On super connectedness and super restricted edge-connectedness of total graphs *

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Abstract A graph G is *super-connected*, *super-κ*, for short if every minimum vertex-cut isolates a vertex of G. Call G *super restricted edge-connected*, in short, *super-λ'*, if every minimum restricted edge-cut isolates an edge. We view the *total graph* $T(G)$ of G as the disjoint of G and the line graph $L(G)$, together with the lines of the subdivision graph $S(G)$; for each line $l = (u, v)$ in G there are two lines in $S(G)$, namely $(\hat{l}, u)$ and $(\hat{l}, v)$. In this paper, we prove that $T(G)$ is super-κ if G is a super-κ graph with $\delta(G) \geq 4$. We also show that $T(G)$ is super-λ' if G is a k-regular graph with $\kappa(G) \geq 3$. Furthermore, we give examples which illustrate that the results are best possible.
Keywords: Super connected; Super edge-connected; Total graph

1 Introduction

A network can be conveniently modeled as a graph $G = (V, E)$, with vertices representing nodes and edges representing links. A classic measure of network reliability is the connectivity $\kappa(G)$ and the edge-connectivity $\lambda(G)$. In general, the larger $\kappa(G)$ (or $\lambda(G)$) is, the more reliable the network is. For $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G, a graph G with $\kappa(G) = \delta(G)$ ($\lambda(G) = \delta(G)$) is naturally said to be *maximally connected* (*maximally edge-connected*), or *κ-optimal* (*λ-optimal*) for simplicity. As more refined indices of reliability than maximal connectivity and maximal edge-connectivity, super connectivity and super

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edge-connectivity were proposed in [1, 3]. A graph G is *super-connected*, *super-κ*, for short (resp. *super edge-connected*, *super-λ*, for short) if every minimum vertex-cut (resp. edge-cut) isolates a vertex of G.

For further study, Esfahanian and Hakimi introduce the concept of restricted edge-connectivity [8]. The concept of restricted edge-connectivity is one kind of conditional edge-connectivity proposed by Harary in [9], and has been successfully applied in the further study of tolerance and reliability of networks, see [7, 12, 13]. Let F be a set of edges in G. Call F a *restricted edge-cut* if $G - F$ is disconnected and contains no isolated vertices. The minimum cardinality over all restricted edge-cuts is called *restricted edge-connectivity* of G, and denoted by $\lambda'(G)$. It was shown by Wang and Li that the larger $\lambda'(G)$ is, the more reliable the network is [13]. In [8], the authors proved that if a connected graph G of order $n \geq 4$ is not a star $K_{1,n-1}$, then $\lambda'(G)$ is well-defined and $\lambda(G) \leq \lambda'(G) \leq \xi(G)$, where $\xi(G) = min\{d_G(u) + d_G(v) - 2 : uv \in E(G)\}$ is the minimum edge degree of G. A graph G with $\lambda'(G) = \xi(G)$ is called a *λ'-optimal graph*. Call G *super restricted edge-connected*, in short, *super-λ'*, if every minimum restricted edge-cut isolates an edge, that is, every minimum restricted edge-cut is a set of edges adjacent to a certain edge with minimum edge degree in G. By the definitions, a super-λ' graph must be a λ'-optimal graph. However, the converse is not true since there are many λ'-optimal graphs not to be super-λ'. For example, C_n ($n \geq 6$), the cycle of length n is a trivial counterexample.

It should be pointed out that if $\delta(G) \geq 3$, then a λ'-optimal graph must be super-λ. In fact, a graph G is super-λ if and only if $\lambda(G) < \lambda'(G)$ [12]. Thus, the concepts of λ-optimal graph, super-λ graph, λ'-optimal graph and super-λ' graph describe reliable interconnection structure for graphs at different levels.

The *line graph* $L(G)$ of G is that graph whose point set can be put in one-to-one correspondence with the line set of G, such that two points of $L(G)$ are adjacent if and only if the corresponding lines of G are adjacent. We view the *total graph* $T(G)$ of G as the disjoint of G and $L(G)$, together with the lines of the subdivision graph $S(G)$; for each line $l = (u, v)$ in G there are two lines in $S(G)$, namely $(\hat{l}, u)$ and $(\hat{l}, v)$. For convenience, the points of $T(G)$ belonging to $L(G)$ will be called *linear points* of $T(G)$. We simplify notation by setting $\kappa = \kappa(G)$, $\kappa_L = \kappa(L(G))$, $\kappa_T = \kappa(T(G))$, and similarly for λ and δ. Except where noted we follow the definitions and notations of Bondy [4].

In 1969, Chartrand and Stewart [5] proved that (i) $\kappa(L(G)) \geq \lambda(G)$, if $\lambda(G) \geq 2$; (ii) $\lambda(L(G)) \geq 2\lambda(G) - 2$. Furthermore, Hellwig et. [10] showed that

Theorem 1.1. *Let G be a graph with $|V(G)| \geq 4$ and G is not a star. Then $\kappa(L(G)) = \lambda'(G)$.*

By Theorem 1.1, we easily have

Corollary 1.2. *Let G be a graph with $|V(G)| \geq 4$ and G is not a star. If G is a super-λ graph, then $\kappa(L(G)) = \lambda'(G) > \lambda(G)$.*

In [2], the authors showed the following two theorems.

Theorem 1.3. *If $\lambda(G) \geq 2$, then $T(G)$ is maximally edge-connected.*

Theorem 1.4. $\kappa(G) + \lambda(G) \leq \kappa_T(G) \leq 2\lambda(G)$.

By Theorem 1.4, we obtain the following two corollaries.

Corollary 1.5. *If $\kappa(G) = \lambda(G)$, then $\kappa_T(G) = 2\lambda(G)$.*

Corollary 1.6. *If G is maximally connected, then $T(G)$ is maximally connected.*

Chen and Meng [6] proved that for all but a few exceptions, the total graph $T(G)$ of G is super-λ.

Theorem 1.7. *For a given connected graph G, $T(G)$ is super-λ if and only if one of the following two conditions applies:*
(i) If $\lambda(G) \geq 2$ and G has a cut vertex u with $d_G(u) = \delta(G)$, then there are at least three edges between u and any component of $G \setminus u$.
(ii) If $\lambda(G) = 1$ and $e = uv$ is a bridge, then $min\{d_G(u), d_G(v)\} \geq 2\delta(G)$.

In this paper, we prove that $T(G)$ is super-κ if G is a super-κ graph with $\delta(G) \geq 4$ in section 2. We also show that $T(G)$ is super-λ' if G is a k-regular graph with $\kappa(G) \geq 3$ in section 3. Furthermore, we give examples which illustrate that the results are best possible.

2 Super connected total graphs

In [14], Xu et. proved the following theorem.

Theorem 2.1. *Let G be a super-κ graph with $\delta(G) \geq 4$. Then G is super-λ.*

Now, we prove the following result.

Theorem 2.2. *Let G be a super-κ graph with $\delta(G) \geq 4$. Then $T(G)$ is super-κ, and thus $T(G)$ is super-λ.*

Proof. Suppose to the contrary that $T(G)$ is not super-κ. Then there exists a minimum vertex-cut S of $T(G)$ such that $|S| \leq \delta(T(G)) = 2\delta(G)$ and every component of $T(G) - S$ has at least two vertices. Let $X_1, X_2, \cdots, X_t$ (t$\geq$ 2) be the components of $T(G) - S$. Then we have $|V(X_i)| \geq 2$ for $i = 1, 2, \cdots, t$. We consider three cases.

Case 1. There is one component X_i $(1 \leq i \leq t)$ such that $V(X_i) \subseteq V(G)$.

If $|V(X_i)| = 2$, then $|S| \geq 2\delta - 1 + min\{\kappa(G), n-2\} > 2\delta = \delta(T(G))$, a contradiction. If $|V(X_i)| \geq 3$, then $|S| \geq 3\delta - 3 + min\{\kappa(G), n - |V(X_i)|\} > 2\delta = \delta(T(G))$, also a contradiction.

Case 2. There is one component X_i $(1 \leq i \leq t)$ such that $V(X_i) \subseteq E(G)$.

By Case 1, we can assume that $V(X_j) \cap E(G) \neq \emptyset$ for $j = 1, 2, \cdots, t$. Let $Y = V(G[X_i])$. Then $|Y| \geq 3$ by $|V(X_i)| \geq 2$. Since G is super-κ and $\delta(G) \geq 4$, we have that G is super-λ by Theorem 2.1, thus $\kappa(L(G)) > \lambda(G)$ by Corollary 1.2. If $|Y| \geq \delta(G)$, then $|S| \geq |Y| + \kappa(L(G)) \geq |Y| + \delta(G) + 1 > 2\delta(G)$, a contradiction. Thus, we assume that $3 \leq |Y| \leq \delta(G) - 1$. Then $|S| \geq |Y|\delta(G) - |Y|(|Y|-1) + |Y| = |Y|(\delta(G) + 2 - |Y|) \geq 3(\delta(G) - 1) > 2\delta(G)$ by $\delta(G) \geq 4$, also a contradiction.

Case 3. $V(X_i) \cap E(G) \neq \emptyset$ and $V(X_i) \cap V(G) \neq \emptyset$ for $i = 1, 2, \cdots, t$.

Since $S \cap V(G)$ is a vertex cut of G and $S \cap E(G)$ is a vertex cut of $L(G)$, we obtain that $|S| \geq \kappa(G) + \kappa(L(G)) \geq \delta + (\delta + 1) > 2\delta$ (The inequality $\kappa(L(G)) \geq \lambda(G) + 1 = \delta(G) + 1$ is obtained by the proof of Case 2), contradicting to $|S| \leq 2\delta$. □

We present a class of graphs, which show that Theorem 2.2 is best possible, in the sense that $\kappa(G) = \delta(G)$ does not guarantee that $T(G)$ is super-κ.

Example 2.3. *Let n and δ be arbitrary integers with $n \geq 2\delta \geq 8$. Furthermore, let $H_1 \cong K_\delta$ with vertex set $V(H_1) = \{x_1, x_2, \cdots, x_\delta\}$ and let $H_2 \cong K_{n-\delta}$ with vertex set $V(H_2) = \{y_1, y_2, \cdots, y_{n-\delta}\}$. We define the graph G as the union of H_1 and H_2 together with the δ edges $x_1y_1, x_2y_2, \cdots, x_\delta y_\delta$. Then $n(G) = n, \delta(G) = \delta$, and $\kappa(G) = \delta(G)$. But we have that $T(G)$ is not super-κ, since $V(H_1) \cup \{x_1y_1, x_2y_2, \cdots, x_\delta y_\delta\}$ is a vertex-cut of $T(G)$ with cardinality $\delta(T(G))$.*

The following example shows that the condition $\delta(G) \geq 4$ is necessary in Theorem 2.2.

Example 2.4. *The graphs defined in Fig.2.1 are super-κ, and $\delta(G_i) = i$ for $i = 1, 2, 3$. But we verify that $T(G_i)$ is not super-κ for $i = 1, 2, 3$, since $\{v_{r+1}\} \cup \{v_{r+1}v_{r+2}\}$ is a vertex-cut of $T(G_1)$ with cardinality 2, $\{v_3, v_4\} \cup$*

$\{v_1v_3, v_2v_4\}$ *is a vertex-cut of* $T(G_2)$ *with cardinality 4, and* $\{v_4, v_5, v_6\} \cup \{v_1v_4, v_2v_5, v_3v_6\}$ *is a vertex-cut of* $T(G_3)$ *with cardinality 6.*

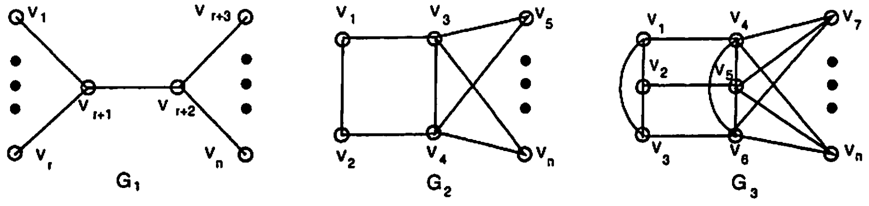

Fig.2.1.

3 Super restricted-edge-connected total graph-s

For a vertex $x \in V(G)$ and two vertex sets $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$, let $E_G(x) = \{e \in E(G) : e \text{ is incident with } x\}$, $[X, Y] = \{e = xy \in E(G) : x \in X, y \in Y\}$ and $E_G(X) = [X, \overline{X}]$.

In the proof of Theorem 3.2, we will use the following theorem, which was given by König [11] in 1931.

Theorem 3.1. *If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.*

Now, we are ready to prove the following main result.

Theorem 3.2. *Let G be a k-regular graph with* $\kappa(G) \geq 3$. *Then* $T(G)$ *is super-*λ'.

Proof. Suppose to the contrary that $T(G)$ is not super-λ'. Then there exists a minimum restricted edge-cut F such that $|F| \leq \xi(T(G)) = 4k - 2$ and each component of the two components of $T(G) - F$ has at least three vertices. Let X and Y be the two vertex sets of the two components of $T(G) - F$. We consider three cases.

Case 1. One of X and Y belongs to $V(G)$.

Assume, without loss of generality, that $X \subseteq V(G)$. If $|X| = 3$, then $|F| \geq 3k + (3k - 6) = 6(k - 1) > 4k - 2$, a contradiction. If $|X| \geq 4$, then $|F| \geq 4k > 4k - 2$, also a contradiction.

Case 2. One of X and Y belongs to $E(G)$.

Assume, without loss of generality, that $X \subseteq E(G)$. Let $Z = V(G[X])$. Since $|X| \geq 3$, we have $|Z| \geq 3$ and equality holds if and only if $G[X] \cong K_3$.

If $|Z| \geq 4$, then there are at least four vertices $u_1, u_2, u_3, u_4 \in V(G) \subseteq Y$, such that $E_G(u_i) \cap X \neq \emptyset$ for i=1,2,3,4. Let $|E_G(u_i) \cap X| = n_i \geq 1$ and $|E_G(u_i) \cap Y| = d_G(u_i) - n_i$. For any $e_i \in E_G(u_i) \cap X$ and $e_j \in E_G(u_i) \cap Y$, e_i and e_j are adjacent in $T(G)$, they contribute one edge to F. Thus,

$$|F| \geq \sum_{i=1}^{4}[n_i + n_i(d_G(u_i) - n_i)] = \sum_{i=1}^{4} n_i(k+1-n_i) \geq \sum_{i=1}^{4} k > 4k-2,$$

a contradiction.

If $|Z| = 3$, then $G[X] \cong K_3$, and let $Z = \{v_1, v_2, v_3\} \subseteq V(G) \subseteq Y$. It is easy to see that $|E_G(v_i) \cap X| = n_i = 2$ and $|E_G(v_i) \cap Y| = k - n_i = k-2$. Thus $|F| \geq \sum_{i=1}^{3} n_i(k - n_i + 1) = 6(k-1) > 4k-2$, also a contradiction.

Case 3. $X \cap V(G) \neq \emptyset$, $X \cap E(G) \neq \emptyset$, $Y \cap V(G) \neq \emptyset$ and $Y \cap E(G) \neq \emptyset$.

Let $V_1 = V(G) \cap X$, $V_2 = V(G) \cap Y$, $E_1 = E(G) \cap X$ and $E_2 = E(G) \cap Y$.

Subcase 3.1. $|V_1| = 1$ or $|V_2| = 1$.

Assume, without loss of generality, that $|V_1| = 1$. Let $V_1 = \{u\}$, $N(u) = \{u_1, u_2, \cdots,$
$u_k\}$, $e_i = uu_i$ $(i = 1, 2, \cdots, k)$ and $t = |\{e_1, e_2, \cdots, e_k\} \cap E_1|$. Then $|E_G(u) \cap E_2| = n_u = k - t$ and $|E_G(u) \cap E_1| = k - n_u = t$.

If $t = 1$, assume that $\{e_1, e_2, \cdots, e_k\} \cap E_1 = \{e_1\}$. Since $|X| \geq 3$ and $T(G)[X]$ is connected, there exists a vertex $u_{k+1} \in N(u_1) \setminus \{u\}$ such that $e_{k+1} = u_1u_{k+1} \in E_1$. Let $|E_G(u_1) \cap E_1| = n_1 \geq 1$, $|E_G(u_1) \cap E_2| = k - n_1$, $|E_G(u_{k+1}) \cap E_1| = n_{k+1} \geq 1$ and $|E_G(u_{k+1}) \cap E_2| = k - n_{k+1}$. Then

$$|F| \geq n_1 + n_1(k-n_1) + n_{k+1} + n_{k+1}(k-n_{k+1}) + n_u + n_u(k-n_u) + |N_G(u)|$$

$$\geq k + k + 2(k-1) + k = 5k-2 > 4k-2,$$

a contradiction.

If $2 \leq t \leq k-1$, assume, without loss of generality, that $\{e_1, e_2, \cdots, e_k\} \cap E_1 = \{e_1, e_2, \cdots, e_t\}$. Let $|E_G(u_i) \cap E_1| = n_i \geq 1$ and $|E_G(u_i) \cap E_2| = k - n_i$ for $i = 1, 2, \cdots, t$. Then

$$|F| \geq \sum_{i=1}^{t}[n_i + n_i(k-n_i)] + n_u + n_u(k-n_u) + |N_G(u)| \geq tk + k + k > 4k-2,$$

a contradiction.

If $t = k$, then $E_G(u_i) \cap E_1 \neq \emptyset$ for $i = 1, 2, \cdots, k$. Let $|E_G(u_i) \cap E_1| = n_i \geq 1$ and $|E_G(u_i) \cap E_2| = k - n_i$ for $i = 1, 2, \cdots, k$. Then

$$|F| \geq \sum_{i=1}^{k} [n_i + n_i(k - n_i)] + |N_G(u)| \geq k \times k + k > 4k - 2,$$

also a contradiction.

Subcase 3.2. $|V_1| = 2$ and $|V_2| \geq 2$, or $|V_1| \geq 2$ and $|V_2| = 2$.

Suppose that $|V_1| = 2$ and $|V_2| \geq 2$. Denote $V_1 = \{u, v\}$.

Subcase 3.2.1. $uv \notin E(G)$.

Let $N(u) = \{u_1, u_2, \cdots, u_k\}$ and $e_i = uu_i$ for $i = 1, 2, \cdots, k$. Since $|X| \geq 3$ and $T(G)[X]$ is connected, assume, without loss of generality, that $e_1 \in E_1$.

If e_2 is also in E_1, then $E_G(u_i) \cap E_1 \neq \emptyset$ for $i = 1, 2$, and let $|E_G(u_i) \cap E_1| = n_i \geq 1$ and $|E_G(u_i) \cap E_2| = k - n_i$ for $i = 1, 2$. Thus $|F| \geq \sum_{i=1}^{2} [n_i + n_i(k - n_i)] + |E_G(\{u, v\})| \geq 2k + 2k > 4k - 2$, a contradiction.

If e_2 is not in E_1, then $E_G(u) \cap E_2 \neq \emptyset$, and let $|E_G(u) \cap E_2| = n_u \geq 1$ and $|E_G(u) \cap E_1| = k - n_u$. Since $E_G(u_1) \cap E_1 \neq \emptyset$, let $|E_G(u_1) \cap E_1| = n_1 \geq 1$ and $|E_G(u_1) \cap E_2| = k - n_1$. Thus $|F| \geq n_u + n_u(k - n_u) + n_1 + n_1(k - n_1) + |E_G(\{u, v\})| \geq 2k + 2k > 4k - 2$, also a contradiction.

Subcase 3.2.2. $uv \in E(G)$.

Let $N(u) = \{v, u_1, \cdots, u_{k-1}\}$ and $N(v) = \{u, v_1, \cdots, v_{k-1}\}$. If $e = uv \in E_2$, assume, without loss of generality, that $e_1 = uu_1 \in E_1$ by $|X| \geq 3$ and $T(G)[X]$ is connected. Let $|E_G(u) \cap E_2| = n_u \geq 1$, $|E_G(v) \cap E_2| = n_v \geq 1$ and $|E_G(u_1) \cap E_1| = n_1 \geq 1$. Then $|F| \geq n_u + n_u(k - n_u) + n_v + n_v(k - n_u) + n_1 + n_1(k - n_1) + |E_G(\{u, v\})| \geq 3k + 2k - 2 > 4k - 2$, a contradiction. Therefore, we assume that $e = uv \in E_1$ in the following.

If $|X| = 3$, then $|E_G(u) \cap E_2| = n_u = k-1$ and $|E_G(v) \cap E_2| = n_v = k-1$. Thus $|F| \geq n_u + n_u(k - n_u) + n_v + n_v(k - n_v) + |E_G(\{u, v\})| = 2(k - 1) + 2(k - 1) + 2(k - 1) > 4k - 2$, also a contradiction.

If $|X| \geq 4$, then we can assume, without loss of generality, that $e_1 = uu_1 \in E_1$ by $T(G)[X]$ is connected. Let $|E_G(u_1) \cap E_1| = n_1 \geq 1$ and $|E_G(u_1) \cap E_2| = k - n_1$.

We claim that $E_G(v) \cap E_2 = \emptyset$. Suppose that $E_G(v) \cap E_2 \neq \emptyset$, let $|E_G(v) \cap E_2| = n_v \geq 1$ and $|E_G(v) \cap E_1| = k - n_v$. If $E_G(u) \cap E_2 \neq \emptyset$, let $|E_G(u) \cap E_2| = n_u \geq 1$ and $|E_G(u) \cap E_1| = k - n_u$. Thus $|F| \geq n_u + n_u(k - n_u) + n_v + n_v(k - n_v) + n_1 + n_1(k - n_1) + |E_G(\{u, v\})| \geq 3k + 2(k - 1) > 4k - 2$, a contradiction. If $E_G(u)\cap E_2=\emptyset$, then $e_2=uu_2\in E_1$ and $E_G(u_2)\cap E_1\neq\emptyset$, let $|E_G(u_2)\cap E_1|=n_2\geq 1$ and $|E_G(u_2)\cap E_2|=k-n_2$. Thus $|F|\geq n_v+n_v(k-n_v)+n_1+n_1(k-n_1)+n_2+n_2(k-n_2)+|E_G(\{u,v\})|\geq 3k+2(k-1)>4k-2$, also a contradiction. Thus, we assume that $E_G(v)\cap E_2=\emptyset$.

If $N(v)\setminus\{u,u_1,u_2\}\neq\emptyset$, let $v_1\in\{u,u_1,u_2\}$. Thus $E_G(v_1)\cap E_1\neq\emptyset$, and let $|E_G(v_1)\cap E_1|=n_{v_1}\geq 1$ and $|E_G(v_1)\cap E_2|=k-n_{v_1}$. If $E_G(u)\cap E_2\neq\emptyset$, let $|E_G(u)\cap E_2|=n_u\geq 1$ and $|E_G(u)\cap E_1|=k-n_u$. Thus $|F|\geq n_u+n_u(k-n_u)+n_{v_1}+n_{v_1}(k-n_{v_1})+n_1+n_1(k-n_1)+|E_G(\{u,v\})|\geq 3k+2(k-1)>4k-2$, a contradiction. If $E_G(u)\cap E_2=\emptyset$, then $e_2=uu_2\in E_1$ and $E_G(u_2)\cap E_1\neq\emptyset$, let $|E_G(u_2)\cap E_1|=n_2\geq 1$ and $|E_G(u_2)\cap E_2|=k-n_2$. Thus $|F|\geq n_{v_1}+n_{v_1}(k-n_{v_1})+n_1+n_1(k-n_1)+n_2+n_2(k-n_2)+|E_G(\{u,v\})|\geq 3k+2(k-1)>4k-2$, also a contradiction. Thus, we assume that $N(v)=\{u,u_1,u_2\}$. Since $\kappa(G)\geq 3$, we obtain that $G\cong K_4$. It is easy to verify that $|F|>10$, a contradiction.

Subcase 3.3. $|V_1|\geq 3$ and $|V_2|\geq 3$.

We consider the graph $H=G[[V_1,V_2]_G]$, which is induced by the edge set $[V_1,V_2]_G$ in G. If H has a matching M which contains at least four edges, assume that $v_{11}v_{21},v_{12}v_{22},\cdots,$
$v_{1t}v_{2t}$ $(t\geq 4)$ are t edges in M. Then either $E_G(v_{1i})\cap E_2\neq\emptyset$, or $E_G(v_{2i})\cap E_1\neq\emptyset$ for $i=1,2,\cdots,t$. Assume, without loss of generality, that $E_G(v_{1i})\cap E_2\neq\emptyset$ for $i=1,\cdots,r$, and $E_G(v_{2j})\cap E_1\neq\emptyset$ for $j=r+1,\cdots,t$. Let $|E_G(v_{1i})\cap E_2|=n_{1i}\geq 1$ and $|E_G(v_{1i})\cap E_1|=k-n_{1i}$ for $i=1,\cdots,r$, $|E_G(v_{2j})\cap E_1|=n_{2j}\geq 1$ and $|E_G(v_{2j})\cap E_2|=k-n_{2j}$ for $j=r+1,\cdots,t$. Then $|F|\geq\sum_{i=1}^{r}[n_{1i}+n_{1i}(k-n_{1i})]+\sum_{j=r+1}^{t}[n_{2j}+n_{2j}(k-n_{2j})]\geq rk+(t-r)k>4k-2$, a contradiction. If the maximum size of matching $\alpha'(H)\leq 2$, then the minimum size of vertex cover $\beta(H)=\alpha'(H)\leq 2$ by Theorem 3.1. Since a vertex cover of H with cardinality at least two is a vertex-cut of G, thus $\kappa(G)\leq 2$, which contradicts to $\kappa(G)\geq 3$. Therefore, we assume that $\beta(H)=\alpha'(H)=3$ in the following.

Since G is k-regular, $|[V_1,V_2]_G|$ is an even number. Thus $|E(H)|$ is an even number and $|E(H)|\geq 4$. If the number of vertices in V_1 which are adjacent to E_2 in $T(G)$ and vertices in V_2 which are adjacent to E_1 in $T(G)$ is at least 4, then we have that $|F|\geq 4k>4k-2$ by a similar argument as above, a contradiction. Since the vertices in V_1 which are adjacent to E_2 in $T(G)$ and the vertices in V_2 which are adjacent to E_1 in $T(G)$ constitute a vertex cover of H, $\beta(H)=\alpha'(H)=3$ and $\kappa(G)\geq 3$, we can assume that there are exactly three vertices v_1,v_2,v_3 such that if $v_i\in V_1$, then $E_G(v_i)\cap E_2\neq\emptyset$, and if $v_j\in V_2$, then $E_G(v_j)\cap E_1\neq\emptyset$. It is easy to verify that $\{v_1,v_2,v_3\}$ is a minimum vertex cover of H. Assume, without loss of generality, that $v_1, v_2, v_3 \in V_1$, or $v_1, v_2 \in V_1$ and $v_3 \in V_2$.

Subcase 3.3.1. $v_1, v_2, v_3 \in V_1$.

Let $|E_G(v_i) \cap E_2| = n_i \geq 1$ and $|E_G(v_i) \cap E_1| = k - n_i$ for $i = 1, 2, 3$. Since $v_1, v_2, v_3 \in V_1$, we have that the linear point set $[\{v_1, v_2, v_3\}, V_2]_G$ of $T(G)$ is contained in E_2. If there exists a n_i such that $2 \leq n_i \leq k-1$, assume, without loss of generality that $2 \leq n_1 \leq k-1$, then $n_1 + n_1(k - n_1) \geq 2(k-1)$. Thus $|F| \geq \sum_{i=1}^{3}[n_i + n_i(k-n_i)] + |E(H)| \geq 2(k-1) + 2k + 4 > 4k-2$, a contradiction. Otherwise, there is a n_j such that $n_j = k$. Assume, without loss of generality that $n_2 = k$, then $|N(v_2) \cap V_2| \geq k-2$ since $N(v_2) \cap V_1 \subseteq \{v_1, v_3\}$. We also have $|N(v_1) \cap V_2| \geq 1$ and $|N(v_3) \cap V_2| \geq 1$. Therefore, $|F| \geq \sum_{i=1}^{3}[n_i + n_i(k - n_i)] + |E(H)| \geq 3k + k - 2 + 2 > 4k - 2$, also a contradiction.

Subcase 3.3.2. $v_1, v_2 \in V_1$ and $v_3 \in V_2$.

Let $|E_G(v_i) \cap E_2| = n_i \geq 1$ and $|E_G(v_i) \cap E_1| = k - n_i$ for $i = 1, 2$, $|E_G(v_3) \cap E_1| = n_3 \geq 1$ and $|E_G(v_3) \cap E_2| = k - n_3$. If there exists a n_i such that $2 \leq n_i \leq k-1$, then we can obtain a contradiction by a similar argument as the proof of the subcase 3.3.1. Otherwise, there exists a n_j such that $n_j = k$. Since $T(G)[Y]$ is connected and there exists exactly one vertex v_3 in V_2 such that $E_G(v_3) \cap E_1 \neq \emptyset$, we have that $n_3 \neq k$. Thus $n_1 = k$ or $n_2 = k$. Assume, without loss of generality that $n_1 = k$, then $|N(v_1) \cap V_2| \geq k-1$ since $N(v_1) \cap V_1 \subseteq \{v_2\}$. We also have $|N(v_2) \cap V_2| \geq 1$. Therefore, $|F| \geq \sum_{i=1}^{3}[n_i + n_i(k - n_i)] + |E(H)| \geq 3k + k - 1 + 1 > 4k - 2$, also a contradiction. □

If G is a connected 1-regular graph, then $G \cong K_2$ and $T(G) \cong K_3$. If G is a connected 2-regular graph, then $T(G)$ is not super-λ' since $E_{T(G)}(\{u, v, e\})$ is a restricted edge-cut with cardinality 6 for every edge $e = uv \in E(G)$. If G is a $k(\geq 3)$-regular with $\kappa(G) = 1$, then k is even. The following examples show that the condition $\kappa(G) \geq 3$ is necessary in the Theorem 3.2.

Example 3.3. *Let k be an even integer with $k = 2t \geq 4$. Furthermore, let $G_1, G_2, \cdots, G_t$ be t connected k-regular graphs, and $e_1 = u_1v_1 \in E(G_1), e_2 = u_2v_2 \in E(G_2), \cdots, e_t = u_tv_t \in E(G_t)$. We define the graph G as the union of $G_1 - e_1, G_2 - e_2, \cdots, G_t - e_t$ together with the k edges $wu_1, wv_1, wu_2, wv_2, \cdots, wu_t, wv_t$, where w is an other vertex not in $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_t)$. Then G is a k-regular graph with $\kappa(G) = 1$. But $T(G)$ is not super-λ' since $E_{T(G)}(V_1 \cup E(G_1) \setminus \{e_1\})$ is a restricted edge-cut with cardinality $2k + 2$.*

Example 3.4. *Let G_1 and G_2 be two $k(\geq 3)$-regular graphs with $\kappa(G_1) \geq 2$ and $\kappa(G_2) \geq 2$, $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$. We define the graph G as the union of $G_1 - e_1$ and $G_2 - e_2$ together with the two edges u_1u_2 and v_1v_2. Then G is a k-regular graph with $\kappa(G) = 2$. But $T(G)$ is not super-λ' since $E_{T(G)}(V(G_1) \cup E(G_1) \setminus \{e_1\})$ is a restricted edge-cut with cardinality $2k + 2$.*

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