The cordiality of the complement of a graph

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abstract A necessary and sufficient condition of the complement to be cordial and its application are obtained.

Keyword Cordial Complement

1 Introduction

We shall consider finite, undirected, simple graphs and $0-1$ labeling only.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We define a labeling f on $V(G)$ by giving each $v \in V(G)$ a label $f(v) = 0 or 1$, and denote $V_i = \{v : v \in V(G), f(v) = i\}$, $i = 0, 1$. From a labeling on $V(G)$, we derive a $0-1$ labeling on $E(G)$ by giving each $uv \in E(G)$ a label $f(uv) = |f(u) - f(v)|$, and denote $E_i = \{uv : uv \in E(G), |f(u) - f(v)| = i\}$, $i = 0, 1$. The vertex x is called i vertex, if $f(x) = i$. The edge uv is called i edge, if $f(uv) = i$. We denote by $v_i = v_i(G)$ and $e_i = e_i(G)$ the number of elements of $V_i(G)$ and $E_i(G)$, respectively.

If there exists a labeling f on $V(G)$ such that $|v_0 - v_1| \le 1$ and $|e_0 - e_1| \le 1$, then G is said to be cordial [1] and f is said to be a cordial labeling of G. The cordiality of the union graphs, join graphs, Cartesian product graphs and so on have been discussed [2]. We shall consider the cordiality of the complement of a graph.

2 Fundmental lemmas

Lemma 1 Let f be a labeling of K_{2n} or K_{2n+1} such that $|v_0 - v_1| \le 1$, then $e_1 = e_0 + n$.

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One can calculate the e_0 and e_1 easily. We omit the proof.

Lemma 2 Let $|G| = 2n$ *or* $2n+1$ and f be a labeling on $V(G)$, then f is a cordial labeling of complement $\overline{G}$ if and only if $|v_0(f,G) - v_1(f,G)| \leq 1$, $|e_1(f,G) - e_0(f,G) - n| \leq 1$.

Proof Let $|G| = m$. Note that $v_i(f,G) = v_i(f,\overline{G}), i = 0,1$, and $e_i(G) + e_i(\overline{G}) = e_i(K_m)$. We have $|e_1(\overline{G}) - e_0(\overline{G})| = |e_1(K_m) - e_1(G) - [e_0(K_m) - e_0(G)]| = |e_0(G) - e_1(G) - [e_0(K_m) - e_1(K_m)]| = |e_1(G) - e_0(G) - n|$. It implies the statement in lemma 2.

Example 1 The cordiality on $\overline{P_2 \times P_n}$.

Case 1. When n is odd, let $n = 2m+1$. Let the labels of the vertices in the first line of the grid $P_2 \times P_n$ $1,1,0,0$ alternatively in order, the labels of the vertices in the second line $0,0,1,1$ alternatively in order. We can get $v_0 = v_1 = 2m+1$, $e = 2 \times 2 \times (2m+1) - 2 - (2m+1) = 6m+1$, $e_0 = 2m$, $e_1 = 6m+1-2m = 4m+1$, so $|e_1 - e_0 - 2 \times (2m+1)/2| = |4m+1-2m-(2m+1)| = 0 \leq 1$. By lemma 2, when n is odd, $\overline{P_2 \times P_n}$ is cordial.
Case 2. When n is even, let $n = 2m$. Let the labels of the vertices in the first line of the grid $P_2 \times P_n$ $0,1$ alternatively in order except the last two, the labels of the last two $1,0$ in order, the labels of the vertices in the second line $0,1$ alternatively in order. We can get $v_0 = v_1 = 2m$, $e_0 = 2m-1$, $e_1 = 2 \times (2 \times 2m - 2m - 2 - (2m-1)) = 4m-1$, so $|e_1 - e_0 - v/2| = 0$. By lemma 2, when n is even, $\overline{P_2 \times P_n}$ is cordial.

Example 2 The cordiality on $\overline{P_2 \times C_n}$.

Case 1. When n is odd, let $n = 2m+1$. We can see $\overline{P_2 \times C_n}$ is even degree graph and we can get $e = (4m+2-4) \times (4m+2)/2 = (4m-2)(2m+1) \equiv 2(mod4)$, by proposition 1 of [4], $\overline{P_2 \times C_n}$ is not cordial.
Case 2. When n is even, let $n = 2m$. Let the labels of the vertices in the outer circle of the grid $P_2 \times c_n$ $0,1$ alternatively in order, the labels of the vertices in the inner circle are same to the labels of the vertices in the circle of outside. We can get $e_0 = 2m$, $e_1 = 6m - 2m = 4m$, $|e_1 - e_0 - 2m| = 0$, By lemma 2, when n is even, $\overline{P_2 \times C_n}$ is cordial.

3 The cordiality of $\overline{\bigcup_{i=1}^{m} P_{n_i}}$

Lemma 3 For every path P_n of order n and $k \in \{1, 2, \cdots, n-1\}, n \geq 2$, there exists a labeling f of P_n such that $|v_0(P_n) - v_1(P_n)| \leq 1$ and $e_1(P_n) = k$.

Proof When $n = 2$, the statement is clear. Assume that the statement is true for $n \leq m$. Put $n = m+1$. If $k \in \{1, 2, \cdots, m-1\}$, we give a labeling for P_m such that $|v_0(P_m) - v_1(P_m)| \leq 1$ and $e_1(P_m) = k$. Choosing an edge $uv \in E(P_m)$ with $f(uv) = 1$. By adding a vertex w in uv, and putting $f(w) = 0$ if $v_0(P_m) \leq v_1(P_m)$ or $f(w) = 1$, if $v_0(P_m) > v_1(P_m)$. We obtain a desired labeling f of P_{m+1}. If $k = m$, we give a labeling with 0,1 alternately from the first vertex of P_{m+1} to the end. It is just the labeling with $|v_0 - v_1| \leq 1$ and $e_1 = m$.

Lemma 4 If $n_i = 2$, $G = \bigcup_{i=1}^{m} P_{n_i}$, $m \geq 1$, $i = 1, 2, \cdots, m$, then for each $k \in \{m, m+1, \cdots, \sum_{i=1}^{m} n_i - m\}$, there exists a labeling f of G such that $|v_0(G) - v_1(G)| \leq 1$ and $e_1(G) = k$.

Proof Since $k \in \{m, m+1, \cdots, \sum_{i=1}^{m} n_i - m\}$, it is easily to see that there are numbers $r_1, r_2, \cdots, r_m$ such that $k = r_1 + r_2 + \cdots + r_m$, $1 \leq r_i \leq n_i - 1$. Note that when we change the label of each vertex of the path P_{n_i}, the label of each edge in P_{n_i} keep with same. Hence by lemma 3, we can give a labeling to each P_{n_i}, such that $e_1(P_{n_i}) = r_i$ and $|v_0(\bigcup_{i=1}^{m} P_{n_i}) - v_1(\bigcup_{i=1}^{m} P_{n_i})| \leq 1$. The compound labeling is desired.

Theorem 1 Every $\overline{G} = \overline{\bigcup_{i=1}^{m} P_{n_i}}$ is cordial, $n_i \geq 2$.

Proof Note that $e_1(G) + e_0(G) = \sum_{i=1}^{m} n_i - m$ for every labeling of $G = \bigcup_{i=1}^{m} P_{n_i}$. It implies that when $e_1(G)$ runs throughout $\{m, m+1, \cdots, \sum_{i=1}^{m} n_i - m\}$, $e_1(G) - e_0(G)$ runs throughout $\{3m - \sum_{i=1}^{m} n_i, 3m - \sum_{i=1}^{m} n_i + 2, \cdots, \sum_{i=1}^{m} n_i - m - 2, \sum_{i=1}^{m} n_i - m\}$. Since $3m - \sum_{i=1}^{m} n_i \leq (\sum_{i=1}^{m} n_i)/2 \leq \sum_{i=1}^{m} n_i - m$, there exists a labeling of $\bigcup_{i=1}^{m} P_{n_i}$ such that $|v_0(\bigcup_{i=1}^{m} P_{n_i}) - v_1(\bigcup_{i=1}^{m} P_{n_i})| \leq 1$ and $|e_1 - e_0 - [(\sum_{i=1}^{m}(n_i))/2]| \leq 1$. By lemma 2, $\overline{\bigcup_{i=1}^{m} P_{n_i}}$ is cordial.

4 The cordiality of $\overline{C_n}$

Lemma 5 Let C_k be a cycle of order k
(1) For any labeling of C_n, e_1 is even.
(2) If $n \geq 2$, for each $k \in \{2, 4, \cdots, 2n\}$, there exits a labeling f of C_{2n} such that $v_0 = v_1$ and $e_1 = k$.
(3) If $n \geq 1$, for each $k \in \{2, 4, \cdots, 2n\}$, there exits a labeling f of C_{2n+1} such that $|v_0 - v_1| = 1$ and $e_1 = k$.

Proof (1) Note that changing the label of any vertex u of C_n, it just makes the labels of the two edges which incident to u changed. On the other hand, when every vertex of C_n has label 0, $e_1 = 0$ is even. Hence (1) is clear.
(2)The statement is trivial for $n = 2$. Suppose the statement holds for $2n$. If $k \in \{2, 4, \cdots, 2n\}$, we give a labeling f as desired to C_{2n}. Choosing an edge $uw \in E(C_{2n})$ with $f(uw) = 1$. Adding two vertices x, y in the edge uw adjacent to u and w respectively, we give x and y two labels such that $f(x) = f(u)$, $f(y) = f(w)$, thus C_{2n+2} has a desired labeling. If $k = 2n+2$, let the labels of the vertices in C_{2n+2} 0, 1 alternatively in order. This is a desired labeling for C_{2n+2}.
(3)The proof is similar to (2).

Theorem 2 $\overline{C_n}$ is not cordial iff $n \equiv 4(mod8)$ or $n \equiv 7(mod8)$.

Proof (1)By lemma 5(2). There is a labeling f of C_{8m} such that $v_0 = v_1$ and $e_1(C_{8m}) = 6m$. Hence $e_1 - e_0 = 4m$. By lemma 2, we know that f is a cordial label of $\overline{C_{8m}}$.
(2)By lemma 5(3). There is a labeling f of C_{8m+1} such that $v_0 = v_1$ and $e_1(C_{8m+1}) = 6m$. Obviously $e_1 - e_0 = 4m - 1$. By lemma 2, f is a cordial label of $\overline{C_{8m+1}}$.
(3)By lemma 5(2). There is a labeling f of C_{8m+2} such that $v_0 = v_1$ and $e_1(C_{8m+2}) = 6m + 2$. We obtain that $e_1 - e_0 = 4m$, then f is a cordial label of $\overline{C_{8m+2}}$.
(4) By lemma 5(3) and lemma 2 we can obtain a cordial labeling of $\overline{C_{8m+3}}$.
(5)Since $v(\overline{C_{8m+4}}) + e(\overline{C_{8m+4}}) = 8m + 4 + (4m + 2)(8m + 1) \equiv 2(mod4)$ and [3]. We know that $\overline{C_{8m+4}}$ is not cordial.
(6)By lemma 5(3). There is a labeling f of C_{8m+5} such that $|v_0 - v_1| = 1$ and $e_1(C_{8m+5}) = 6m + 4$. It implies that $e_1 - e_0 = 4m + 3$. By lemma 2, f is a cordial labeling of $\overline{C_{8m+5}}$.
(7)By lemma 5(2). There is a labeling f of C_{8m+6} such that $v_0 = v_1$ and $e_1(C_{8m+6}) = 6m + 4$. Obviously, $e_1 - e_0 = 4m + 2$. By lemma 2, this is a cordial labeling.
(8)Since $v(\overline{C_{8m+7}}) + e(\overline{C_{8m+7}}) = 8m + 7 + (8m + 7)(8m + 4)/2$ the degree of each vertex of $\overline{C_{8m+7}}$ is even. By proposition 1 of [4], we know that $\overline{C_{8m+7}}$ is not cordial.

5 The cordiality of $\overline{F_n}$

The fan F_n of order n is the join of a vertex w and a path $P_{x_1 \cdots x_{n-1}}$.
Lemma 6 (1)For each $k \in \{n + 1, n + 2, \cdots, 3n - 1\}$. There exists a labeling f of the fan F_{2n+1} of order $2n + 1$ such that $|v_0 - v_1| = 1$ and $e_1 = k$.
(2)For each $k \in \{n, n + 1, \cdots, 3n - 2\}$, there exists a labeling f of the fan F_{2n} of order $2n$.

Proof (1)Let F_{2n+1} be the join of vertex w and a path P_{2n}. For any

$k \in \{n+1, n+2, \cdots, 3n-1\}$ then $k-n \in \{1, 2, \cdots, 2n-1\}$. By lemma 3, there is a labeling f of path P_{2n} such that $v_0(P_{2n}) = v_1(P_{2n})$ and $e_1(P_{2n}) = k-n$. Put $f(w) = 0$, then the fan F_{2n+1} has a labeling f. It is just desired.
(2)The proof is similar to (1).

Theorem 3 Every $\overline{F_n}$ is cordial for $n \geq 3$.

Proof (1)$n = 2m$
Subcase1 m is odd. Since $m \leq (5m-3)/2 \leq 3m-2$, by lemma 6(2), there exists a labeling f of F_{2m} such that $v_0 = v_1$ and $e_1 = (5m-3)/2$. From $e(F_{2m}) = 4m-3$, we have $e_1 - e_0 = m$. By lemma 2, $\overline{F_{2m}}$ is cordial.
Subcase2 m is even. Since $m \geq 2$, we have $m \leq (5m-4)/2 \leq 3m-2$. By lemma 6(2), there exists a labeling f of F_{2m} such that $v_0 = v_1$ and $e_1 = 5m/2 - 2$. From $e(F_{2m}) = 4m-3$, we have $e_1 - e_0 = m-1$. By lemma 2, $\overline{F_{2m}}$ is cordial.
(2)$n = 2m+1$
Subcase1 m is odd. Since $m+1 \leq (5m-1)/2 \leq 3m-1$. By lemma 6(1), there is a labeling f of F_{2m+1} such that $|v_0 - v_1| = 1$ and $e_1 = (5m-1)/2$. From $e(F_{2m+1}) = 4m-1$, we have $e_1 - e_0 = m$. By lemma 2, $\overline{F_{2m+1}}$ is cordial.
Subcase2 m is even. Since $m+1 \leq 5m/2 \leq 3m-1$. By lemma6(1), there is a labeling f of F_{2m+1} such that $|v_0 - v_1| = 1$ and $e_1 = 5m/2$. From $e_1 + e_0 = 4m-1$, we have $e_1 - e_0 = m+1$. By lemma 2, $\overline{F_{2m+1}}$ is cordial.

6 The cordiality of $\overline{W_{n+1}}$

Let W_{n+1} be a wheel of order $n+1$. It is the join of a vertex w and a cycle C_n. Denote the join by $W_{n+1} = W + C_n$, w is the center of wheel W_{n+1}.
Lemma 7 Suppose $n \geq 2$, then
(1) For each $k \in \{n+2, n+3, \cdots, 3n-1, 3n\}$, there is a labeling of W_{2n+1} such that $|v_0 - v_1| = 1$ and $e_1 = k$.
(2)For each $k \in \{n+2, n+4, \cdots, 3n-2\}$, there is a labeling of W_{2n} such that $v_0 = v_1$ and $e_1 = k$.

Proof (1)Let $W_{2n+1} = W + C_{2n}$. If $k \in \{n+2, n+4, \cdots, 3n-2, 3n\}$, then $k - n \in \{2, 4, \cdots, 2n\}$. By lemma 5(2), there is a labeling of C_{2n} such that $v_0(C_{2n}) = v_1(C_{2n})$ and $e_1(C_{2n}) = k-n$. Put $f(w) = 0$, we have $v_0(W_{2n+1}) = v_0(C_{2n})+1 = v_1(C_{2n})+1 = v_1(W_{2n+1})+1$ and $e_1(W_{2n+1}) = e_1(C_{2n}) + n = k - n + n = k$. If $k \in \{n+3, n+5, \cdots, 3n-1\}$, then $k-1 \in \{n+2, n+4, \cdots, 3n-2\}$. From above statement, there is a labeling of W_{2n+1} with $f(w) = 0$, $v_0(C_{2n}) = v_1(C_{2n})$ and $e_1(W_{2n+1}) = k-1 \leq 3n-2$, then $e_1(C_{2n}) \leq 2n-2$, $e_0(C_{2n}) \geq 2$. By the lemma 1 of [5], we see that, there is an edge $x_1x_2 \in E(C_{2n})$ such that $f(x_1) = f(x_2) = 0$. Obviously, we can assume that x_2 has another neighbor $y \in V(C_{2n})$ with $f(y) = 1$. By changing the label of x_2 from 0 to 1, then we obtain a labeling with $v_1(W_{2n+1}) = v_0(W_{2n+1}) + 1$ and $e_1(W_{2n+1}) = k - 1 + 1 = k$.
(2)Let $W_{2n} = W + C_{2n-1}$. If $k \in \{n+2, n+4, \cdots, 3n-2\}$, then $k - n \in$

$\{2, 4, \cdots, 2n-2\}$. By lemma 5(3), there is a labeling f of C_{2n-1} such that $v_0(C_{2n-1}) = v_1(C_{2n-1}) - 1$ and $e_1(C_{2n-1}) = k - n$. Put $f(w) = 0$, it yields the desired labeling of W_{2n}.

Theorem 4 For $n \geq 4$, $\overline{W_n}$ is not cordial iff $n \equiv 0(mod 8)$.

Proof If $n = 8m$, then the degree of each vertex of $\overline{W_{8m}}$ is even. $e(\overline{W_{8m}}) = (8m-1)(8m-4)/2 \equiv 2(mod4)$. By the proposition 1 of [4], $\overline{W_{8m}}$ is not cordial.
If $n \neq 8m$. We distinguish the following.
(1)$n = 4m+1$. Since lemma 7(1), there is a labeling f of W_{4m+1} such that $|v_0 - v_1| = 1$ and $e_1 = 5m$. Note that $e(\overline{W_{4m+1}}) = 8m$. It implies that $e_1 - e_0 = 2m$. By lemma 2, we know that $\overline{W_{4m+1}}$ is cordial.
(2)$n = 4m+3$. Since lemma 7(1), there is a labeling of W_{4m+3} such that $|v_0 - v_1| = 1$ and $e_1 = 5m+2$. Note that $e(\overline{W_{4m+3}}) = 8m+4$. It implies that $e_1 - e_0 = 2m$. By lemma 2, we know that $\overline{W_{4m+3}}$ is cordial.
(3)$n = 8m+2$. Since lemma 7(2), we can see that there is a labeling of W_{8m+2} such that $v_0 = v_1$ and $e_1 = 10m+2$. Note that $e(\overline{W_{8m+2}}) = 16m+2$. It implies that $e_1 - e_0 = 4m+2$. By lemma 2, we see that $\overline{W_{8m+2}}$ is cordial.
(4)$n = 8m+4$. Since lemma 7(2), we can see that there is a labeling of W_{8m+4} such that $v_0 = v_1$ and $e_1 = 10m+4$. Note that $e(\overline{W_{8m+4}}) = 16m+6$. It implies that $e_1 - e_0 = 4m+2$. By lemma 2, we see that $\overline{W_{8m+4}}$ is cordial.
(5)$n = 8m+6$. Since lemma 7(2), we can see that there is a labeling of W_{8m+6} such that $v_0 = v_1$ and $e_1 = 10m+6$. Note that $e(\overline{W_{8m+6}}) = 16m+10$. It implies that $e_1 - e_0 = 4m+2$, $|e_1 - e_0 - v(W_{8m+6})/2| = 1$. Hence $\overline{W_{8m+6}}$ is cordial.

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