3-CONNECTED SIMPLE GRAPHS WITH THE SAME GENUS DISTRIBUTIONS

YUNSHENG ZHANG AND YICHAO CHEN

ABSTRACT. The paper construct infinite classes of non-isomorphic 3-connected simple graphs with the same total genus polynomial, using overlap matrix, symmetry and Gustin representation. This answers a problem (Problem 3 of Page 38) of L.A. McGeoch in his PHD thesis. The result is helpful for firms to make marketing decisions by calculating the graphs of user demand relationships of different complex ecosystems of platform products and comparing genus polynomials.

1. INTRODUCTION

1.1. Background. In graph theory, an isomorphism of graphs G and H is a bijection between the vertex sets of G and H

$$f : V(G) \longrightarrow V(H)$$

such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. If an isomorphism exists between two graphs, then the graphs are called isomorphic and we write $G \cong H$. Intuitively, two graphs are isomorphic if we can re-draw one of them so that it looks exactly like the other. Genus distributions problems have been attracted a lot of attention in the past quarter century, since the topic was inaugurated by Gross and Furst [18]. A natural problem is: whether two non-isomorphic graphs have the same genus distribution? The pioneer work in using genus distributions of graphs to test non-isomorphic graphs due to Gross and his coauthors, see [1, 2, 14] etc.

In [23], McGeoch found a way to construct many 2-edge connected multigraphs with the same genus distributions, and he also posed the following problem in his PHD thesis.

2000 *Mathematics Subject Classification.* Primary: 05C10;

Key words and phrases. graph isomorphism; graph embedding; genus distribution; overlap matrix.

The first author's work was supported by National Natural Science Foundation of China (No. 71072134, 71273085, 71373074) and Programs for New Century Excellent Talents in University (No. NCET-12-0162).

The work of the second author was partially supported by NNSFC under Grant No. 10901048.

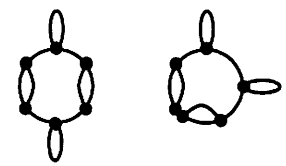

FIGURE 1. Two non-isomorphic necklaces with genus polynomial 64+512x

Problem 1.1. *Are there two non-isomorphic, simple, 3-edge-connected graphs with the same genus distribution?*

Gross, Klein and Rieper [17] constructed many non-isomorphic 2-edge connected pseudographs (see Figure 2) and 3-edge connected multigraphs with the same genus distributions. In the same paper, the authors claimed that Rieper has constructed many simple graphs with the same distributions by using method of Jackson [12]. However, he did not publish his work in the same paper. There are two reasons that we write the current paper. One is that Rieper did not publish his work and most published papers that using Jackson's method to calculate genus distributions are small **diameter graphs which must contain multiple edges or loops,** (In [10, 11], Jackson and Sloss called them **central enumerative problems**) see also [16], [20], [27], [28] etc. The other is the method used here is the overlap matrix, which was introduced by Mohar [24]. For other papers concerning graphs with the same genus distributions, see [22, 25] etc.

1.2. **Total genus polynomial.** It is assumed that the reader is somewhat familiar with the basics of topological graph theory, as found in Gross and Tucker [19]. All graphs considered in this paper are connected. A ***graph*** is often denoted by $G = (V, E)$, it is ***simple*** if it contains neither multiple edges nor self-loops. If a graph does not contain self-loops but contains multiple edges, we call it a ***multigraph***, otherwise if it contains self-loops, we call it a ***pseudograph.*** The graph with only one vertex and no edges is called the ***trivial*** graph. The ***vertex-connectivity*** $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G results in a disconnected or trivial graph. The ***edge-connectivity*** $\kappa_1(G)$ of G is the minimum number of edges whose removal from G results in a disconnected or trivial graph. A ***surface*** is a compact closed 2-dimensional manifold without boundary. In topology, surfaces are classified into the *orientable surfaces* S_g, with g handles ($g \geq 0$), and the *nonorientable surfaces*, N_k with k crosscaps ($k > 0$). A graph embedding into a surface means a *cellular embedding.* For any spanning tree of G, the number of co-tree edges is called the ***Betti number*** of G, and is denoted by $\beta(G)$.

A ***rotation at a vertex*** v of a graph G is a cyclic order of all edge-ends (or equivalently, half-edges) incident with v. A ***pure rotation system*** ρ

of a graph G is the collection of rotations at all vertices of G. An embedding of G into an oriented surface S induces a pure rotation system as follows: the rotation at v is the cyclic permutation corresponding to the order in which the edge-ends are traversed in an orientation-preserving tour around v. Conversely, by the *Heffter-Edmonds principle*, every rotation system induces a unique embedding (up to homeomorphism) of G into some orientable surface S. The bijection of this correspondence implies that the total number of orientable embeddings is

$$\prod_{v \in V(G)} (d_v - 1)!,$$

where d_v is the degree of vertex v.

A ***general rotation system*** is a pair (ρ, λ), where ρ is a pure rotation system and λ is a mapping $E(G) \to \{0, 1\}$. The edge e is said to be *twisted* (respectively, *untwisted*) if $\lambda(e) = 1$ (respectively, $\lambda(e) = 0$). It is well-known that every oriented embedding of a graph G can be described by a general rotation system (ρ, λ) with $\lambda(e) = 0$ for all $e \in E(G)$. By allowing λ to take non-zero values, we can describe the nonorientable embeddings of G. For any spaning tree T, a ***T-rotation system*** (ρ, λ) of G is a general rotation system (ρ, λ) such that $\lambda(e) = 0$, for all $e \in E(T)$.

By the ***total genus polynomial*** of G, we shall mean the polynomial

$$I_G(x, y) = \sum_{i=0}^{\infty} g_i x^i + \sum_{i=1}^{\infty} f_i y^i,$$

where g_i is the number of embeddings (up to equivalence) of G into the orientable surface O_i and f_i is the number of embeddings (up to equivalence) of G into the nonorientable surface N_i. We call the first (respectively, second) part of $I_G(x, y)$ the ***genus polynomial*** (respectively, ***crosscap number polynomial***) of G and denoted by $g_G(x) = \sum_{i=0}^{\infty} g_i x^i$ (respectively, $f_G(y) = \sum_{i=1}^{\infty} f_i y^i$). Clearly, $I_G(x, y) = g_G(x) + f_G(y)$.

1.3. **Overlap matrices.** Mohar [24] introduced an invariant that has subsequently been used numerous times in the calculation of distributions of graph embeddings, including non-orientable embeddings. The contributions include [3, 5, 7, 8] etc. We use Mohar's invariant here in our construction of graphs with the same genus distributions.

Let T be a spanning tree of a graph G and let (ρ, λ) be a T-rotation system. Let $e_1, e_2, \ldots, e_{\beta(G)}$ be the cotree edges of T, where $\beta(G)$ is the cycle rank of G. The ***overlap matrix*** of (ρ, λ) is the $\beta(G) \times \beta(G)$ matrix

$M = [m_{ij}]$ over $\mathbb{Z}_2$ such that

$$m_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } e_i \text{ is twisted;} \\ 1, & \text{if } i \neq j \text{ and the restriction of the underlying pure} \\ & \text{rotation system to the subgraph } T + e_i + e_j \text{ is nonplanar;} \\ 0, & \text{otherwise.} \end{cases}$$

When the restriction of the underlying pure rotation system to the subgraph $T + e_i + e_j$ is nonplanar, we say that edges e_i and e_j ***overlap***. The importance of overlap matrix is indicated by this theorem of Mohar [24]:

Theorem 1.2. *Let (ρ, λ) be a general rotation system for a graph, and let M be the overlap matrix. Then the rank of M equals twice the genus of the corresponding embedding surface, if that surface is orientable, and it equals the crosscap number otherwise. It is independent of the choice of a spanning tree.*

2. 3-EDGE-CONNECTED SIMPLE GRAPHS WITH THE SAME GENUS DISTRIBUTION

For drawing a planar representation of a rotation system on a cubic graph, we adopt the graphic convention introduced by Gustin [21], and used extensively by Ringel and Youngs (see [26]) in their solution to the Heawood map-coloring problem. There are two possible cyclic orderings of each trivalent vertex. Under this convention, we color a vertex *black*, if the rotation of the edge-ends incident on it is *clockwise*, and we color it *white* if the rotation is *counterclockwise*. We call any drawing of a graph that uses this convention to indicate a rotation system a ***Gustin representation*** of that rotation system.

In a Gustin nomogram, an edge is called ***matched*** if it has the same color at both endpoints; otherwise, it is called ***unmatched***. In Figure 2, we have indicated our choice of a spanning tree for graphs H_1, H_2 by thicker lines, so that the cotree edges are $e, a_1, a_2, \ldots, a_6$, and our partial choice of rotations at the vertices. It easy to see that they are non-isomorphic. We will show that H_1, H_2 have the same genus distributions.

In the Gustin nomogram of Figure 2, we observe the following properties:

Property 2.1. *Two cotree edges e and a_i overlap if and only if the edge d_i is unmatched, for $i = 1, 2, 3, 4$.*

Property 2.2. *Two cotree edges e and a_5 overlap if and only if the edge d_1 is unmatched.*

Property 2.3. *Two cotree edges e and a_6 overlap if and only if the edge d_4 is unmatched.*

Property 2.4. *The cotree edges a_1 and a_2 overlap if and only if the vertices v_2 and v_7 are colored differently.*

Property 2.5. *The cotree edges a_1 and a_5 overlap if and only if the edge a_5 is unmatched.*

Property 2.6. *The cotree edges a_2 and a_3 overlap if and only if the edge c is unmatched.*

Property 2.7. *The cotree edges a_2 and a_5 overlap if and only if the vertices v_2 and v_{12} are colored differently.*

Property 2.8. *The cotree edges a_3 and a_4 overlap if and only if the vertices v_4 and v_9 are colored differently.*

Property 2.9. *The cotree edges a_3 and a_6 overlap if and only if the vertices v_4 and v_{13} are colored differently.*

Property 2.10. *The cotree edges a_4 and a_6 overlap if and only if the edge a_6 is unmatched.*

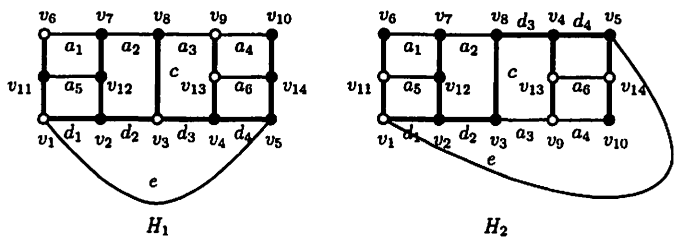

FIGURE 2. Two non-isomorphic graphs H_1 and H_2

Let $X = (x_e, x_1, x_2, \ldots, x_6)$, $Y = (y_1, y_2, \ldots, y_7)$, and $Z = (z_1, z_2, \ldots, z_4)$. It follows that the overlap matrix $M^{X,Y,Z}$ of H_1 and H_2, respectively, have the following form:

$$M^{X,Y,Z} = \begin{pmatrix} x_e & z_1 & z_2 & z_3 & z_4 & z_1 & z_4 \\ z_1 & x_1 & y_1 & & & y_4 & \\ z_2 & y_1 & x_2 & y_2 & & y_5 & \\ z_3 & & y_2 & x_3 & y_3 & & y_7 \\ z_4 & & & y_3 & x_4 & & y_6 \\ z_1 & y_4 & y_5 & & & x_5 & \\ z_4 & & & y_7 & y_6 & & x_6 \end{pmatrix}$$

Note that $x_e = 1$ if and only if the edge e is twisted.
Also note that

- $x_i = 1$ if and only if the edge a_i is twisted, for all $i = 1, 2, \ldots, 6$;

- $z_i = 1$ if and only if d_i is unmatched, for all $j = 1, 2, \ldots, 4$;

Moreover, we have $y_1 = 1$ if and only if the colors of vertices v_2 and v_7 are different, $y_2 = 1$ if and only if the edge c is unmatched, $y_3 = 1$ if and only if colors of vertices v_4 and v_9 are different, $y_4 = 1$ if and only if the edge a_5 is unmatched, $y_5 = 1$ if and only if the colors of vertices v_2 and v_{12} are different, $y_6 = 1$ if and only if the edge a_6 is unmatched, and we have $y_7 = 1$ if and only if the colors of vertices v_4 and v_{13} are different.

Property 2.11. *For each fixed matrix of the form $M^{X,Y,Z}$, there are exactly two different T-rotation systems of the graph H_i ($i = 1, 2$)for which $M^{X,Y,Z}$is the overlap matrix.*

Proof. Given a matrix $M^{X,Y,Z}$, the values of $x_e, x_1, x_2, \ldots, x_6; y_1, y_2, \ldots, y_7;$ $z_1, z_2, \ldots, z_4$ are determined. We know that there is one-to-one correspondence between the values of $x_e, x_1, x_2, \ldots, x_6$ and the cotree edges of the T-rotations system (ρ, λ) of H_i. Now we will show that every values of $y_1, y_2, \ldots, y_5; z_1, z_2, \ldots, z_4$ correspond two pure rotation system ρ.

Note that when the values of $x_e, x_1, x_2, \ldots, x_6; y_1, y_2, \ldots, y_7; z_1, z_2, \ldots, z_4$ are given, by Property 2.1-2.10, all the colors of vertices of $V(H_i) - \{v_6, v_{10}\}$ are determined by the coloring of v_1. Since there are two colorings of v_1 (black and white), this means the values of $y_1, y_2, \ldots, y_5; z_1, z_2, \ldots, z_4$ correspond two vertex coloring of H_i. That is, all the rotations at vertices of H_i are determined ($i = 1, 2$). □

Theorem 2.12. *The two graphs H_1 and H_2 have the same total genus polynomial.*

Proof. Note that the graphs H_1 and H_2 have the same overlap matrix $M^{X,Y,Z}$, by Property 2.11, the theorem follows. □

Actually, we have $I_{H_1}(x, y) = I_{H_2}(x, y) = 2 + 134x + 1592x^2 + 2368x^3 + 46y + 578y^2 + 6216y^3 + 33672y^4 + 117504y^5 + 209600y^6 + 152576y^7$.

Note that if we smooth all 2-degree vertices of H_1 and H_2, the resulted graphs are also simple and 3-edge connected, furthermore they have the same genus distributions. Thus McGeoch's open problem is answered. Since a 3-regular simple graph is 3-connected if only if it is 3-edge connected. This answers McGeoch's Question affirmatively for 3−connected simple graphs.

Remark 2.13. *Actually, we can add edges that are parallel to a_5 or a_6 of H_i ($i = 1, 2$), the resulted graphs also have the same total genus polynomial. See Figure 3.*

Remark 2.14. *We call the graphs H_1 and H_2* ***H type*** *graphs. Also we can add more "H" to graph H_i, the resulted graphs also have the same total genus polynomial. See Figure 4.*

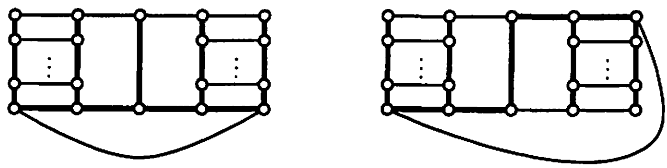

FIGURE 3

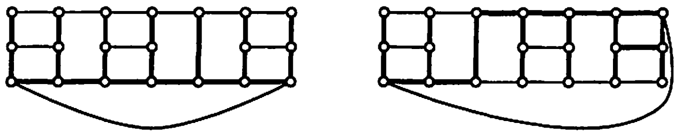

FIGURE 4. Two non-isomorphic graphs H type graphs

By the same consideration, we can also construct many other type of classes of graphs with same embedding distributions, for example ***X type*** et al.

Remark 2.15. *The anonymous referee tell us that Poshni's paper* [25] *produced a general method for constructing 2-connected graphs (simple or nonsimple) with the same genus distributions. Furthermore Poshni's theorem may construct 3-connected simple graphs with the same genus distributions, if Poshni's Theorems 3.2, 3.3, and 3.4 for edge-amalgamation is generalized to self-edge-amalgamation.*

3. K-EDGE-CONNECTED MULTIGRAPHS WITH THE SAME GENUS DISTRIBUTION

A *fan graph* $F_{(1,n)}$ is defined as the graph K_1+P_n, where K_1 is the empty graph on one vertex and P_n is the path graph on n vertices. A Fan-type graph $F_{t_1,t_2,\ldots,t_n}$ is defined as the graph K_1 connect t_j edges to the vertex v_j of P_n, $t_j \geq 1$, $j = 1, 2, \ldots, n$. A dipole graph D_n is a multigraph consisting of two vertices connected with n edges. Let the sequence $s_1, s_2, \ldots, s_n$ be the ***permutation*** of the set $\{t_1, t_2, \ldots, t_n\}$. In [6], proved that the genus polynomial of Fan-type graphs $F_{t_1,t_2,\ldots,t_k}$ equals to a constant times the genus polynomial of the diple graph D_n where $n = \sum_{i=1}^{k} t_i$. This means $F_{t_1,t_2,\ldots,t_n}$ and $F_{s_1,s_2,\ldots,s_n}$ have the same genus distribution. Also, let C_n be the cycle graph on n vertices. A Wheel-type graph $W_{t_1,t_2,\ldots,t_n}$ is defined as the graph K_1 connect t_j edges to the vertex v_j of C_n, $t_j \geq 1$, $j = 1, 2, \ldots, n$. Actually, $W_{t_1,t_2,\ldots,t_n}$ and $W_{s_1,s_2,\ldots,s_n}$ also have the same genus distributions.

For example

$$g_{W_{1,2,2,1}}(x) = g_{W_{1,2,1,2}}(x) = 8 + 1312x + 11192x^2 + 4768x^3;$$

$$g_{W_{2,2,2,1,1}}(x) = g_{W_{1,2,2,1,2}}(x) = 16 + 13680x + 550032x^2 + 2834480x^3 + 956352x^4;$$

$$g_{W_{1,3,3,1}}(x) = g_{W_{1,3,1,3}}(x) = 72 + 39024x + 1493352x^2 + 7553376x^3 + 2526336x^4.$$

4. Some discussions

It remains an interested problem to find other k- connected (k-edge-connected) simple graphs with the same genus distributions for $k \geq 4$. Another problem is to find which type of graphs are unique determined by their genus distributions. Actually in [9], the authors study the embedding distributions of cubic outerplanar graphs, they proved that two non-isomorphic cubic graphs have the same embedding distribution if their characteristic tree (or weak dual) are isomorphic, However it is still an open problem to prove that whether the embedding distribution of a cubic outer planar graph is unique determined by its characteristic tree.

The result in the article also indicates that the two non-isomorphic graphs can have the same genus polynomial. However, each graph has the unique genus polynomial and two different graphs have the same genus polynomial are "look alike". It would be an interesting problem to investigate this property. Firms are curious about the demand situations of analogous products existing in market when implementing the market entry strategies of new products. Many kinds of demand relationships exist among users, such as group-buying, direct network externalities, indirect network externalities, users are defined as nodes of a graph, and the demand relationships are edges, the complex graphs of user demand relationships would be calculated by mass data in markets. On account of the same genus polynomials between the graphs of analogous products existing in market and the new product, there are some similarities or universalities between the complex demand relationships of these products, which provides objective evidences for new product to make pricing or promotion decisions by imitating the successful marketing strategies of analogous products existing in market. A complex system of platform products is a coexistence and co-evolution product ecosystem which is comprised of so many components and parts, optional products, complementary products. All the products in a platform system belong to the same technological paradigm and are complementary in use, the interdependent relationships in technology and the complementary relationships in use can be translated into a complex graph by calculating mass data in patent databases and market demand data. Some similarities or universalities will be discovered when genus polynomials are identical between the graphs of different platform systems. Thus, a decision-making basis can be provided for new platform system to imitate the technology licensing strategies and product pricing tactics of platform systems existing in market, which is helpful for new platform system to offer some cooperative partners or users technical assistance or price subsidies, and offer others monopolistic prices.

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BUSINESS SCHOOL, HUNAN UNIVERSITY, 410082 CHANGSHA, CHINA
E-mail address: yunsheng01@yahoo.com.cn

COLLEGE OF MATHEMATICS AND ECONOMETRICS, HUNAN UNIVERSITY, 410082 CHANGSHA, CHINA
E-mail address: ycchen@hnu.edu.cn