The zero divisor graph of partially ordered set with respect to a semi-ideal

Hongxing Liu
School of Mathematical Sciences, Shandong Normal University,
250014, Jinan, P. R. China
E-mail: lhxshanda@163.com

Abstract: In this paper, we investigate the zero divisor graph $G_I(P)$ of a poset P with respect to a semi-ideal I. We show that the girth of $G_I(P)$ is 3, 4 or ∞. In addition, it is shown that the diameter of such a graph is either 1, 2 or 3. Moreover, we investigate the properties of a cut vertex in $G_I(P)$ and study the relation between semi-ideal I and the graph $G_I(P)$ (Theorem 3.9).
Key words: zero divisor graph; poset; semi-ideal; cut vertex; diameter; girth
Mathematics Subject Classifications (2010): 05C25; 06A06

1 Introduction

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. In this area, many works on associating a graph to an algebraic structure have been published. In 1988, Beck [5] introduced the concept of zero divisor graph to study the interplay between ring theory and graph theory. He defined $\Gamma_0(R)$ to be the graph as follows: The vertices of $\Gamma_0(R)$ are the elements of R and two distinct vertices x, y are adjacent if and only if $xy = 0$. Since then, the concept of the zero divisor graph is studied in many algebraic structures such as rings, semigroups, semirings (see [2], [3],[4], [5], [6], [7], [8], [9], [14], [13] etc.). In [10], Halaš and Jukl introduced the zero divisor graph of a poset. The study of the zero-divisor graphs of posets was then continued by many authors, see [1], [14] and [15]. Recently, Joshi [12] introduced the zero divisor graph of a poset with respect to an ideal. It is shown that the graph is connected with its diameter ≤ 3. Also, the complete bipartite zero divisor graphs are characterized in [12].

Motivated by the paper [12], we give the definition of the zero divisor graph with respect to a semi-ideal of a poset and study the properties of these graphs. The definition may be viewed as a generalization of the definition in [12]. Also, Theorem 2.4 and Theorem 3.3 in this paper generalize Theorem 2.4 in [12]. The paper is constructed as follows. We investigate the girth of $G_I(P)$ and characterize the bipartite graph of these graphs in Section 2. In Section 3, we study the diameter and a cut vertex of the graph $G_I(P)$ and prove that if I and J are two semi-ideals of a poset P, then $G_I(P) = G_J(P)$ and $p(I) = p(J)$ if and only if $I = J$.

Given a partially ordered set $(P, \leq)$ (a poset, in brief) and $X \subseteq P$, the set $L(X) = \{y \in P | y \leq x \text{ for all } x \in X\}$ is called the *lower cone* of X. Dually, the set $U(X) = \{y \in P | y \geq x \text{ for all } x \in X\}$ is called the *upper cone* of X.

We shall write $L(x_1, \cdots, x_n)$ or $U(x_1, \cdots, x_n)$ instead of $L(X)$ or $U(X)$ whenever $X = \{x_1, \cdots, x_n\}$.

Given a poset $(P, \leq)$ and a nonempty subset I of P, I is called a *semi-ideal* of P if

$$\text{for all } x, y \in P, x \in I, y \leq x \Longrightarrow y \in I.$$

A proper semi-ideal I of P is called *prime* if

$$\text{for all } x, y \in P, L(x, y) \subseteq I \Longrightarrow x \in I \text{ or } y \in I.$$

A proper semi-ideal I of P is called *n-prime*$(n \geq 2)$ if for all $x_1, x_2, \cdots, x_n \in P, L(x_1, x_2, \cdots, x_n) \subseteq I$ yields $x_i \in I$ for some $1 \leq i \leq n$.

A proper semi-ideal Q of a poset P is called a *prime semi-ideal belonging to a semi-ideal* I, if Q is a prime semi-ideal and $I \subseteq Q$.

A nonempty subset I of P is called an *ideal* of P if

$$\text{for all } x, y \in P \Longrightarrow LU(x, y) \subseteq I.$$

Obviously, every ideal is a semi-ideal.

All posets in the paper have the smallest element 0.

Now, we give the definition of zero divisor graph of a poset with respect to a semi-ideal, which generalize the definition in [12].

Definition 1.1[12] *Let I be a semi-ideal of a poset P. We define a graph $G_I(P)$ with vertices $V(G_I(P)) = \{x \in P \backslash I | L(x, y) \subseteq I$, for some $y \in P \backslash I\}$, where distinct vertices x and y are adjacent if and only if $L(x, y) \subseteq I$. We call $G_I(P)$ the zero divisor graph of P with respect to I.*

By Definition 1.1, $G_{\{0\}}(P)$ is the zero divisor graph of P defined in [1].

For a graph G, the *girth* of G is the length of a shortest cycle in G and is denoted by *girth*(G). If G has no cycles, we define *girth*(G) $= \infty$. A *bipartite graph* is one whose vertices can be partitioned into two disjoint subsets such that the two end vertices of each edge lie in distinct subsets.

A *complete bipartite graph* is a bipartite graph such that every vertex is adjacent to every vertex that is not in the same subset. Let $K_{m,n}$ be the complete bipartite graph with exactly two subsets of size m and n. The graph $K_{1,n}$ is usually called *star graph*. It is well known that a graph is bipartite if and only if it contains no cycle of odd length.

Throughout the paper, all graphs are simple graphs and not null graphs.

2 Basic properties of the graph $G_I(P)$

In this section, we get a result characterizing the girth of $G_I(P)$. Also, we study the bipartite graph of $G_I(P)$. Firstly, we give the following lemma which is useful to prove Theorem 2.4.

Lemma 2.1 *Let I be a semi-ideal of a poset P. If P_5 is a subgraph of $G_I(P)$, then it contains a cycle of length ≤ 4, i.e. girth$(G_I(P)) \leq 4$.*

Proof Denote by $a_1 - a_2 - a_3 - a_4 - a_5$ the path P_5 in $G_I(P)$.

Case I: If $a_1 - a_4$ is an edge in $G_I(P)$, then we have a cycle $a_1 - a_2 - a_3 - a_4 - a_1$, whose length is 4.

Case II: If there is no edge joining a_1 and a_4, then $L(a_1, a_4)$ is not contained in I. Hence, there exists $0 \neq x \in L(a_1, a_4)$ and $x \notin I$. We claim that $x \neq a_2, a_3, a_5$. If $x = a_2$, then we have $a_2 \in L(a_1, a_2)$. This is a contradiction to $L(a_1, a_2) \subseteq I$, since $a_1 - a_2$ is an edge in $G_I(P)$ and $a_2 \notin I$. Similarly, we can prove that $x \neq a_3, a_5$.

1) Suppose $x \neq a_1, a_4$. Since $L(x, a_2) \subseteq L(a_1, a_2) \subseteq I$ and $L(x, a_3) \subseteq L(a_4, a_3) \subseteq I$, we get that $x - a_2$ and $x - a_3$ are edges in $G_I(P)$. Thus, we obtain a cycle $a_2 - a_3 - x - a_2$, whose length is 3.

2) Suppose $x = a_1$. Since $L(a_1, a_3) \subseteq L(a_4, a_3) \subseteq I$, we get that $a_1 - a_3$ is an edge in $G_I(P)$. So, we obtain a cycle $a_1 - a_2 - a_3 - a_1$, whose length is 3.

3) Suppose $x = a_4$. The discussion is similar to part 2), and we can obtain a cycle $a_2 - a_3 - a_4 - a_2$, whose length is 3.

In any case, if there is a subgraph P_5 in $G_I(P)$, we can obtain a cycle of length ≤ 4. □

By Lemma 2.1, we can get the following.

Corollary 2.2 *The cycle on n $(n \geq 5)$ vertices cannot be realized as a zero divisor graph of a poset.*

The symbol $||$ used to denote non-comparability. Denote by $GA_I = \{a \in P \backslash I | \text{if } y < a, \text{then } y \in I\}$. If $a \in GA_I$, denote by $\overline{U}(a) = \{y \in P | y \geq a \text{ and } y || b \text{ for all } b \in GA_I \backslash \{a\}\}$.

Lemma 2.3 *Let P be a poset and I is a semi-ideal of P. We have the followings.*

1) if $|GA_I| \leq 1$, then $G_I(P)$ is the null graph;
2) if $|GA_I| = 2$, then $G_I(P)$ is a complete bipartite graph;
3) if $|GA_I| \geq 3$, then $G_I(P)$ contains a cycle.

Proof 1) is obviously.

2) Assume $|GA_I| = 2$ and $GA_I = \{a_1, a_2\}$. If $x \geq a_1$ and $x \geq a_2$, then x is not a vertex of $G_I(P)$. If $u, v \in \overline{U}(a_i)$, where $i = 1, 2$, then there is no edge between u and v. For all $x \in \overline{U}(a_1)$ and $y \in \overline{U}(a_2)$, by Definition 1.1, $x - y$ is an edge in $G_I(P)$. Therefore, $G_I(P)$ is a complete bipartite graph

3) If $|GA_I| \geq 3$ and $a_1, a_2, a_3 \in GA_I$, then $a_1 - a_2 - a_3 - a_1$ is a cycle in $G_I(P)$. This proves the result. □

Theorem 2.4 *Let P be a poset and I is a semi-ideal of P. We have the followings.*
1) If $G_I(P)$ contains a cycle, then girth$(G_I(P)) \leq 4$.
2) $G_I(P)$ contains no cycle, i.e. girth$(G_I(P)) = \infty$ if and only if $G_I(P)$ is a star graph.

Proof 1) If $G_I(P)$ contains a cycle of length ≤ 4, we have nothing to prove. If $G_I(P)$ contains a cycle of length ≥ 5, then it also contains P_5. By Lemma 2.1, we get that girth$(G_I(P)) \leq 4$.

2) $\Rightarrow$: By Lemma 2.3, we have that $G_I(P)$ is a complete bipartite graph. Suppose that $G_I(P)$ has two parts $\overline{U}(a_1)$ and $\overline{U}(a_2)$. Since $G_I(P)$ contains no cycle, then we have $|\overline{U}(a_1)| = 1$ or $|\overline{U}(a_2)| = 1$. Hence, $G_I(P)$ is a star graph.

$\Leftarrow$: Obviously. □

Theorem 2.5 *girth$(G_I(P)) = 4$ if and only if $G_I(P)$ is a complete bipartite graph but not a star graph.*

Proof $\Rightarrow$: If girth$(G_I(P)) = 4$, then $G_I(P)$ contains no triangle and $|GA_I| = 2$. Similarly to part 2) of the preceeding theorem, we get that $G_I(P)$ must be a complete bipartite graph. Since it also contains a cycle, it is not a star graph.

$\Leftarrow$: Obviously. □

Let $I = \{0\}$ and using Theorem 2.4 and 3.4, we can rediscover Theorem 4.2 of paper [1].

Theorem 2.6 *[1] Let P be a poset. We have the following statements.*
(1) girth$(G(P)) = 3, 4,$ or ∞;
(2) girth$(G(P)) = 3 \Leftrightarrow G(P)$ has a cycle of odd length;
(3) girth$(G(P)) = 4 \Leftrightarrow G(P)$ is a bipartite but not a star graph;
(4) girth$(G(P)) = \infty \Leftrightarrow G(P)$ is a star graph.

Now, we give the following theorem characterizing the bipartite graph.

Theorem 2.7 *Let I be a semi-ideal of a poset P. Then the graph $G_I(P)$ is bipartite if and only if it contains no triangles.*

Proof $\Rightarrow$: Since any bipartite graph contains no cycle of odd length, we can obviously get the desired result.

$\Leftarrow$: Suppose that $G_I(P)$ is not bipartite. Then $G_I(P)$ contains a cycle of odd length. Let $a_0 - a_1 - \cdots - a_{2n} - a_0$ be a cycle in $G_I(P)$, where $n \geq 1$.

1) If $n = 1$, then $G_I(P)$ contains a triangle, which is a contradiction.

2) If $n \geq 2$, we will prove that $G_I(P)$ also contains a triangle.

Case I: If there exists an edge joining a_i and a_j, which is not in the cycle. Without loss of generality, assume a_0 is adjacent to a_k, where $k \neq 1, 2n$. Then we can get two cycles $a_0 - a_1 - \cdots - a_k - a_0$ and $a_k - a_{k+1} - \cdots - a_{2n} - a_0 - a_k$. Obviously, one of the two cycles is of length $2n_1 + 1$, where $n_1 < n$.

Case II: If any edge joining a_i and a_j is in the cycle, then there is no edge joining a_0 and a_2 and so $L(a_0, a_2)$ is not contained in I. Hence, there exists an element $0 \neq x \in L(a_0, a_2)$ and $x \notin I$. Since $x \leq a_2$, we have $L(x, a_3) \subseteq L(a_2, a_3) \subseteq I$, i.e. there exists an edge joining x and a_3. Similarly, we can prove that $x - a_{2n}$ and $x - a_1$ are edges in $G_I(P)$. This conclude that $x \notin \{a_0, a_1, \cdots, a_{2n}\}$. So, we get a cycle $a_{2n} - x - a_3 - \cdots - a_{2n}$, whose length is $2(n-1)+1$.

By induction on n, we get that $G_I(P)$ contains a triangle, a contradiction. □

In the following, we shall characterize, when $G_I(P)$ is bipartite. The following proposition is useful to prove Theorem 2.9.

Proposition 2.8 *Let G be a complete r-partite graph, then G can be realized as a zero divisor graph of a poset.*

Proof Let $G = \bigcup_{i=1}^{r} A_i$ be a complete r-partite graph with parties A_i. Well order the elements in A_i, for all $1 \leq i \leq r$. For all $i \in I$ and $x \in A_i$, let $0 \leq x \leq 1$. If $x \in A_i, y \in A_j$, where $i \neq j$, then $x||y$. Let P be a poset consisting of the vertices in A_i with 0 and 1. Then G can be realized as the graph $G_{\{0\}}(P)$. □

Theorem 2.9 *1) If $G_I(P)$ is a bipartite graph, then $G_I(P)$ is a complete bipartite graph or a star graph.*
2) Complete bipartite graph and star graph can be realized as a zero divisor graph of a poset.

Proof 1) If $G_I(P)$ is a bipartite graph, then $G_I(P)$ contains no triangles by Theorem 2.7. Hence, the girth of $G_I(P)$ is 4 or ∞ and $G_I(P)$ must be a complete bipartite graph or a star graph by Theorem 2.6.

2) By Proposition 2.8, complete bipartite graph can be realized as a zero divisor graph of a poset. Suppose G is a star graph and a is the center. Let $P = V(G) \bigcup \{0, 1\}$. Well order the vertices in $V(G) \backslash \{a\}$. If $x \in V(G) \backslash \{a\}$, let $x||a$. For all $x \in V(G)$, let $1 \geq x \geq 0$. Then G can be realized as the graph $G_{\{0\}}(P)$. □

3 Diameter, cut vertex and semi-ideal

Let I be a semi-ideal of a poset P. It is easy to see that $(P\backslash I)\bigcup\{0\}$ is also a poset, denoted by P_I. We have $G_I(P) \cong G_{\{0\}}(P_I)$ ([12]).

Let I be a semi-ideal of a poset P. For $\emptyset \neq A \subseteq P$, the *relative annihilator* of I with respect to A is the set $(I : A) = \{p \in P | L(a,p) \subseteq I, \forall a \in A\}$. It is easy to see that $(I : A)$ is a semi-ideal. If $A = \{a\}$, we usually write $(I : a)$.

For $x \in P$, the annihilator of x, denoted by $\text{Ann}(x)$, is defined to be $\{y \in P | L(x,y) = \{0\}\}$.

Lemma 3.1 *If $J = (0 : x)$ is an annihilator semi-ideal in P_I, then $I \cup J = (I : x)$ is also an annihilator semi-ideal in P, i.e. the annihilator semi-ideals in P_I correspond to the relative annihilator semi-ideals of I in P.*

Proof We obviously have $I \cup J \subseteq (I : x)$. Let $y \in (I : x)$ in P and $y \notin I$. Since $xy \in I$, we get $xy = 0$ in P_I, i.e. $y \in (0 : x) = J$ in P_I. So, we obtain that $I \cup J = (I : x)$. □

Let $Z(P)$ be the set of zero-divisors of P. While, let $\text{Min}(P)$ be the set of minimal elements of P. We obviously have the following lemma.

Lemma 3.2 *Let I be a semi-ideal of a poset P. Then the following assertions hold:*
1) $a \in GA_I$ if and only if $a \in Min(P_I)$.
2) $a \in Z(P_I)\backslash\{0\}$ if and only if there exists $b \in P\backslash I$ such that $L(a,b) \subseteq I$.

Theorem 3.3 in [1] states that *for a poset P, the following assertions hold:*
(1) $\Gamma(P)$ is a connected graph and $diam(\Gamma(P)) \leq 3$;
(2) $diam(\Gamma(P)) = 1 \Leftrightarrow Min(P\backslash\{0\}) = Z(P)\backslash\{0\}$.
(3) $diam(\Gamma(P)) = 2 \Leftrightarrow Z(P)\backslash Min(P) \neq \emptyset$ and for all $x,y \in Z(P)\backslash Min(P)$ such that $L(x,y) \neq \{0\}$, we have $Ann(x) \cap Ann(y) \neq \{0\}$;
(4) $diam(\Gamma(P)) = 3 \Leftrightarrow Z(P)\backslash Min(P) \neq \emptyset$ and for some $x,y \in Z(P)\backslash Min(P)$ such that $L(x,y) \neq \{0\}$, we have $Ann(x) \cap Ann(y) = \{0\}$.

By Lemma 3.1, 3.2 and Theorem 3.3 in [1], we get the following theorem.

Theorem 3.3 *Let I be a semi-ideal of a poset P. Then the following assertions hold:*
(1) $G_I(P)$ is a connected graph with $diam(G_I(P)) \leq 3$.
(2) $diam(G_I(P)) = 1$ if and only if $V((G_I(P)) = GA_I$.
(3) $diam(G_I(P)) = 2$ if and only if $V((G_I(P))\backslash GA_I \neq \emptyset$ and $(I : x) \cap (I : y) \neq I$ for every $x, y \in V((G_I(P))\backslash GA_I$ such that $L(x,y)$ is not contained in I.
(4) $diam(G_I(P)) = 3$ if and only if $V((G_I(P))\backslash GA_I = \emptyset$ and $(I : x) \cap (I : y) = I$ for some $x, y \in V((G_I(P))\backslash GA_I$ such that $L(x,y)$ is not contained

in I.

A vertex a of a graph G is called a *cut vertex* if there exist vertices b and c distinct from a such that a is in every path from b to c.

Proposition 3.4 *Let I be a semi-ideal of a poset P. If a is a cut vertex in $G_I(P)$, then $I \cup \{a\}$ is also an semi-ideal, i.e. $a \in GA_I$.*

Proof For any $y < a$, we have to show that $y \in I$. On the contrary, suppose $y \notin I$. Since a is a cut vertex, there exist vertices b and c distinct from a such that a is in every path from b to c. By Theorem 3.3, the shortest path from b to c is of length 2 or 3.

I: Suppose the shortest path from b to c is of length 2 and $b - a - c$ is a path from b to c. As $y < a$, we have $L(b, y) \subseteq L(b, a) \subseteq I$ and $L(c, y) \subseteq L(c, a) \subseteq I$. Consequently, we have a path $b - y - c$, which is a contradiction.

II: Suppose the shortest path from b to c is of length 3. Without loss of generality, let $b - d - a - c$ is a path from b to c. Similar to part 1), we have a path $b - d - y - c$, a contradiction. □

By Proposition 3.4, we know that a cut vertex of $G_I(P)$ must be contained in GA_I. Denote by $\overline{U}(GA_I \backslash \{a\}) = \{y \in P | \text{for all } x \in GA_I \backslash \{a\},\ y \geq x \text{ and } y || a\}$. The following theorem gives a characterization of a cut vertex of $G_I(P)$.

Theorem 3.5 *Let I be a semi-ideal of a poset P. Then $a \in GA_I$ is a cut vertex in $G_I(P)$ if and only if $\overline{U}(GA_I \backslash \{a\}) \neq \emptyset$.*

Proof $\Rightarrow$: Without loss of generality, Suppose that $x - a - y$ is a path of shortest length from x to y. Then $x || a$ and $y || a$. If $\overline{U}(GA_I \backslash \{a\}) = \emptyset$, then there exist $b, c \in GA_I$ such that $x || b$ and $y || c$. If $b \neq c$, we obtain a path $x - b - c - y$. If $b = c$, we obtain a path $x - b - y$. In both cases, we get a contradiction.

$\Leftarrow$: If $x \in \overline{U}(GA_I \backslash \{a\})$, then a is the only vertex connected with x. Therefore, a is a cut vertex. □

Let x be a vertex of a graph G. Denote by $N(x)$ the set of vertices which are adjacent to x. The degree of x is equal to $|N(x)|$.

Theorem 3.6 *Let I be a semi-ideal of a poset P and J_k be the set of vertices in $G_I(P)$ whose degree are more than or equal to k together with 0. Then $\{J_k\}$ is a descending chain semi-ideals in $G_I(P)$.*

Proof Let $x \in J_k$ and $y \leq x$. Assume $a - x$ is an edge in $G_I(P)$. Since $L(a, y) \subseteq L(a, x) \subseteq I$, we obtain that $a - y$ is also an edge in $G_I(P)$. Consequently, we get $N(x) \subseteq N(y)$. Therefore, the degree of y is greater than or equal to k and so $y \in J_k$. This conclude that J_k is an semi-ideal of P and we easily get the result. □

Let I be a proper semi-ideal of P. Denote by

$$q(I) = \{a \in P | \text{for all } r \in P, L(r, a) \subseteq I \Rightarrow r \in I\}.$$

Let $p(I) = P \backslash q(I)$. Then we have

$$p(I) = \{a \in P | \text{there exists an element } r \in P \backslash I \text{ such that } L(r,a) \subseteq I\}.$$

Proposition 3.7 *Let I be a proper semi-ideal of a poset P, where P has the largest element* 1. *Then*
1) $p(I)$ is a semi-ideal of P and $I \subseteq p(I)$.
2) If $|P| < \infty$ and I is prime, then $p(I)$ is prime.

Proof 1) Let $a \in p(I)$ and $y \leq a$. Assume that $r \notin I$ such that $L(r,a) \subseteq I$. Then we have $L(r,y) \subseteq L(r,a) \subseteq I$ and so $y \in p(I)$.

Let $i \in I$. Since $L(1,i) \subseteq I$ and $1 \notin I$, then $i \in p(I)$ and we have $I \subseteq p(I)$.

2) Assume that $L(x,y) \subseteq p(I)$. Since $|P| < \infty$, $L(x,y)$ has finite elements. Suppose $L(x,y) = \{z_1, z_2, \cdots, z_n\}$. For all $z_i \in L(x,y)$, where $i = 1, 2, \cdots, n$, we have $r_i \notin I$ such that $L(r_i, z_i) \subseteq I$, where $i = 1, 2, \cdots, n$. We claim that $L(r_1, r_2, \cdots, r_n)$ is not contained in I. In fact, since I is prime, I is also n-prime by Lemma 2.5 in [10]. If $L(r_1, r_2, \cdots, r_n) \subseteq I$, we have some $r_i \in I$ for some $i \in \{1, \cdots, n\}$, which is a contradiction. Take $r \in L(r_1, r_2, \cdots, r_n)$ and $r \notin I$. We have $L(r, z_i) \subseteq L(r_i, z_i) \subseteq I$, for all $i \in \{1, \cdots, n\}$. Therefore, we have $L(x, L(r,y)) = L(r,x,y) = L(r, L(x,y)) \subseteq I$. Without loss of generality, suppose $x \notin p(I)$. We get $L(r,y) \subseteq I$ by the definition of $p(I)$. Since $r \notin I$, we must have $y \in p(I)$. So we get that $p(I)$ is prime. □

Lemma 3.8 *Let I be a proper semi-ideal of a poset P, where P has the largest element* 1. *Then $V(G_I(P)) = p(I) \backslash I$.*

Proof Let $a \in G_I(P)$. Then $a \notin I$ and $L(a,b) \subseteq I$ for some $b \notin I$. So $a \in p(I)$ and $V(G_I(P)) \subseteq p(I) \backslash I$. For the reverse inclusion, suppose $a \in p(I) \backslash I$. Then there exists an element $b \in P \backslash I$ such that $L(a,b) \subseteq I$. Consequently, $a \in V(G_I(P))$ and $V(G_I(P)) = p(I) \backslash I$. □

In the following, we give an example of poset P and semi-ideals I and J, such that $G_I(P) = G_J(P)$ but $I \neq J$ and an example of poset P and semi-ideals I and J, such that $p(I) = p(J)$ but $I \neq J$.

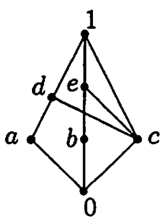

Fig.1

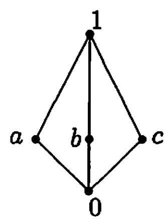

Fig.2

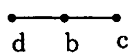

Fig.3

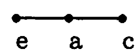

Fig.4

In Fig.1, let $I = \{0, a\}$ and $J = \{0, b\}$ be two semi-ideals of P. Then $p(I) = \{0, a, b, c, d\}$ and $p(J) = \{0, a, b, c, e\}$. Hence $p(I)\backslash I = \{b, c, d\}$ and $p(J)\backslash J = \{a, c, e\}$. We easily get that $G_I(P)$ and $G_J(P)$ are Fig.3 and Fig.4, respectively. Hence, $G_I(P) = G_J(P)$.

In Fig.2, let $I = \{0, a\}$ and $J = \{0, b\}$ be two semi-ideals of P. Then $p(I) = \{0, a, b, c\} = p(J)$.

Using Lemma 3.8 and part 1) of Proposition 3.7 we can easily get the following.

Theorem 3.9 *Let I and J be two semi-ideals of a poset P, where P has the largest element* 1. *Then $G_I(P) = G_J(P)$ and $p(I) = p(J)$ if and only if $I = J$.*

Acknowledgments The author would like to thank the referees for their careful reading, valuable suggestions and useful comments which improved the present paper.

References

[1] M. Alizadeh, A.K. Das, H.R. Maimani, M.R. Pournaki, S. Yassemi, On the diameter and girth of zero-divisor graphs of posets, Discrete Appl. Math. 160 (2012) 1319-1324.

[2] D.F. Anderson, P.S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra 217 (2) (1999) 434-447.

[3] D.D. Anderson, M. Naseer, Beck's coloring of a commutative ring, J. Algebra 159 (2) (1993) 500-514.

[4] M. Axtell, N. Baeth and J. Stickles, Cut vertices in zero-divisor graphs of finite commutative rings, Comm. Alg. 39(6) (2011) 2179-2188.

[5] I. Beck, Coloring of commutative rings, J. Algebra 116 (1) (1988) 208-226.

[6] F.R. DeMeyer, L. DeMeyer, Zero-divisor graphs of semigroups, J. Algebra 283 (2005) 190-198.

[7] F.R. DeMeyer, T. McKenzie, K. Schneider, The zero-divisor graph of a commutative semigroup, Semigroup Forum 65 (2002) 206-214.

[8] D. Dolžan and P. Oblak, The zero-divisor graph of rings and semirings, Int. J. Algebra Comput. 22 (2012) 1250033-1-1250033-20.

[9] S. Ebrahimi Atani, An ideal-based zero-divisor graph of a commutative semiring, Glasnik Matematicki 44(1) (2009) 141-153.

[10] R. Halaš, M. Jukl, On Beck's coloring of posets, Discrete Math. 309 (13) (2009) 4584-589.

[11] R. Halaš, H. Länger, The zero divisor graph of a qoset. Order 27 (2010) 343-351.

[12] V. Joshi, Zero divisor graph of a poset with respect to an ideal, Order 29(2012) 499-506.

[13] D. Lu, T. Wu, On bipartite zero-divisor graphs, Discrete Math. 309 (2009) 755-762.

[14] D. Lu, T. Wu, The zero-divisor graphs of posets and an application to semigroups, Graphs Combin. 26 (6) (2010) 793-804.

[15] Z. Xue, S. Liu, Zero-divisor graphs of partially ordered sets, Appl. Math. Lett. 23 (4) (2010) 449-452.