Distance Two Labeling of Halin Graphs

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Abstract

Let T be a tree with no vertices of degree 2 and at least one vertex of degree 3 or more. A Halin graph G is a plane graph obtained by connecting the leaves of T in the cyclic order determined by the planar drawing of T. Let Δ, $\lambda(G)$, and $\chi(G^2)$ denote, respectively, the maximum degree, the $L(2,1)$-labeling number, and the chromatic number of the square of G. In this paper we prove the following results for any Halin graph G: (1) $\chi(G^2) \leq \Delta + 3$, and moreover $\chi(G^2) = \Delta + 1$ if $\Delta \geq 6$; (2) $\lambda(G) \leq \Delta + 7$, and moreover $\lambda(G) \leq \Delta + 2$ if $\Delta \geq 9$.

Keywords: $L(2,1)$-labeling; Chromatic number; Halin graph

1 Introduction

All graphs considered in this paper are finite and simple graphs. For a graph G, we denote its vertex set, edge set, and order by $V(G)$, $E(G)$, and $|G|$, respectively. For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of neighbors of v and let $d_G(v) = |N_G(v)|$ denote the degree of v in G. A vertex of degree k is called a *k-vertex*. We denote the

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maximum degree of G by $\Delta(G)$ or simply Δ. The *distance* between two vertices u and v is the length of a shortest path connecting them in G. The *square* G^2 of a graph G is the graph defined on the vertex set $V(G)$ such that two distinct vertices are adjacent in G^2 if and only if their distance is at most 2 in G.

A *k-coloring* of a graph G is a mapping σ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\sigma(x) \neq \sigma(y)$ for every edge xy of G. The *chromatic number* $\chi(G)$ of G is the smallest k such that G has a k-coloring.

Wegner [25] first investigated the chromatic number of the square of a planar graph. He proved that $\chi(G^2) \leq 8$ for every planar graph G with $\Delta = 3$ and conjectured that the upper bound could be reduced to 7. In [25], he also proposed the following conjecture.

Conjecture 1 *For a planar graph G,*

$$\chi(G^2) \leq \begin{cases} \Delta + 5, & \text{if } 4 \leq \Delta \leq 7; \\ \lfloor 3\Delta/2 \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

This conjecture remains open. van den Heuvel and McGuinness [14] proved that $\chi(G^2) \leq 2\Delta + 25$ for any planar graph G. The best known result so far is $\chi(G^2) \leq \lceil 5\Delta/3 \rceil + 78$ [19]. Lih, Wang and Zhu [18] established the conjecture for a K_4-minor free graph. It is shown [22, 23] that every outerplanar graph G with $\Delta \geq 3$ has $\chi(G^2) \leq \Delta + 2$, and $\chi(G^2) = \Delta + 1$ if $\Delta \geq 6$.

For positive integers p and q, an *$L(p,q)$-labeling* of a graph G is a function σ from $V(G)$ to the set $\{0, 1, \cdots, k\}$ for some positive integer k such that $|\sigma(x) - \sigma(y)| \geq p$ if x and y are adjacent; and $|\sigma(x) - \sigma(y)| \geq q$ if x and y are at distance 2. The *$L(p,q)$-labeling number* $\lambda_{p,q}(G)$ of G is the smallest k such that G has an $L(p,q)$-labeling with $\max\{\sigma(v) \mid v \in V(G)\} = k$. In particular, we simply write $\lambda(G) = \lambda_{2,1}(G)$. Note that an $L(1,1)$-labeling of G is a proper coloring of the square G^2, and $\lambda_{1,1}(G) = \chi(G^2) - 1$.

The $L(2,1)$-labeling of a graph arose from a variation of the frequency channel assignment problem introduced by Hale [11]. It holds trivially that $\lambda(G) \geq \Delta + 1$ for any graph G. Griggs and Yeh [10] proposed the following conjecture.

Conjecture 2 *For any graph G with $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$.*

In 1996, Chang and Kuo [6] proved that $\lambda(G) \le \Delta^2 + \Delta$ for any graph G. This bound was improved to $\lambda(G) \le \Delta^2 + \Delta - 1$ by Král and Škrekovski [16], and further to $\lambda(G) \le \Delta^2 + \Delta - 2$ by Gonçalves [9]. Using powerful probabilistic method, Havet, Reed and Sereni [13] showed that for any fixed integer p, there is a Δ_p such that every graph G with $\Delta \ge \Delta_p$ has $\lambda_{p,1}(G) \le \Delta^2$. Thus, Conjecture 2 holds for graphs with sufficiently large maximum degree Δ.

Let G be a planar graph. van den Heuvel and McGuinness [14] proved that $\lambda(G) \le 2\Delta + 35$. Molloy and Salavatipour [19] reduced to that $\lambda(G) \le \lceil 5\Delta/3 \rceil + 95$. The result of [14] asserts that Conjecture 2 holds for planar graphs with $\Delta \ge 7$. Further, Bella et al. [1] settled the case $4 \le \Delta \le 6$. Wang and Lih [24] proved that if G contains neither 3-cycles nor 4-cycles, then $\chi(G^2) \le \Delta + 16$ and $\lambda(G) \le \Delta + 21$. Zhu et al. [27] improved this result by showing that if G contains no 4-cycles or no 5-cycles, then $\chi(G^2) \le \Delta + 7$ and $\lambda(G) \le \Delta + 12$. Other related results about this subject can be found in [6, 8, 10, 14, 16, 20]. In particular, [4] and [26] are two nice surveys.

Let T be a tree with no vertex of degree 2 and at least one vertex of degree 3 or more. A vertex of degree 1 of T is called a *leaf*. A *Halin graph* is a plane graph $G = T \cup C$, where C is a cycle connecting the leaves of T in the cyclic order determined by the planar drawing of T. Vertices of C are called *outer vertices* of G and vertices in $V(G) \setminus V(C)$ are called *inner vertices* of G. A Halin graph G is called a *wheel* if G contains only one inner vertex. An inner vertex is called *special* if only one of its neighbors is an inner vertex.

It is easy to see that Halin graphs are 3-connected and planar. Some properties and parameters on Halin graphs have been investigated in [3, 5, 7, 12, 17, 21].

The purpose of this paper is to study the chromatic number of the square and the $L(2,1)$-labeling number of Halin graphs. Let G be a Halin graph. Our main results are:

(1) $\chi(G^2) \le \Delta + 3$, and moreover $\chi(G^2) = \Delta + 1$ if $\Delta \ge 6$;

(2) $\lambda(G) \le \Delta + 7$, and moreover $\lambda(G) \le \Delta + 2$ if $\Delta \ge 9$.

In Section 2, we give structural lemmas and some auxiliary colorings. In Section 3, we establish the proof of (1). The proof of (2) is postponed to Section 4.

2 Preliminaries

The following structural property for Halin graphs appeared in [7].

Lemma 3 (Chen and Wang [7]) *Let $G = T \cup C$ be a Halin graph that is not a wheel. Then C contains a path $P_k = x_1x_2\cdots x_k$ such that one of the following holds (see Fig.1):*

(A1) *$k = 3$ and there exist a special inner 3-vertex u and a vertex v such that $N_G(u) = \{x_1, x_2, v\}$ and $x_3v \in E(G)$.*

(A2) *$k = 4$ and there exist two special inner 3-vertices u_1, u_2 and a vertex v such that $N_G(u_1) = \{x_1, x_2, v\}$ and $N_G(u_2) = \{x_3, x_4, v\}$.*

(A3) *$k \geq 3$ and there exist a special inner $(k+1)$-vertex u and a vertex v such that $N_G(u) = \{x_1, x_2, \cdots, x_k, v\}$.*

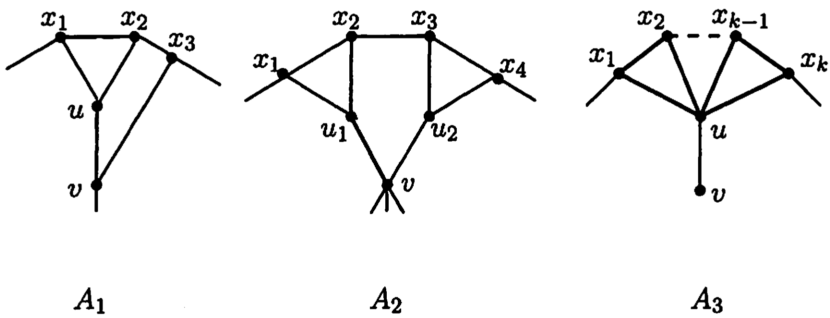


Fig. 1: Three configurations in Lemma 3.

As a special case of Lemma 3, we have obviously the following.

Corollary 4 *Every Halin graph G with $|G| \geq 6$ and $\Delta = 3$ satisfies either* (A1) *or* (A2).

An *$L^*(2,1)$-labeling* of a graph G is defined to be a one-to-one $L(2,1)$-labeling. A function L is called an *assignment* for the graph G if it assigns a list $L(v)$ of possible labels to each vertex v of G. If G has an $L(2,1)$-labeling (or $L^*(2,1)$-labeling, respectively) f such that $f(v) \in L(v)$ for all vertices v, then we say that f is an *L-$L(2,1)$-labeling* (or *L-$L^*(2,1)$-labeling*, respectively) of G. Given a positive integer k, we denote the set $\{k-1, k, k+1\}$ by $\overline{k}$. Moreover, we use $v \Rightarrow \alpha$ to express that a vertex v is labeled with the label α in the given labeling.

Lemma 5 *Let $P = x_1x_2x_3$ be a path. Let L be a list assignment for P such that $|L(x_1)| \geq 2$, $|L(x_3)| \geq 3$, and $|L(x_2)| \geq 5$. Then P has an L-$L^*(2,1)$-labeling.*

Proof. Without loss of generality, we may assume that $|L(x_1)| = 2$, $|L(x_3)| = 3$, and $|L(x_2)| = 5$. Suppose that $L(x_2) = \{a, b, c, d, e\}$ such that $a < b < c < d < e$. If there exists $k \in L(x_1) \setminus L(x_2)$, we first label x_1 with k. Then let $L'(x_2) = L(x_2) \setminus \{k-1, k+1\}$ and $L'(x_3) = L(x_3) \setminus \{k\}$. Thus $|L'(x_2)| \geq 3$ and $|L'(x_3)| \geq 2$. It is easy to see that there exist $i \in L'(x_2)$ and $j \in L'(x_3)$ such that $|i - j| \geq 2$. It suffices to label x_2 with i and x_3 with j. So suppose $L(x_1) \subseteq L(x_2)$. If there exists $p \in L(x_3) \setminus L(x_2)$, implying $p \notin L(x_1)$, we label x_3 with p. Afterwards we define $L'(x_1) = L(x_1)$ and $L'(x_2) = L(x_3) \setminus \{p-1, p+1\}$. Since $|L'(x_1)| = 2$ and $|L'(x_2)| \geq 3$, we can properly label x_1 and x_2. Thus suppose $L(x_3) \subseteq L(x_2)$. If $a \in L(x_1)$ or $e \in L(x_1)$, we first label x_1 with a or e. Then we define similarly $L'(x_2)$ and $L'(x_3)$ to make that $|L'(x_2)| \geq 3$ and $|L'(x_3)| \geq 2$. Hence assume $L(x_1) \subseteq \{b, c, d\}$. By symmetry, we only need to consider two cases as follows.

Assume $L(x_1) = \{b, c\}$. First let $x_1 \Rightarrow c$. Then let $x_2 \Rightarrow a$ and $x_3 \Rightarrow e$ or d if $L(x_3) \neq \{a, b, c\}$, and $x_3 \Rightarrow a$ and $x_2 \Rightarrow e$ otherwise.

Assume $L(x_1) = \{b, d\}$. If $e \in L(x_3)$, we let $x_3 \Rightarrow e$, $x_2 \Rightarrow a$, and $x_1 \Rightarrow d$. If $a \in L(x_3)$, we let $x_3 \Rightarrow a$, $x_2 \Rightarrow e$, and $x_1 \Rightarrow b$. If $L(x_3) = \{b, c, d\}$, we let $x_3 \Rightarrow c$, $x_2 \Rightarrow e$, and $x_1 \Rightarrow b$. □

Lemma 6 *Let $P = x_1x_2\cdots x_k$ be a path with $k \geq 5$. Let L be a list assignment for P with $L(x_2) = L(x_3) = \cdots = L(x_{k-1}) = \{c_1, c_2, \cdots, c_m\}$ such that $m \geq k$ and $c_1 < c_2 < \cdots < c_m$, $|L(x_1)|$, $|L(x_k)| \geq 3$ with $L(x_1), L(x_k) \subseteq L(x_2)$. Then P has an L-$L^*(2,1)$-labeling.*

Proof. Without loss of generality, we suppose that $m = k$ and $|L(x_1)| = |L(x_k)| = 3$. First assume $k = 5$. Since $|L(x_1)| = 3$ and $L(x_1) \subseteq L(x_2)$, we can label x_1 with some label $s \in L(x_1) \setminus \{c_1, c_5\}$.

If $s = c_2$ (if $s = c_4$, we can give a similar argument), we label x_5 with some label $t \in L(x_5) \setminus \{c_1, c_2\}$. For $t = c_3, c_4$ and c_5, we label the vertices (x_2, x_3, x_4), respectively, by (c_4, c_1, c_5), (c_5, c_3, c_1) and (c_4, c_1, c_3). If $s = c_3$, we label x_5 with some label $t \in L(x_5) \setminus \{c_1, c_3\}$. For $t = c_2, c_4$ and c_5, we label the vertices (x_2, x_3, x_4), respectively, by (c_5, c_1, c_4), (c_1, c_5, c_2) and (c_1, c_4, c_2).

Next assume $k = 6$. We label x_1 with some label $s \in L(x_1) \setminus \{c_1, c_6\}$. If $s = c_2$, we further label x_6 with $t \in L(x_6) \setminus \{c_1, c_2\}$. For $t = c_3, c_4, c_5$ and c_6, we label the vertices (x_2, x_3, x_4, x_5), respectively, by (c_6, c_4, c_1, c_5), (c_5, c_1, c_3, c_6), (c_4, c_6, c_1, c_3) and (c_5, c_3, c_1, c_4). If $s = c_3$, we further label x_6 with $t \in L(x_6) \setminus \{c_1, c_3\}$. For $t = c_2, c_4, c_5$ and c_6, we label the vertices (x_2, x_3, x_4, x_5), respectively, by (c_5, c_1, c_6, c_4), (c_5, c_1, c_6, c_2), (c_6, c_2, c_4, c_1) and (c_1, c_5, c_2, c_4). If $s = c_4$, or $s = c_5$, we can give a similar labeling.

Finally assume $k \geq 7$. Let $X' = \{c_i \in L(x_2) | i \equiv 0 (\text{mod } 2),\ i = 1, 2, \cdots, k\}$ and $X'' = L(x_2) \setminus X'$. Thus $|X'| \geq 3$, $|X''| \geq 4$, and for any $c_i, c_j \in X'$, or $c_i, c_j \in X''$, we have $|c_i - c_j| \geq 2$. First assume $X' \cap L(x_1) \neq \emptyset$. (If $X' \cap L(x_k) \neq \emptyset$, we have a similar argument.) If $|X' \cap L(x_k)| \geq 2$, we label, successively, x_1, x_k, $x_2, x_3, \cdots, x_{\lfloor k/2 \rfloor - 1}$ with mutually different labels in X', and $x_{\lfloor k/2 \rfloor}, x_{k-1}, x_{k-2}, \cdots, x_{\lfloor k/2 \rfloor + 1}$ with mutually different labels in X''. If $|X' \cap L(x_k)| \leq 1$, it follows that $|X'' \cap L(x_k)| \geq 2$. We label, successively, $x_1, x_2, \cdots, x_{\lfloor k/2 \rfloor}$ with mutually different labels in X', and $x_k, x_{\lfloor k/2 \rfloor + 1}, x_{\lfloor k/2 \rfloor + 2}, \cdots, x_{k-1}$ with different labels in X''. Now assume $L(x_1) \cup L(x_k) \subseteq X''$. We label, successively, $x_2, x_3, \cdots, x_{\lfloor k/2 \rfloor + 1}$ with mutually different labels in X', and $x_1, x_{\lfloor k/2 \rfloor + 2}, x_k, x_{k-1}, \cdots, x_{\lfloor k/2 \rfloor + 3}$ with mutually different labels in X''. □

The following result is an easy observation.

Lemma 7 *For a wheel* G,

(1) $\chi(G^2) = \Delta + 1$.

(2) $\lambda(G) = 6$ *if* $4 \leq |G| \leq 5$, *and* $\lambda(G) = \Delta + 1$ *if* $|G| \geq 6$.

3 Coloring the square

In what follows, a k-coloring of G^2 is called a *square-k-coloring* of G.

Theorem 8 *If* G *is a Halin graph, then* $\chi(G^2) \leq \max\{7, \Delta + 1\}$.

Proof. Let $K = \max\{7, \Delta+1\}$. The proof is proceeded by induction on the vertex number $|G|$. If $|G| \le 7$, then it holds obviously that $\chi(G^2) \le 7 \le K$, since we may assign different colors to the vertices of G. Let $G = T \cup C$ be a Halin graph with $|G| \ge 8$. If G is a wheel, then the result follows from Lemma 7. Assume that G is not a wheel. By Lemma 3, there exists a path $P_k = x_1x_2\cdots x_k$ in C such that one of (A1) to (A3) holds. In the following argument, we always assume that $y \in N_C(x_1)\backslash\{x_2\}$, $z \in N_C(x_k)\backslash\{x_{k-1}\}$, $N_G(y) = \{x_1, y_1, y_2\}$, and $N_G(z) = \{x_k, z_1, z_2\}$. Let $S = \{1, 2, \cdots, K\}$ denote a set of K colors. We handle separately each of these three cases.

(A1) Let $H = G - \{x_1, x_2\} + \{yu, x_3u\}$. By the induction hypothesis, H has a square-K-coloring f with the color set S. Obviously, $f(u), f(v), f(y), f(x_3)$ are mutually distinct. Thus, we may assume that $f(u) = 1$, $f(v) = 2$, $f(y) = 3$ and $f(x_3) = 4$. Since $|S| = K \ge 7$, we can let $x_1 \Rightarrow a \in \{5, 6, 7\}\backslash\{f(y_1), f(y_2)\}$ and $x_2 \Rightarrow b \in \{5, 6, 7\}\backslash\{a, f(z)\}$.

(A2) Let $H = G - \{x_1, x_2, x_3, x_4\} + \{yu_1, u_1u_2, u_2z\}$. By the induction hypothesis, H has a square-K-coloring f using the color set S. We define

$$\begin{aligned}
L(x_1) &= S\backslash\{f(v), f(u_1), f(y), f(y_1), f(y_2)\},\\
L(x_2) &= S\backslash\{f(v), f(y), f(u_1), f(u_2)\},\\
L(x_3) &= S\backslash\{f(v), f(z), f(u_1), f(u_2)\},\\
L(x_4) &= S\backslash\{f(v), f(u_2), f(z), f(z_1), f(z_2)\}.
\end{aligned}$$

It is easy to inspect that $|L(x_1)| \ge |S| - 5 = K - 5 \ge 7 - 5 = 2$, and similarly $|L(x_4)| \ge 2$, $|L(x_2)| \ge 3$, and $|L(x_3)| \ge 3$. If $|L(x_1)| \ge 3$, we let $x_4 \Rightarrow a \in L(x_4)$, $x_3 \Rightarrow b \in L(x_3)\backslash\{a\}$, $x_2 \Rightarrow c \in L(x_2)\backslash\{a, b\}$, $x_1 \Rightarrow c \in L(x_1)\backslash\{b, c\}$. So suppose $|L(x_1)| = 2$. There is a color $a \in L(x_2)\backslash L(x_1)$. We assign a to x_2, then color x_4 with $b \in L(x_4)\backslash\{a\}$, x_3 with $c \in L(x_3)\backslash\{a, b\}$, and x_1 with a color in $L(x_1)\backslash\{c\}$.

(A3) Let $H = G - \{x_2, x_3, \cdots, x_{k-1}\} + \{x_1x_k\}$. By the induction hypothesis, H has a square-K-coloring f using S. Assume that $f(u) = 1$, $f(v) = 2$, $f(x_1) = 3$ and $f(x_k) = 4$.

If $k = 3$, we can color x_2 with a color in $\{5, 6, 7\}\backslash\{f(y), f(z)\}$.

If $k = 4$, there exist $a \in \{5, 6, 7\}\backslash\{f(y)\}$ and $b \in \{5, 6, 7\}\backslash\{f(z)\}$ such that $a \ne b$. We color x_2 with a and x_3 with b.

If $k = 5$, we first color x_2 with $a \in \{5, 6, 7\}\backslash\{f(y)\}$ and x_4 with $b \in \{5, 6, 7\}\backslash\{f(z)\}$ such that $a \ne b$. Afterwards we color x_3 with a color in $\{5,6,7\}\backslash\{a,b\}$.

If $k \geq 6$, then $\Delta \geq d_G(u) = k+1 \geq 7$, we define

$$L(x_2) = \{5,6,\cdots,K\}\backslash\{f(y)\},$$
$$L(x_i) = \{5,6,\cdots,K\},\ \ i=3,4,\cdots,k-2,$$
$$L(x_{k-1}) = \{5,6,\cdots,K\}\backslash\{f(z)\}.$$

Since $K = \Delta+1 \geq 8$ in this case, we get $|L(x_2)|, |L(x_{k-1})| \geq \Delta - 4 \geq 3$, and $|L(x_i)| = \Delta - 3$ for $i = 3,4,\cdots,k-2$. Note that $|\{x_2,x_3,\cdots,x_{k-1}\}| = k-2 \leq \Delta-1-2 = \Delta-3$. By Lemma 6, there exists $c_i \in L(x_i)$, $i = 2,3,\cdots,k-1$, such that all $c_2, c_3, \cdots, c_{k-1}$ are mutually distinct. Color x_i with c_i for $2 \leq i \leq k-1$ to establish a square-K-coloring of G. □

Theorem 9 *If G is a Halin graph with $\Delta = 3$, then $\chi(G^2) \leq 6$.*

Proof. Since G is 3-regular, $|G|$ is even. If $|G| \leq 6$, the conclusion holds trivially. Let $G = T \cup C$ be a Halin graph with $\Delta = 3$ and $|G| \geq 8$. By Corollary 4, G satisfies (A1) or (A2). Similarly to the proof of Theorem 8, we suppose that $y \in N_C(x_1)\backslash\{x_2\}$, $z \in N_C(x_k)\backslash\{x_{k-1}\}$, $N_G(y) = \{x_1, y_1, y_2\}$, and $N_G(z) = \{x_k, z_1, z_2\}$. Let $S = \{1,2,\cdots,6\}$ denote a set of six colors used in the following.

(A1) Let w denote the neighbor of v different from u and x_3. Let $H = G - \{x_1,x_2,x_3,u\} + \{vy, vz\}$. Then H is a 3-regular Halin graph with $|H| < |G|$. By the induction hypothesis, H has a square-6-coloring f using S such that $f(v) = 1$, $f(w) = 2$, $f(y) = 3$ and $f(z) = 4$. Let $Y = \{f(y_1), f(y_2)\}$ and $Z = \{f(z_1), f(z_2)\}$.

If $2 \notin Y$, let $x_1 \Rightarrow 2$, $x_3 \Rightarrow a \in \{3,5,6\} \setminus Z$, $x_2 \Rightarrow b \in \{5,6\} \setminus \{a\}$, and $u \Rightarrow c \in \{4,5,6\} \setminus \{a,b\}$. If $4 \notin Y$, let $x_1 \Rightarrow 4$, $x_2 \Rightarrow 2$, $x_3 \Rightarrow a \in \{3,5,6\} \setminus Z$, and $u \Rightarrow b \in \{5,6\} \setminus \{a\}$. If $Y = \{2,4\}$, let $x_2 \Rightarrow 2$, $x_3 \Rightarrow a \in \{3,5,6\} \setminus Z$, $x_1 \Rightarrow b \in \{5,6\} \setminus \{a\}$, and $u \Rightarrow c \in \{4,5,6\} \setminus \{a,b\}$.

(A2) Let w denote the neighbor of v in G different from u_1 and u_2. Let $H = G - \{x_1,x_2,x_3,x_4,u_1,u_2\} + \{vy, vz\}$. By the induction hypothesis, H has a square-6-coloring f using S such that $f(v) = 1$, $f(w) = 2$, $f(y) = 3$ and $f(z) = 4$. Similarly, we set $Y = \{f(y_1), f(y_2)\}$ and $Z = \{f(z_1), f(z_2)\}$.

If $Y \neq \{5,6\}$, we let $x_1, u_2 \Rightarrow a \in \{5,6\} \setminus Y$, $x_2 \Rightarrow 4$, $u_1 \Rightarrow b \in \{5,6\}\backslash\{a\}$, $x_4 \Rightarrow c \in \{2,3,5,6\}\backslash(Z \cup \{a\})$, and $x_3 \Rightarrow d \in \{2,3\}\backslash\{c\}$.

If $Y = \{5,6\}$, let $x_1 \Rightarrow 2$, $x_2 \Rightarrow 4$, $u_2 \Rightarrow 3$, $x_4 \Rightarrow a \in \{2,5,6\} \setminus Z$, $x_3 \Rightarrow b \in \{5,6\} \setminus \{a\}$, and $u_1 \Rightarrow c \in \{5,6\} \setminus \{b\}$. □

The following consequences follow from Theorems 8 and 9:

Corollary 10 *For a Halin graph G, we have $\chi(G^2) \le \Delta + 3$.*

Corollary 11 *If G is a Halin graph with $\Delta \ge 6$, then $\chi(G^2) = \Delta+1$.*

4 $L(2,1)$-labeling

In this section, we study the $L(2,1)$-labeling of Halin graphs. We first give an interesting observation. Bondy showed in [2] that Halin graphs are Hamiltonian. Kang showed in [15] that every Hamiltonian graph G with $\Delta = 3$ satisfies $\lambda(G) \le 9$. Combining these two facts, we conclude immediately the following Theorem 12:

Theorem 12 *If G is a Halin graph with $\Delta = 3$, then $\lambda(G) \le 9$.*

Theorem 13 *If G is a Halin graph, then $\lambda(G) \le \max\{11, \Delta + 2\}$.*

Proof. Set $M = \max\{11, \Delta + 2\}$ and let $B = \{0, 1, \cdots, M\}$ denote a set of $M+1$ labels. We make use of induction on $|G|$. The theorem holds trivially for $|G| \le 5$. Suppose that $G = T \cup C$ is a Halin graph with $|G| \ge 6$. If G is a wheel, the result follows from Lemma 7. So assume that G is not a wheel. By Lemma 3, there exists a path $P_k = x_1 x_2 \cdots x_k$ in C such that one of (A1) to (A3) holds. We reduce these three configurations separately in the following.

(A1) Let $H = G - \{x_1, x_2\} + \{yu, x_3u\}$. By the induction hypothesis, H has an $L(2,1)$-labeling f with the label set B. Define the list of assignments

$$L(x_1) = B \backslash \{\overline{f(u)}, \overline{f(y)}, f(v), f(y_1), f(y_2), f(x_3)\},$$
$$L(x_2) = B \backslash \{\overline{f(u)}, \overline{f(x_3)}, f(v), f(y), f(z)\}.$$

Since $M \ge 11$, it follows that $|L(x_1)| \ge M + 1 - 3 - 3 - 4 \ge 2$ and $|L(x_2)| \ge M + 1 - 3 - 3 - 3 \ge 3$. Thus there exist $c_1 \in L(x_1)$ and $c_2 \in L(x_2)$ such that $|c_1 - c_2| \ge 2$. Label x_i with c_i for $i = 1, 2$.

(A2) Let $H = G - \{x_1, x_2\} + \{yu_1, x_3u_1\}$. By the induction hypothesis, let f be an $L(2,1)$-labeling of H with the label set B. Delete the label of x_3 and then define the list of assignments

$$L(x_1) = B\backslash\{\overline{f(u_1)}, \overline{f(y)}, f(v), f(y_1), f(y_2)\},$$
$$L(x_2) = B\backslash\{\overline{f(u_1)}, f(v), f(y), f(u_2), f(x_4)\},$$
$$L(x_3) = B\backslash\{\overline{f(u_2)}, \overline{f(x_4)}, f(v), f(z), f(u_1)\}.$$

It is easy to see that $|L(x_1)| \geq 3$, $|L(x_2)| \geq 5$, and $|L(x_3)| \geq 3$. By Lemma 5, x_1, x_2, x_3 can be labeled.

(A3) Let $H = G - \{x_2\} + \{x_1x_3\}$, and let f be an $L(2,1)$-labeling of H with the label set B. If $k \leq 4$, we delete the labels of x_1 and x_3, and let $P = x_1x_2x_3$. If $k = 3$, we define

$$L(x_1) = B\backslash\{\overline{f(u)}, \overline{f(y)}, f(v), f(y_1), f(y_2)\},$$
$$L(x_2) = B\backslash\{\overline{f(u)}, f(v), f(y), f(z)\},$$
$$L(x_3) = B\backslash\{\overline{f(u)}, \overline{f(z)}, f(v), f(z_1), f(z_2)\}.$$

Then $|L(x_1)|, |L(x_3)| \geq 3$ and $|L(x_2)| \geq 6$. If $k = 4$, L can be defined analogously so that $|L(x_1)| \geq 2$, $|L(x_3)| \geq 4$, and $|L(x_2)| \geq 6$. If $5 \leq k \leq 6$, we take $P = x_2x_3x_4$ after deleting the labels of x_3 and x_4, so that the defined list assignment L satisfies $|L(x_2)| \geq 2$, $|L(x_4)| \geq 3$, and $|L(x_3)| \geq 5$. If $k \geq 7$, we delete the labels of $x_3, x_4, \cdots, x_{k-1}$, and let $P = x_2x_3 \cdots x_{k-1}$. Thus P is a path of length at least 4. Define the list of assignments

$$L(x_2) = B\backslash\{\overline{f(x_1)}, \overline{f(u)}, f(y), f(v), f(x_k)\},$$
$$L(x_{k-1}) = B\backslash\{\overline{f(x_k)}, \overline{f(u)}, f(z), f(v), f(x_1)\},$$
$$L(x_3) = L(x_4) = \cdots = L(x_{k-2}) = B\backslash\{\overline{f(u)}, f(v), f(x_1), f(x_k)\}.$$

Note that $|L(x_2)|, |L(x_{k-1})| \geq 3$, and $|L(x_i)| \geq M + 1 - 3 - 3 \geq \Delta + 2 - 5 \geq k - 2$ for all $i = 3, 4, \cdots, k-2$. If $3 \leq k \leq 6$, P admits an L-$L(2,1)$-labeling by Lemma 5. If $k \geq 7$, P admits an L-$L^*(2,1)$-labeling by Lemma 6. Therefore f can always be extended to an $L(2,1)$-labeling of G. The proof of the theorem is complete. □

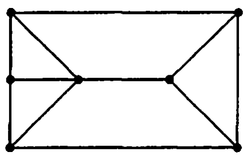
The following consequence follows from Theorems 12 and 13:

Corollary 14 *Let G be a Halin graph. Then $\lambda(G) \leq \Delta + 7$; and moreover $\lambda(G) \leq \Delta + 2$ if $\Delta \geq 9$.*

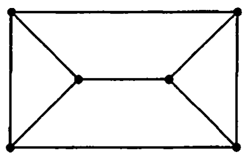
5 Concluding remarks

Corollary 11 asserts that the chromatic number of the square of a Halin graph G with $\Delta \geq 6$ is exactly $\Delta+1$. Corollary 10 shows that if G is a Halin graph with $3 \leq \Delta \leq 5$, then $\chi(G^2) \leq \Delta+3$. The result for the case $3 \leq \Delta \leq 4$ is the best possible in the sense that there exist Halin graphs G such that $\chi(G^2) = \Delta+3$. Observe graphs H_1 and H_2 depicted in Fig. 2. It is easy to see that H_1 is a Halin graph with $\Delta = 4$ and $\chi(H_1^2) = 7 = \Delta+3$, and H_2 is a Halin graph with $\Delta = 3$ and $\chi(H_2^2) = 6 = \Delta+3$. Moreover, Theorem 8 implies that a Halin graph G with $\Delta = 5$ has $\chi(G^2) \leq 7 = \Delta+2$. However, we like to put forward to the following conjecture:

Conjecture 15 *If G is a Halin graph with $\Delta = 5$, then $\chi(G^2) = 6 = \Delta+1$.*



H_1



H_2

Fig. 2: Two Halin graph examples H_1 and H_2.

Problem 16 *Determine the least constant Δ_0 such that every Halin graph G with $\Delta \geq \Delta_0$ has $\lambda(G) \leq \Delta+2$.*

Since K_4 is a Halin graph with $\Delta = 3$ and $\lambda(K_4) = 6 = \Delta+3$, we derive that $4 \leq \Delta_0 \leq 9$ by Corollary 14.

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