\{\, i \mid y_i \neq 0\} & \text{if } Y \neq 0 \\ 0 & \text{if } Y = 0. \end{cases}$$

Extending the definitions of wt_ρ to the class of $m \times s$ matrices as

$$wt_\rho(A) = \sum_{i=1}^{m} wt_\rho(R_i)$$

where $A = \begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_m \end{bmatrix} \in \text{Mat}_{m\times s}(F_q)$ and R_i denotes the i^{th} row of A. Then wt_ρ satisfies $0 \leq wt_\rho(A) \leq n(= ms)\ \forall\ A \in \text{Mat}_{m\times s}(F_q)$ and determines a metric on $\text{Mat}_{m\times s}(F_q)$ known as row-metric or ρ-metric.

It turns out that row weight of a vector is equal to the column weight of transpose of the vector with its component reversed and hence the two metrices viz. row-metric and column-metric give rise to equivalent codes and both the metrices have been known as m-metric or RT-metric.

In this paper, we take distance and weight in the sense of row-metric (or ρ-metric). Throughout this paper, $< x, y >$ will denote the minimum of x and y and $[x]$ as the greatest integer less than equal to x.

3. Blockwise-Bursts in m-Metric Array Codes

We now define blockwise-bursts in m-metric array codes:

Definition 3.1. A blockwise-burst of order1 pr(or $p \times r$)$(1 \leq p \leq m, 1 \leq r \leq s)$ in the space $\text{Mat}_{m\times s}(F_q)$ is an $m \times s$ matrix A such that all the nonzero entries of matrix A are confined to a $p \times r$ submatrix B of it with first and last entry in each of the p rows of B are nonzero.

Remark 3.2. (i) For $p = 1$, the class of blockwise-bursts reduces to the class of classical bursts [5].

(ii) For $r = 1$, every entry in $p \times 1$ column vector B in the definition of blockwise-burst is required to be nonzero

Definition 3.3. A blockwise-burst of order pr or less $(1 \leq p \leq m,\ 1 \leq r \leq s)$ in the space $\text{Mat}_{m\times s}(F_q)$ is a blockwise-burst of order cd (or $c \times d$) where $1 \leq c \leq p \leq m$ and $1 \leq d \leq r \leq s$.

Example 3.4. Consider the linear space $\text{Mat}_{3\times 3}(F_2)$. Then all blockwise-bursts of order 2×3 are given by:

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We now obtain a bound for the correction of blockwise-burst errors in linear m-metric array codes.

Theorem 3.5. *An (n, k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$ that corrects all blockwise-bursts of order $pr(1 \le p \le m, 1 \le r \le s)$ must satisfy*

$$q^{n-k} \ge 1 + \mathcal{B}^{p\times r}_{m\times s}(F_q), \tag{1}$$

where $\mathcal{B}^{p\times r}_{m\times s}(F_q)$ is the number of blockwise-bursts of order $pr(1 \le p \le m, 1 \le r \le s)$ in $Mat_{m\times s}(F_q)$ and is given by

$$\mathcal{B}^{p\times r}_{m\times s}(F_q) = \begin{cases} ms(q-1) & \text{if } p = 1,\ r = 1, \\ (m-p+1)s(q-1)^p & \text{if } p \ge 2, r = 1, \\ (m-p+1)(s-r+1)(q-1)^{2p}q^{(r-2)p} & \text{if } p \ge 1,\ r \ge 2. \end{cases} \tag{2}$$

Proof. Consider a blockwise-burst $A \in \mathrm{Mat}_{m\times s}(F_q)$ of order $pr(1 \le p \le m, 1 \le r \le s)$. Let B be the $p \times r$ nonzero submatrix of A such that all the nonzero entries of A are confined to B with first and last entries in each of the p rows of B are nonzero. There are three cases depending upon the values of p and r.

Case 1. When $p = 1,\ r = 1$.

In this case, the number of starting positions for the 1×1 nonzero submatrix B in $m \times s$ matrix A is ms and these ms positions can be filled by $(q-1)$ nonzero elements from F_q. Therefore, the number of blockwise-bursts of order 1×1 in $\mathrm{Mat}_{m\times s}(F_q)$ is given by

$$\mathcal{B}^{1\times 1}_{m\times s}(F_q) = ms(q-1).$$

Case 2. When $p \geq 2, \ r = 1$.

In this case, the number of starting positions for the $p \times 1$ nonzero column submatrix B in $m \times s$ matrix A is $(m-p+1)s$ and entries in the $p \times 1$ submatrix B can be selected in $(q-1)^p$ ways. Therefore, the number of blockwise-bursts of order $p \times 1$ in $\mathrm{Mat}_{m\times s}(F_q)$ is given by

$$\mathcal{B}^{p\times 1}_{m\times s}(F_q) = (m-p+1)s(q-1)^p.$$

Case 3. When $p \geq 1, \ r \geq 2$.

In this case, the number of starting positions for the $p \times r$ nonzero submatrix B is $(m-p+1)(s-r+1)$ and entries in B can be selected in $(q-1)^{2p}q^{(r-2)p}$ ways. Therefore, the number of blockwise-bursts of order $p \times r$ $(p \geq 1, \ r \geq 2)$ in $\mathrm{Mat}_{m\times s}(F_q)$ is given by

$$\mathcal{B}^{p\times r}_{m\times s}(F_q) = (m-p+1)(s-r+1)(q-1)^{2p}q^{(r-2)p}.$$

Combining the three cases, we get (2).

Now, since the linear m-metric array code $V \subseteq \mathrm{Mat}_{m\times s}(F_q)$ corrects all blockwise-bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$, therefore, all the blockwise-bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ including the null $m \times s$ matrix must belong to different cosets of the standard array. Since number of available cosets $= q^{n-k}$. Therefore, we must have

$$q^{n-k} \geq 1 + \mathcal{B}^{p\times r}_{m\times s}(F_q)$$

where $\mathcal{B}^{p\times r}_{m\times s}(F_q)$ is given by (2) and we get (1). □

Remark 3.6.

(i) Take $m = s = 3, p = 2, r = 3$ and $q = 2$ in $\mathcal{B}^{p\times r}_{m\times s}(F_q)$ computed in (2). We get $\mathcal{B}^{2\times 3}_{3\times 3}(F_2) = 2 \times 4 = 8$ and these 8 blockwise-bursts of order 2×3 in $\mathrm{Mat}_{3\times 3}(F_2)$ are listed in Example 3.4.

(ii) Take $m = s = 3, p = 1, r = 2$ and $q = 2$ in $\mathcal{B}^{p\times r}_{m\times s}(F_q)$ computed in (2). We get $\mathcal{B}^{1\times 2}_{3\times 3}(F_2) = 3 \times 2 = 6$ and these 6 blockwise-bursts of order 1×2 in $\mathrm{Mat}_{3\times 3}(F_2)$ are listed below:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Now, we prove Fire's bound in linear m-metric array codes for blockwise-burst error correction.

Theorem 3.7. *(Fire's bound) The number of parity check digits required for an (n, k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$, that corrects all blockwise-bursts of order pr or less $(1 \le p \le m, 1 \le r \le s)$ is at least*

$$log_q\left[1 + \sum_{c=1}^{p}\sum_{d=1}^{r}\mathcal{B}_{m\times s}^{c\times d}(F_q)\right],$$

where $\mathcal{B}_{m\times s}^{c\times d}(F_q)$ is given by (2).

Proof. Follows directly from Theorem 3.5 and Definition 3.3. □

4. Blockwise-Bursts with Weight Constraint in m-Metric Array Codes

In this section, we obtain a lower bound on the number of parity check digits required to correct all blockwise-bursts of order pr or less $(1 \le p \le m, 1 \le r \le s)$ in $\text{Mat}_{m\times s}(F_q)$ having weight (or ρ-weight) w or less $(1 \le w \le ms)$.

The bound obtained is analogous to the Hamming bound for random error correction [13]. We first prove a lemma that enumerates the number of blockwise-bursts of order $pr(1 \le p \le m, 1 \le r \le s)$ having ρ-weight w or less.

Lemma 4.1. *The number of blockwise-bursts of order $pr(1 \le p \le m, 1 \le r \le s)$ in $Mat_{m\times s}(F_q)$ having ρ-weight w or less $(1 \le w \le ms)$ is given by*

$$\mathcal{B}_{m\times s}^{p\times r}(F_q, w) = \begin{cases} m \times \min(w, s) \times (q-1) & \text{if } p = r = 1, \\ m \times \min(w - r + 1, s - r + 1)\times \\ \times (q-1)^2 q^{r-2} & \text{if } p = 1, r \ge 2, \\ (m - p + 1) \times \min([\frac{w}{p}], s) \times (q-1)^p & \text{if } p \ge 2, r = 1, \\ (m - p + 1) \times L \times (q-1)^{2p} \times q^{(r-2)p} & \text{if } p \ge 2, r \ge 2, \end{cases} \quad (3)$$

where

$$L = \max\left(0, \min\left(\left[\frac{w}{p}\right] - r + 1, s - r + 1\right)\right). \tag{4}$$

Proof. Consider a blockwise-burst $A = \begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ A_m \end{bmatrix}$ where $A_i = (a_{i_1}, a_{i_2}, \cdots, a_{i_s})$, of order $pr (1 \leq p \leq m, 1 \leq r \leq s)$ having ρ-weight w or less $(1 \leq w \leq ms)$. Let B be the $p \times r$ nonzero submatrix of A such that all the nonzero entries of A are confined to B with first and last entries in each of the p rows of B are nonzero. There are four cases depending upon the values of p and r.

Case 1. When $p = 1,\ r = 1$.

In this case, the number of starting positions for the 1×1 nonzero submatrix B in $m \times s$ matrix A is $m \times \min(w, s)$ and these $m \times \min(w, s)$ positions can be filled by $(q-1)$ nonzero elements from F_q. Therefore, the number of blockwise-bursts of order 1×1 having ρ-weight w or less in $\mathrm{Mat}_{m \times s}(F_q)$ is given by

$$\mathcal{B}^{1\times1}_{m\times s}(F_q, w) = m \times \min(w, s) \times (q-1).$$

Case 2. When $p = 1,\ r \geq 2$.

In this case, the number of starting positions for the $1 \times r$ nonzero submatrix B in $m \times s$ matrix A is $m \times \min(w-r+1, s-r+1)$ and entries in the $1 \times r$ submatrix B can be selected in $(q-1)^2 q^{r-2}$ ways as the first and last components of the single rowed submatrix B can be chosen in $(q-1)^2$ ways and intermediate $(r-2)$ components can be chosen in q^{r-2} ways. Therefore, the number of blockwise-bursts of order $1 \times r$ having ρ-weight w or less in $\mathrm{Mat}_{m \times s}(F_q)$ is given by

$$\mathcal{B}^{1\times r}_{m\times s}(F_q, w) = m \times \min(w - r + 1, s - r + 1) \times (q-1)^2 q^{r-2}.$$

Case 3. When $p \geq 2,\ r = 1$.

In this case, the $p \times 1$ nonzero column vector B can have (i, j) as its starting positions in $m \times s$ matrix A where i can vary from 1 to $(m-p+1)$

and j can vary from 1 to $\min\left(\left[\frac{w}{p}\right], s\right)$. With (i,j) as the starting position of $p \times 1$ nonzero column matrix B, entries in B can be filled in $(q-1)^p$ ways. Therefore, number of blockwise-bursts of order $p \times 1$ having ρ-weight w or less in $\mathrm{Mat}_{m\times s}(F_q)$ is given by

$$\mathcal{B}^{p\times 1}_{m\times s}(F_q, w) = (m-p+1) \times \min\left(\left[\frac{w}{p}\right], s\right) \times (q-1)^p.$$

Case 4. When $p \geq 2,\ r \geq 2$.

In this case, the number of starting positions for the $p \times r$ nonzero submatrix B in $m \times s$ matrix A is $(m-p+1) \times L$ where L is given by (4) and entries in submatrix B can be filled in $(q-1)^{2p} q^{(r-2)p}$ ways. Therefore, the number of blockwise-bursts of order $p \times r$ having ρ-weight having w or less in $\mathrm{Mat}_{m\times s}(F_q)$ is given by

$$\mathcal{B}^{p\times r}_{m\times s}(F_q, w) = (m-p+1) \times L \times (q-1)^{2p} q^{(r-2)p},$$

where L is given by (4). □

Remark 4.2. For $w = ms$, the expression for $\mathcal{B}^{p\times r}_{m\times s}(F_q, w)$ computed in (3) reduces to $\mathcal{B}^{p\times r}_{m\times s}(F_q)$ computed in (2).

Example 4.3. Take $m = s = 3, p = r = 2, q = 2$ and $w = 3$ in Lemma 4.1. Then number of blockwise-bursts of order 2×2 having ρ-weight 3 or less in $\mathrm{Mat}_{3\times 3}(F_2)$ is given by :

$$\mathcal{B}^{2\times 2}_{3\times 3}(F_2, 3) \quad = \quad 2 \times \max(0, \min(0,2)) \times 1 = 2 \times 0 = 0$$

Thus, there is no blockwise-burst of order 2×2 having ρ-weight 3 or less in $\mathrm{Mat}_{3\times 3}(F_2)$.

Example 4.4. Take $m = s = 3, p = 2, r = 3, q = 2$ and $w = 6$ in Lemma 4.1. Then $\mathcal{B}^{2\times 3}_{3\times 3}(F_2, 4)$ is given by:

$$\mathcal{B}^{2\times 3}_{3\times 3}(F_2, 3) \quad = \quad 2 \times \max(0, \min(1,1)) \times 2^2 = 8.$$

These 8 blockwise-bursts of order 2×3 having ρ-weight 6 or less in $\mathrm{Mat}_{3\times 3}(F_2)$ are listed in Example 3.4.

Now, we obtain a lower bound on the number of parity check digits for the correction of blockwise-bursts of order pr(or less) having ρ-weight w or less $(1 \leq w \leq ms)$.

Theorem 4.5. *An (n,k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$ that corrects all blockwise-bursts of order pr $(1 \leq p \leq m, 1 \leq r \leq s)$ having ρ-weight w or less $(1 \leq w \leq ms)$ must satisfy*

$$q^{n-k} \geq 1 + \mathcal{B}^{p\times r}_{m\times s}(F_q, w)$$

where $\mathcal{B}^{p\times r}_{m\times s}(F_q, w)$ is given by (3) in Lemma 4.1.

Proof. The proof follows from the fact that the number of available cosets must be greater than or equal to the number of correctable error matrices including the null matrix. □

Theorem 4.6. *An (n,k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$ that corrects all blockwise-bursts of order pr or less $(1 \leq p \leq m, 1 \leq r \leq s)$ having ρ-weight w or less $(1 \leq w \leq ms)$ must satisfy*

$$q^{n-k} \geq 1 + \sum_{c=1}^{p} \sum_{d=1}^{r} \mathcal{B}^{c\times d}_{m\times s}(F_q, w)$$

where $\mathcal{B}^{c\times d}_{m\times s}(F_q, w)$ is given by Lemma 4.1.

Proof. Follows directly from Theorem 4.5 and Definition 3.3. □

5. Construction Bounds for Blockwise-Burst Error Detection and Correction in Linear m-Metric Array Codes

In this section, we obtain construction bounds for blockwise-burst error detection and correction. To obtain the desired bounds, we shall identify the space $\text{Mat}_{m\times s}(F_q)$ with the space F_q^{ms} i.e. an $m \times s$ matrix over F_q is considered as an ms-tuple over F_q arranged into m groups of s elements each. Each group of s elements in an ms-tuple is called a block. Also, s is called the block length or block size and m is the number of blocks. Each block of an ms-tuple has a ρ-weight and sum of ρ-weights of all the m blocks of an ms-tuple is the ρ-weight of that ms-tuple. Also, columns of generator matrix G and parity check matrix H of a linear m-metric array code V are grouped into m blocks of s columns each. Therefore, generator matrix G and parity check matrix H of a linear m-metric array code V are represented as $G = [G_1, G_2, \cdots, G_m], H = [H_1, H_2, \cdots, H_m]$ where G_i and H_i are the i^{th} block $(1 \leq i \leq m)$ of generator and parity check matrix respectively of the code V and are given by

$$G_i = [G_{i1}, G_{i2}, \cdots, G_{is}],$$

and

$$H_i = [H_{i1}, H_{i2}, \cdots, H_{is}],$$

where each $G_{ij}(1 \leq i \leq m, 1 \leq j \leq s)$ is a $k \times 1$ column vector and each $H_{ij}(1 \leq i \leq m, 1 \leq j \leq s)$ is an $(ms - k) \times 1$ column vector.

Throughout our discussion, we use the following terminology:

Definition 5.1. A vector $v \in F_q^n$ is said to be a *strict linear combination* of the vectors $v_1, v_2, \cdots, v_m$ *from the left hand side* (respectively *right hand side*) if

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m, \quad \alpha_i \in F_q$$

where $\alpha_1 \neq 0$(respectively $\alpha_m \neq 0$).

Definition 5.2. A vector $v \in F_q^n$ is said to be a *strict linear combination* of the vectors $v_1, v_2, \cdots, v_m$ *from both sides* if

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m, \quad \alpha_i \in F_q$$

where $\alpha_1, \alpha_m \neq 0$.

Definition 5.3. A vector $v \in F_q^n$ is said to be a *strict linear combination* of the vectors $v_1, v_2, \cdots, v_m$ if

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m,$$

where $\alpha_i \in F_q/\{0\}$.

Now, we obtain a construction (upper) bound for blockwise-burst error detection in linear m-metric array codes.

Theorem 5.4. *Let q be prime or power of prime and m, s, p, r, k be positive integers satisfying $1 \leq p \leq m, 1 \leq r \leq s$ and $1 \leq k \leq ms$, then there exists an $[m \times s, k]$ linear m-metric array code over F_q i.e. a linear m-metric array code with m as the number of blocks and s as the block size, that has no blockwise-burst of order pr or less as a code array provided*

$$q^{ms-k} > 1 + (q-1)^{p-1} + \sum_{j=2}^{s}((q-1)^2 q^{<j-2,r-2>})^{p-1}(q-1)q^{<j-2,r-2>}). \qquad (5)$$

Proof. The existence of such a code will be proved by constructing a suitable $(ms-k) \times ms$ parity check matrix H for the desired code. To detect any blockwise-burst of order pr or less, it is necessary and sufficient that no strict linear combination from both sides involving r (or fewer) consecutive columns in p(or fewer) consecutive blocks should be zero. Suppose that $i-1 (1 \leq i \leq m)$ blocks $H_1, H_2, \cdots, H_{i-1}$ have been chosen suitably. To add the j^{th} column $(1 \leq j \leq s)$ in the i^{th} block, we consider following two cases:

Case 1. When $j = 1$.

In this case, the j^{th} column (i.e. the first column) in the i^{th} block may be added provided it is not a strict linear combination (i.e. all the scalars are non-zero) of the first column from the immediately preceeding $< i - 1, p - 1 >$ blocks. Therefore, column H_{i1} in the i^{th} block can be added to H provided that

$$H_{i1} \neq \sum_{g=i-<i,p>+1}^{i-1} \alpha_{g,1} H_{g,1} \quad \text{where } \alpha_{g,1} \neq 0 \ \forall \ g. \qquad (6)$$

The number of linear combinations occuring in (6) is given by

$$(q-1)^{<i-1,p-1>}. \qquad (7)$$

Case 2. When $2 \leq j \leq s$.

In this case, the j^{th} column $(2 \leq j \leq s)$ in the i^{th} block may be added provided it is not a strict linear combination from both sides of $l_j^{th}, (l_j +$

$1)^{th}, \cdots, j^{th}$ columns from the immediately preceding $< i-1, p-1 >$ blocks (where $l_j =< 1, j-r+1 >$) together with strict linear combination from left hand side of $l_j^{th}, (l_j+1)^{th}, \cdots, (j-1)^{th}$ columns in the i^{th} block. Therefore, column $H_{ij} (2 \leq j \leq s)$ in the i^{th} block can be added to H provided

$$\begin{aligned} H_{ij} \neq & \sum_{g=i-<i,p>+1}^{i-1} (\alpha_{g,l_j} H_{g,l_j} + \alpha_{g,l_j+1} H_{g,l_j+1} + \cdots + \alpha_{g,j} H_{g,j}) \\ & + \alpha_{i,l_j} H_{i,l_j} + \alpha_{i,l_j+1} H_{i,l_j+1} + \cdots + \alpha_{i,j-1} H_{i,j-1} \end{aligned} \tag{8}$$

where $\alpha_{g,l_j}, \alpha_{g,j} \neq 0 \ \forall \ g$ and also $\alpha_{i,l_j} \neq 0$.

Note that summation in (8) will not run at all if the lower limit of the summation is greater than the upper limit and this will occur when $< i, p >= 1$ and in this case value of the summation is assumed to be zero.

Now, the number of linear combinations occuring in (8) is given by

$$((q-1)^2 q^{<j-2,r-2>})^{<i-1,p-1>} (q-1) q^{<j-2,r-2>}. \tag{9}$$

Therefore, i^{th} block can be added to H provided the summation of number of linear combinations enumerated in (7) and (9) for $1 \leq j \leq s$ including the pattern of all zeros is less than the total number of $(ms-k)$-tuples. Therefore, i^{th} block H_i can be added to H provided that

$$\begin{aligned} q^{ms-k} > 1 + (q-1)^{<i-1,p-1>} & + \sum_{j=2}^{s} (q-1)^2 (q^{<j-2,r-2>})^{<i-1,p-1>} \times \\ & \times (q-1) q^{<j-2,r-2>}. \end{aligned} \tag{10}$$

For the existence of an $[m \times s, k]$ linear m-metric array code, inequality (10) should hold for $i = m$ so that it is possible to add up to the m^{th} block to form an $(ms-k) \times ms$ parity check matrix and we get (5). (Note that $1 \leq p \leq m$ gives $< m-1, p-1 > \ = p-1$). □

Example 5.5. Take $m = s = 3, p = r = 2, k = 4$ and $q = 2$.

Then

$$\begin{aligned} \text{R.H.S. of (5)} &= 1 + 1 + \sum_{j=2}^{3} (2^{<j-2,0>})^1 2^{<j-2,0>} \\ &= 1 + 1 + 1 + 1 = 4 \end{aligned}$$

Also, L.H.S. of (5) $= q^{ms-k} = 2^{9-4} = 2^5 = 32$.

Therefore, L.H.S. of (5) $= 32 > 4 =$ R.H.S. of (5).

Thus, sufficient condition of Theorem 5.4. is satisfied for the chosen parameters and hence there exists a $[3 \times 2, 2]$ linear m-metric array code over F_2 detecting all blockwise-bursts of order 2×2 or less. Consider the following $(3\times 2-2)\times(3\times 2) = 4\times 6$ parity check matrix of a $[3\times 2, 2]$ linear m-metric array code over F_2 constructed by the algorithm discussed in the proof of Theorem 5.4.

$$H = \begin{bmatrix} 1 & 0 & \vdots & 0 & 0 & \vdots & 1 & 1 \\ 0 & 1 & \vdots & 0 & 0 & \vdots & 0 & 1 \\ 0 & 0 & \vdots & 1 & 0 & \vdots & 1 & 0 \\ 0 & 0 & \vdots & 0 & 1 & \vdots & 0 & 0 \end{bmatrix}_{4\times 6}$$

The generator matrix of the code corresponding to the parity check matrix H is given by

$$G = \begin{bmatrix} 1 & 0 & \vdots & 1 & 0 & \vdots & 1 & 0 \\ 1 & 1 & \vdots & 0 & 0 & \vdots & 0 & 1 \end{bmatrix}_{2\times 6}$$

The four code arrays of the code $V \subseteq \mathrm{Mat}_{3\times 2}(F_2)$ with G as generator matrix and H as parity check matrix are given by

$$v_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, v_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We note that none of the code arrays is a blockwise-burst of order 2×2 or less over F_2. Therefore, construction condition (5) is verified.

Now, we obtain a construction (upper) bound for blockwise-burst error correction analogous to Campopiano's bound [2].

Theorem 5.6. *Let q be prime or power of prime and m, s, p, r, k be positive integers satisfying $1 \le p \le [m/2], 1 \le r \le s$ and $1 \le k \le ms$, then a sufficient condition for the existence of an $[m \times s, k]$ linear m-metric array code over F_q that corrects all blockwise-bursts of order pr or less is given* *by*

$$q^{ms-k} > 1+\Bigg((q-1)^{p-1}+\Bigg(\sum_{j=2}^{s}((q-1)^2q^{<j-2,r-2>})^{p-1}\times$$
$$\times(q-1)q^{<j-2,r-2>}\Bigg)\Bigg)\Bigg(\sum_{c=1}^{p}\sum_{d=1}^{r}\mathcal{B}^{c\times d}_{(m-p)\times s}(F_q)\Bigg). \quad (11)$$

where $\mathcal{B}^{c\times d}_{(m-p)\times s}(F_q)$ *is given by (1).*

Proof. The existence of such a code will be proved as in previous theorem by constructing a suitable parity check matrix for the code. To correct all blockwise-bursts of order pr or less, it is necessary and sufficient that no code array consist of the sum of two blockwise-bursts of order pr or less. Thus, no strict linear combination from both sides involving two sets of r (or fewer) consecutive columns in p (or fewer) consecutive blocks should be zero. Suppose that $m-1$ blocks $H_1, H_2, \cdots, H_{m-1}$ of the parity check matrix H have been chosen suitably. Then j^{th} column $(1 \le j \le s)$ in the m^{th} block may be added, provided that it is not a strong linear combination from both sides of $l_j^{th}, (l_j+1)^{th}, \cdots, j^{th}$ columns from the immediately preceding $p-1$ blocks (where $l_j =< 1, j-r+1 >$) together with strong linear combination from left of $l_j^{th}, (l_j+1)^{th}, \cdots, (j-1)^{th}$ columns in the m^{th} block and any set of r (or fewer) consecutive columns in p (or fewer) consecutive blocks among the first $(m-p)$ blocks which from a blockwise-burst of order pr or less. In other words, column $H_{mj} (1 \le j \le s)$ in the m^{th} block can be added to H provided that

$$\begin{aligned} H_{mj} &\ne \sum_{g=m-p+1}^{m-1}(\alpha_{g,l_j}H_{g,l_j}+\alpha_{g,l_j+1}H_{g,l_j+1}+\cdots,\alpha_{g,j},H_{g,j}) \\ &\quad + \alpha_{m,l_j}H_{m,l_j}+\alpha_{m,l_j+1}H_{m,l_j+1}+\cdots,\alpha_{m,j-1},H_{m,j-1} \\ &\quad + \text{strong linear combination from both sides which form a} \\ &\quad \text{blockwise-burst of order } pr \text{ or less among the first } (m-p) \text{ blocks} \\ &= G_j + P. \end{aligned}$$

where

$$G_j = \sum_{g=m-p+1}^{m-1}(\alpha_{g,l_j}H_{g,l_j}+\alpha_{g,l_j+1}H_{g,l_j+1}+\cdots+\alpha_{g,j}H_{g,j})$$

$$+\alpha_{m,l_j}H_{m,l_j} + \alpha_{m,l_j+1}H_{m,l_j+1} + \cdots + \alpha_{m,j-1}H_{m,j-1},$$

P = strong linear combinations from both sides which form a blockwise-burst of order pr or less among the first $(m-p)$ blocks.

Also, as in Theorem 5.4, the value of G_j $(1 \le j \le s)$ is given by

$$G_j = \begin{cases} (q-1)^{p-1} & \text{for } j = 1, \\ ((q-1)^2 q^{<j-2,r-2>})^{p-1}(q-1)q^{<j-2,r-2>} & \text{for } 2 \le j \le s. \end{cases} \tag{12}$$

The number of strong linear combinations from both sides which form a blockwise-burst of order pr or less in the space of $(m-p)\times s$ matrices is given by

$$\sum_{c=1}^{p}\sum_{d=1}^{r} \mathcal{B}^{c\times d}_{(m-p)\times s}(F_q). \qquad \text{(refer Theorems 3.5 and 3.7)} \tag{13}$$

To add all the s columns in the m^{th} block, the number of available $(ms-k)$-tuples must be greater $1+R$ where R is obtained by adding G_j for $i \le j \le s$ from (12) and then multiplying by (13) and is given by

$$R = \left((q-1)^{p-1} + \left(\sum_{j=2}^{s}((q-1)^2 q^{<j-2,r-2>})^{p-1}(q-1)q^{<j-2,r-2>}\right)\right) \times$$
$$\times\left(\sum_{c=1}^{p}\sum_{d=1}^{r}\mathcal{B}^{c\times d}_{(m-p)\times s}(F_q)\right).$$

At worst, all these linear combinations might yield a distinct sum. Therefore, m^{th} block H_m can be added to H provided

$$q^{ms-k} > 1 + \left((q-1)^{p-1} + \left(\sum_{j=2}^{s}((q-1)^2 q^{<j-2,r-2>})^{p-1} \times\right.\right.$$
$$\left.\left.(q-1)q^{<j-2,r-2>}\right)\right)\left(\sum_{c=1}^{p}\sum_{d=1}^{r}\mathcal{B}^{c\times d}_{(m-p)\times s}(F_q)\right).$$

Thus we conclude that if (11) is satisfied, then it is possible to construct an $(ms-k)\times ms$ parity check matrix of an $[m\times s, k]$ linear m-metric array code which corrects all blockwise-bursts of order pr or less. □

Further, for a given $r (1 \le r \le s)$, let p_g^r be the largest value of p satisfying inequality (11). Then for $p = p_g^r + 1$, the opposite inequality is satisfied and the following theorem giving another upper bound on the number of parity checks holds:

Theorem 5.7. *There exists an $[m \times s, k]$ linear m-metric array code over F_q that corrects any single blockwise-burst of order $p_g^r \times r$ or less where $1 \leq r \leq s, 1 \leq p_g^r < [m/2]$, for which the following inequality is satisfied:*

$$ms - k \leq log_q\Bigg(1 + \bigg((q-1)^{p_g^r} + \bigg(\sum_{j=2}^{s}((q-1)^2 q^{<j-2,r-2>})^{p_g^r} \times \times (q-1)q^{<j-2,r-2>}\bigg)\bigg)\bigg(\sum_{c=1}^{p_g^r+1}\sum_{d=1}^{r} \mathcal{B}_{(m-p_g^r-1)\times s}^{c\times d}(F_q)\bigg)\Bigg).$$

Example 5.8. Take $m = s = 3, p = 1, r = 2, q = 2$ and $k = 3$, Then

$$\begin{aligned} \text{R.H.S. of (11)} &= 1 + \bigg(1 + \bigg(\sum_{j=2}^{3}(2^{<j-2,0>})^0 2^{<j-2,0>}\bigg)\bigg) \times \\ &\quad \times \bigg(\sum_{c=1}^{1}\sum_{d=1}^{2} \mathcal{B}_{2\times 3}^{c\times d}(F_2)\bigg) \\ &= 1 + (1+1+1) \times \Big(\mathcal{B}_{2\times 3}^{1\times 1}(F_2) + \mathcal{B}_{2\times 3}^{1\times 2}(F_2)\Big) \\ &= 1 + 3(6 + 2\times 2) = 31 \end{aligned}$$

Also, L.H.S. of (11) $= 2^{ms-k} = 2^{9-3} = 2^6 = 64$.

Therefore, L.H.S. of (11) $= 64 > 31$ =R.H.S. of (11) and hence by Theorem 5.6, there exists a $[3 \times 3, 3]$ linear m-metric array code over F_2 that corrects any blockwise-burst of order 1×2 or less.

Consider the following $(3 \times 3 - 3) \times (3 \times 3) = 6 \times 9$ parity check matrix of a $[3 \times 3, 3]$ linear m-metric array code over F_2 constructed by the synthesis procedure outlined in the proof of Theorem 5.6.

$$H = \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 1 & 0 & 1 \\ 0 & 1 & 0 & \vdots & 0 & 0 & 0 & \vdots & 1 & 0 & 1 \\ 0 & 0 & 1 & \vdots & 0 & 0 & 0 & \vdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \vdots & 1 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 & \vdots & 1 & 0 & 0 \end{bmatrix}_{6\times 9}$$

We now claim that the code $V \subseteq \text{mat}_{3\times3}(F_2)$ which is the null subspace of H corrects all blockwise-bursts of order 1×2 or less. The claim is verified from Table 5.1 which shows that syndromes of all blockwise-burst errors of order 1×2 or less are all distinct.

Table 5.1

Blockwise-burst Errors of order 1×2 or less	Syndromes
$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ = (110 000 000)	(110000)
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ = (011 000 000)	(011000)
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ = (000 110 000)	(000110)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ = (000 011 000)	(000011)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ = (000 000 110)	(111101)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ = (000 000 011)	(110100)

Table contd.

Blockwise-burst Errors of order 1×2 or less	Syndromes
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (100\ 000\ 000)$	(100000)
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (010\ 000\ 000)$	(010000)
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (001\ 000\ 000)$	(001000)
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (000\ 100\ 000)$	(000100)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (000\ 010\ 000)$	(000010)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = (000\ 001\ 000)$	(000001)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (000\ 000\ 100)$	(110001)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (000\ 000\ 010)$	(001100)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (000\ 000\ 001)$	(111000)

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References

[1] M. Blaum, P.G. Farrell and H.C.A. van Tilborg, *Array Codes*, in Handbook of Coding Theory, (Ed.: V. Pless and Huffman), Vol. II, Elsevier, North-Holland, 1998, pp.1855-1909.

[2] C.N. Campopiano, *Bounds on Burst Error Correcting Codes*, IRE. Trans., IT-8 (1962), 257-259.

[3] S.T. Dougherty and M.M. Skriganov, *MacWilliams duality and the Rosenbloom-Tsfasman metric*, Moscow Mathematical Journal, 2 (2002), 83-99.

[4] S.T. Dougherty and M.M. Skriganov, *Maximum Distance Separable Codes in the p-metric over Arbitrary Alphabets*, Journal of Algebraic Combinatorics, 16 (2002), 71-81.

[5] P. Fire, *A Class of Multiple-Error-Correcting Binary Codes for Non-Independent Errors*, Sylvania Reports RSL-E-2, 1959, Sylvania Reconnaissance Systems, Mountain View, California.

[6] E.M. Gabidulin and V.V. Zanin, *Matrix codes correcting array errors of size* 2×2, International Symp. on Communication Theory and Applications, Ambleside, U.K., 11-16 June, 1993.

[7] S. Jain, *Bursts in m-Metric Array Codes*, Linear Algebra and Its Applications, 418 (2006), 130-141.

[8] S. Jain, *Campopiano-Type Bounds in Non-Hamming Array Coding*, Linear Algebra and Its Applications, 420 (2007), 135-159.

[9] S. Jain, *An Algorithmic Approach to Achieve Minimum* ρ*-Distance at least* d *in Linear Array Codes*, Kyushu Journal of Mathematics, 62 (2008), 189-200.

[10] S. Jain, *CT Bursts- From Classical to Array Coding*, Discrete Mathematics, 308 (2008), 1489-1499.

[11] S. Jain, *On a Class of Cyclic Bursts in Array Codes*, to appear in Ars Combinatoria.

[12] W.W. Peterson and E.J. Weldon, Jr., Error Correcting Codes, 2nd Edition, MIT Press, Cambridge, Massachusetts, 1972.

[13] M. Yu. Rosenbloom and M.A. Tsfasman, *Codes for m-metric*, Problems of Information Transmission, 33 (1997), 45-52.

[14] I. Siap, *The Complete Weight Enumerator for Codes over* $M_{m\times s}(F_q)$, Lecture Notes on Computer Sciences, 2260 (2001), 20-26.