Antidirected Hamilton cycles in the Cartesian product of directed cycles

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Abstract

We prove that the Cartesian product of two directed cycles of lengths n_1, n_2 contains an antidirected Hamilton cycle, and hence it is decomposable into antidirected Hamilton cycles, if and only if $gcd(n_1, n_2) = 2$. For the Cartesian product of $k > 2$ directed cycles we give new sufficient conditions for the existence of an antidirected Hamilton cycle.

Keywords: Cartesian product, antidirected Hamilton cycle, graph decomposition, simple digraph.

1 Introduction

By cycles in this paper we mean either directed or antidirected cycles. An antidirected cycle on k vertices in G is a cycle consisting of k arcs that does not contain an induced directed path on three vertices. The Cartesian product $G = C_{n_1}\square C_{n_2}\square\cdots\square C_{n_k}$ of k directed cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ is a digraph such that the vertex set $V(G)$ equals the Cartesian product $V(C_{n_1}) \times V(C_{n_2}) \times \cdots \times V(C_{n_k})$ and there is an arc in G from vertex $u = (u_1, u_2, \ldots, u_k)$ to vertex $v = (v_1, v_2, \ldots, v_k)$ if and only if there exists $1 \leq r \leq k$ such that there is an arc (u_r, v_r) in C_{n_r} and $u_i = v_i$ for all $i \neq r$. Antidirected Hamilton cycles were studied for the special families of graphs [1,3,5]. Here we study them with respect to the Cartesian product of directed cycles.

Trotter and Erdös [6] proved that the Cartesian product of two directed cycles $C_{n_1}\square C_{n_2}$ contains a Hamilton cycle if and only if there exist positive integers s_1 and s_2 such that $gcd(n_1, n_2) = s_1 + s_2$ and $gcd(n_1, s_1) = gcd(n_2, s_2) = 1$. Later, Keating [4] showed that the Cartesian product of two directed cycles $C_{n_1}\square C_{n_2}$ can be decomposed into directed Hamilton cycles if and only if there exist positive integers s_1 and s_2 such that $gcd(n_1, n_2) = s_1 + s_2$ and $gcd(n_1 n_2, s_1 s_2) = 1$. We give the corresponding results in Section 2 for $G = C_{n_1}\square C_{n_2}$ in respect to antidirected Hamilton cycles H_G and arc-disjoint $\overline{H}_G$ in G. In particular, we show that G has H_G if and only if $gcd(n_1, n_2) = 2$. Furthermore, we prove that G can be decomposed into H_G and $\overline{H}_G$ if and only if $gcd(n_1, n_2) = 2$.

Curran and Witte [2] extended the result of Trotter and Erdös [6] and showed that there exists a Hamilton cycle in the Cartesian product of more than two nontrivial directed cycles. In Section 3 of this paper we focus on the corresponding problem related to an antidirected Hamilton cycle in the Cartesian product of three or more directed cycles. In particular, we give some sufficient conditions for the existence of an antidirected Hamilton cycle in $G = C_{n_1}\square C_{n_2}\square\cdots\square C_{n_k}$, for $k \geq 3$. Based on these results we give a conjecture for the necessary and sufficient conditions for the existence of an antidirected Hamilton cycle in G.

2 Cartesian product of two cycles

Let $C^i = v_0^i v_1^i \ldots v_{n_1-2}^i v_{n_1-1}^i v_0^i$ be i'th cycle generated by C_{n_1} in $G = C_{n_1}\square C_{n_2}$, where $n_2 \geq i \geq 1$. Then a shortest antidirected path $P_{j_1,j_2}^i = v_{j_1}^i v_{j_1+1(mod\ n_1)}^i \cdots v_{j_2}^i$ is illustrated in Fig. 1, where $j_1 + 1 \pmod{n_1} \neq j_2$.

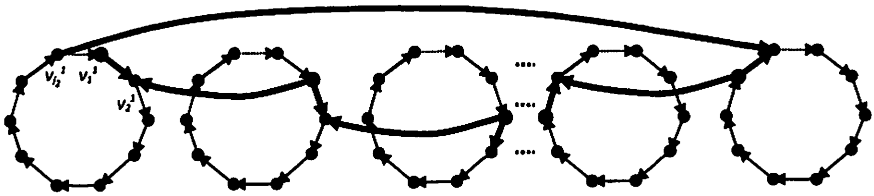

Figure 1: Illustration of $P^i_{j_1,j_2} = P^1_{1,j_2}$ in H_G

Theorem 2.1 *The Cartesian product $G = C_{n_1} \square C_{n_2}$ contains an antidirected Hamilton cycle if and only if $gcd(n_1, n_2) = 2$.*

Proof. Antidirected Hamilton cycle H_G exists in G only if there is a closed walk of form $W = a_1b_1a_2b_2 \dots a_{\frac{n_1n_2}{2}} b_{\frac{n_1n_2}{2}}$ on a 2-dimensional torus, where each a_j corresponds to an arc in a cycle induced by C_{n_1} (i.e., an arc on a C^i cycle in Fig. 1), and each b_j corresponds to a reversed arc in a cycle induced by C_{n_2}, for $\frac{n_1n_2}{2} \geq j \geq 1$. Hence, H_G exists only if $in_1 + jn_2 = n_1n_2$ and $in_1 = jn_2$ for some positive integers i, j, where $n_2 > i \geq 1$ and $n_1 > j \geq 1$. This implies $n_1 = 2j$ and $n_2 = 2i$, which means that $\gcd(n_1, n_2) \geq 2$. By prime factorization $n_1 = p_1p_2 \cdots p_r = 2p_2 \cdots p_r$ and $n_2 = p'_1p'_2 \cdots p'_s = 2p'_2 \cdots p'_s$, where $p_{i+1} \geq p_i$ for $r > i \geq 1$, $p'_{j+1} \geq p'_j$ for $s > j \geq 1$. Suppose $\gcd(n_1, n_2) > 2$. Then based on the above prime factorization there exists positive integers i', j', q that satisfy $i'n_1 + j'n_2 = q < n_1n_2$ and $i'n_1 = j'n_2$. This means that there exists a shorter closed walk $W' = a_1b_1a_2b_2 \dots a_{\frac{q}{2}} b_{\frac{q}{2}}$, which means that $W \neq H_G$ – a contradiction that proves the necessary conditions. On the other hand, $\gcd(n_1, n_2) = 2$ implies that W is the shortest closed walk since according to the above prime factorization of n_1, n_2 there are no positive integers i', j', q that satisfy $i'n_1 + j'n_2 = q < n_1n_2$ and $i'n_1 = j'n_2$. □

Corollary 2.2 *The Cartesian product $G = C_{n_1} \square C_{n_2}$ can be decomposed into two antidirected Hamilton cycles if and only if $gcd(n_1, n_2) = 2$.*

Proof. Let H_G denote an antidirected Hamilton cycle in G. By Theorem 2.1 G has H_G if and only if $gcd(n_1, n_2) = 2$. Let $C^i_{n_j}$ be ith cycle in G induced by C_{n_j}, where $j = 1$ or $j = 2$. We denote $C^i_{n_j}$ by a sequence of vertices $C^i_{n_j} = v^i_0v^i_1...v^i_{n_j-1}v^i_0$. Clearly, H_G induces 1-factor in every $C^i_{n_j}$. By isomorphism $v^i_x \to v^i_{x+1(mod\ n_j)}$ we can replace every arc $(v^i_x, v^i_{x+1(mod\ n_j)})$ with $(v^i_{x+1(mod\ n_j)}, v^i_{x+2(mod\ n_j)})$ in every $C^i_{n_j}$, which produces an arc-disjoint antidirected Hamilton cycle with respect to H_G. □

3 Cartesian product of more than two cycles

Since the Cartesian product of cycles is commutative then without loss of generality we can assume that C_{n_1}, C_{n_2} represent any two cycles in $G = C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}$. This allows simplifying the statements of the next two theorems.

Theorem 3.1 *Let G be a Cartesian product of k cycles $C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}$, where $n_i \geq 2$ and $k \geq 2$. Then G contains an antidirected Hamilton cycle if $gcd(n_1, n_2, \ldots, n_k) = 2$ and $gcd(n_2, n_3, \ldots, n_k) = n_2$.*

Proof. We proceed by induction on k. For $k = 2$ $gcd(n_2) = n_2$ and if $gcd(n_1, n_2) = 2$ then G contains an antidirected Hamilton cycle according to Theorem 2.1. Suppose $G = C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}$ contains an antidirected Hamilton cycle H_G, $gcd(n_1, n_2, \ldots, n_k) = 2$, and $gcd(n_2, n_3, \ldots, n_k) = n_2$ for $k \geq 2$. Let $G' = G \square C_{n_{k+1}}$ and $gcd(n_2, n_3, \ldots, n_{k+1}) = n_2$. So, there are n_{k+1} pairwise disjoint antidirected cycles $H^1, H^2, \ldots, H^{n_{k+1}}$ in G' corresponding to H_G. By the property of Cartesian product of directed cycles, each H^i must contain at least one arc corresponding to C_{n_2}. Let $(v_j^i, v_{j+1(mod\ n_2)}^i)$ be an arc induced by C_{n_2} in H^i. Let P_j^i be an antidirected path in G' defined by removal of arc $(v_j^i, v_{j+1(mod\ n_2)}^i)$ from H^i.

By assumption, $gcd(n_2, n_{k+1}) = n_2$ assures that the following antidirected cycle exists in G':

$$C_{G'} = v_1^1 v_2^1 v_2^2 v_3^2$$

$$v_3^3 v_4^3 v_4^4 v_5^4$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$v_{n_2-3}^{n_k-3} v_{n_2-2}^{n_k-3} v_{n_2-2}^{n_k-2} v_{n_2-1}^{n_k-2}$$

$$v_{n_2-1}^{n_k-1} v_0^{n_k-1} v_0^{n_k} v_1^{n_k} v_1^1.$$

We can substitute each $(v_j^i, v_{j+1(mod\ n_2)}^i)$ in $C_{G'}$ with P_j^i that produces the following antidirected Hamilton cycle $H_{G'}$ (Fig. 2):

$$H_{G'} = P_1^1 P_2^2$$

$$P_3^3 P_4^4$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_{n_2-3}^{n_{k+1}-3} P_{n_2-2}^{n_{k+1}-2}$$

$$P_{n_2-1}^{n_{k+1}-1} P_0^{n_{k+1}} v_1^1.$$

This completes the proof of Theorem 3.1. □

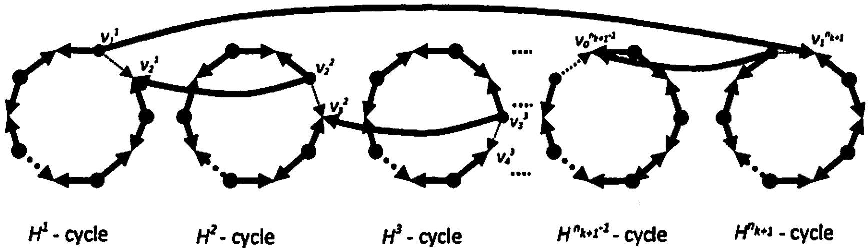

Figure 2: Illustration of $H_{G'}$

As a special case of Theorem 3.1 we obtain:

Corollary 3.2 *Let G be a Cartesian product of even cycles with at least one cycle of length 2. Then G contains an antidirected Hamilton cycle.* □

In general, $gcd(n_1, n_2, \ldots, n_k) = 2$ is not necessary for $C_{n_1} \Box C_{n_2} \Box \cdots \Box C_{n_k}$ to contain an antidirected Hamilton cycle. This case is demonstrated in the next result.

Theorem 3.3 *Let G be a Cartesian product of k cycles $C_{n_1} \Box C_{n_2} \Box \cdots \Box C_{n_k}$, where $n_1 = 2$, n_2 odd, n_i even for $i \geq 3$, and $k \geq 3$. Then G contains an antidirected Hamilton cycle if* $\gcd(2n_2, n_i) = 2n_2$ *for some $i \geq 3$.*

Proof. Consider first a Cartesian product of two cycles $G = C_2 \Box C_{n_2}$ for n_2 odd. Let two cycles corresponding to C_{n_2} in G be defined as

$$C_G^0 = v_0(0)v_1(0) \ldots v_{n_2-1}(0)v_0(0)$$

and

$$C_G^1 = v_0(1)v_1(1) \ldots v_{n_2-1}(1)v_0(1).$$

Let $P_i(j)$ denote an antidirected Hamilton path in G with end-vertex $v_i(j)$. Clearly, there is an antidirected Hamilton path

$$P_0(0) = v_0(0)v_0(1)v_1(1)$$

$$v_1(0)v_2(0)v_2(1)v_3(1)$$

$$v_3(0)v_4(0)v_4(1)v_5(1)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$v_{n_2-4}(0)v_{n_2-3}(0)v_{n_2-3}(1)v_{n_2-2}(1)$$

$$v_{n_2-2}(0)v_{n_2-1}(0)v_{n_2-1}(1)$$

in G. Furthermore, by vertex transitivity of G there is an antidirected Hamilton path

$$P_i(j) = v_i(j)v_i(j+1(mod\ 2))v_{i+1(mod\ n_2)}(j+1(mod\ 2))$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$v_{i+n_2-1(mod\ n_2)}(j+1(mod\ 2)).$$

Consider now $G' = G\Box C_{n3} = C_2\Box C_{n_2}\Box C_{n3}$, where $gcd(2n_2, n_3) = 2n_2$. So, there are n_3 pairwise disjoint antidirected paths $P_i^1(j), P_i^2(j), \dots, P_i^{n_3}(j)$ in G'. Then there is the following antidirected path P in G'.

$$P = P_0^1(0)P_{n_2-1}^2(1)$$

$$P_{n_2-1}^3(0)P_{n_2-2}^4(1)$$

$$P_{n_2-2}^5(0)P_{n_2-3}^6(1)$$

$$P_{n_2-3}^7(0)P_{n_2-4}^8(1)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$P_2^{n_3-3}(0)P_1^{n_3-2}(1)$$

$$P_1^{n_3-1}(0)P_0^{n_3}(1).$$

Furthermore, for $gcd(2n_2, n_3) = 2n_2$, $Pv_0^{n_3}(0)$ represents an antidirected Hamilton path, and $Pv_0^{n_3}(0)v_0^1(0)$ forms an antidirected Hamilton cycle $H_{G'}$ in G'.

The rest of the proof follows by induction on k. Suppose that $G = C_{n_1}\Box C_{n_2}\Box C_{n_3}\cdots\Box C_{n_k}$ has an antidirected Hamilton cycle for $n_1 = 2$, n_2 odd, n_i even for $i \geq 3$, and $k \geq 3$. Let $G' = G\Box C_{n_{k+1}}$ and n_{k+1} be even. So, there are n_{k+1} pairwise disjoint antidirected cycles $H^1, H^2, \dots, H^{n_{k+1}}$ in G' corresponding to H_G as in Fig. 2. By the property of Cartesian product of directed cycles, each H^i must contain at least one arc corresponding to C_{n_1}. Let $(v_j^i, v_{j+1(mod\ 2)}^i)$ be an arc induced by C_{n_1} in H^i. Let P_j^i be an antidirected path in G' defined by removal of arc $(v_j^i, v_{j+1(mod\ 2)}^i)$ from H^i.

By assumption, n_{k+1} even assures that the following antidirected cycle exists in G':

$$C_{G'} = v_1^1 v_0^1 v_0^2 v_1^2$$

$$v_1^3 v_0^3 v_0^4 v_1^4$$

$$\dots\dots\dots\dots\dots\dots$$

$$v_1^{n_k-3} v_0^{n_k-3} v_0^{n_k-2} v_1^{n_k-2}$$

$$v_1^{n_k-1} v_0^{n_k-1} v_0^{n_k} v_1^{n_k} v_1^1.$$

We can now substitute each $(v_j^i, v_{j+1(mod\ 2)}^i)$ in $C_{G'}$ with P_j^i that produces the antidirected Hamilton cycle $H_{G'}$ in Fig. 2, which completes the proof. □

Finally, the natural question arises whether or not Theorems 3.1 and 3.3 can be generalized to necessary conditions as well. It's straightforward to prove that $C_{n_1} \square C_{n_2}$ does not contain an antidirected Hamilton path if $n_1 > 2$ and n_2 is odd. Hence, by discarding a cycle of length two in a Cartesian product of cycles, and taking into account Theorem 3.1 we conjecture the following.

Conjecture 3.4 *Let G be a Cartesian product of k cycles $C_{n_1} \square C_{n_2} \cdots \square C_{n_k}$, where each $n_i > 2$ and $k \geq 3$. Then G contains an antidirected Hamilton cycle if and only if $gcd(n_1, n_2, \ldots, n_k) = 2$ and there exist two distinct i, j ($k \geq i \geq 1$, $k \geq j \geq 1$) for which $gcd(n_1, n_2, \ldots, n_{i-1}, n_{i+1}, n_{i+2}, \ldots, n_k) = n_j$.*

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Pseudo-outerplanar graphs and chromatic conjectures*

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Abstract

In this paper, we verify the list edge coloring conjecture for pseudo-outerplanar graphs with maximum degree at least 5 and the equitable Δ-coloring conjecture for all pseudo-outerplanar graphs.

Keywords: *pseudo-outerplanar graph, list edge coloring, equitable coloring*

1 Introduction

All graphs considered in this paper are finite, simple and undirected. By $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$, we denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. By $d_G(v)$, or $d(v)$ for brevity, we denote the degree of a vertex v in G. For other undefined notations, we refer the readers to [1].

For each edge $uv \in E(G)$, assign it a set $L(uv)$ of colors, called a *list* of uv. An edge coloring φ is an *edge L-coloring*, if $\varphi(xy) \in L(uv)$ for each edge $xy \in E(G)$. If $|L(xy)| = k$ for every $xy \in E(G)$, then an edge L-coloring is a *list edge k-coloring* and we say that G is *edge k-choosable*. The minimum integer k for which G has a list edge k-coloring, denoted by $\chi'_l(G)$, is the *list chromatic index* of G. It is trivial that $\chi'(G) \le \chi'_l(G)$. As far as the list edge coloring is concerned,

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Vizing, Gupta, Abertson and Collins, and Bollobás and Harris (see [14] for details) independently posed the following conjecture, which is well-known as List Edge Coloring Conjecture (LECC).

Conjecture 1.1. $\chi'_l(G) = \chi'(G)$ *for every graph G.*

As far as we know, Conjecture 1.1 has been proved for a few special cases, such as bipartite graphs [6], planar graphs with maximum degree at least 12 [3], series-parallel graphs [16] and outerplanar graphs [22]. On the other hand, Vizing's theorem implies that if G is a graph with maximum degree Δ, then $\Delta \le \chi'(G) \le \Delta + 1$. Thus, Vizing [21] posed a weaker conjecture than LECC, which is named Weak List Edge Coloring Conjecture (WLECC).

Conjecture 1.2. $\chi'_l(G) \le \Delta(G) + 1$ *for every graph G.*

Up to now, Conjecture 1.2 was confirmed for graphs with $\Delta(G) \le 4$ [12, 15] and some special graphs such as graphs with girth at least $8\Delta(G)(\log\Delta(G) + 1.1)$ [18], planar graphs with maximum degree at least 9 [2], and planar graphs with maximum degree $\Delta(G) \ne 5$ and without adjacent 3-cycles or with maximum degree $\Delta(G) \ne 5, 6$ and without 7-cycles [13]. However, the above two conjectures on list edge coloring remain very open.

A proper vertex coloring is *equitable* if the sizes of any two color classes differ by at most one, thus an equitable vertex coloring (or equitable coloring for short) is indeed a partition of vertices among the different colors so that they are as evenly as possible. The *equitable chromatic number* of a graph G, denoted by $\chi_{eq}(G)$, is the smallest number k such that G has an equitable coloring with k colors. Note that an equitably k-colorable graph may admit no equitably k'-colorings for some $k' > k$ (the balanced complete k-partite graph with n vertices is such an example), therefore, another chromatic parameter for equitable coloring of graphs is defined naturally. We call the smallest k such that G has equitable k'-colorings for every integer $k' \ge k$ the *equitable chromatic threshold* of G, denoted by $\chi^*_{eq}(G)$. In 1970, Hajnal and Szemerédi [7] answered a question of Erdős by proving every graph G with $\Delta(G) \le r$ has an equitable $(r+1)$-coloring, which implies $\chi^*_{eq}(G) \le \Delta(G) + 1$ for every graph G. Three years later, Meyer [19] considered an equitable version of Brooks' Theorem and made the following Equitable Coloring Conjecture (ECC).

Conjecture 1.3. *For any connected graph G, except the complete graph and the odd cycle,* $\chi_{eq}(G) \le \Delta(G)$.

In 1994, Chen, Lih and Wu [4] posed the following Equitable Δ-coloring Conjecture (EΔCC), which is stronger than Conjecture 1.3, since $\chi^*_{eq}(G) \ge \chi_{eq}(G)$,

Conjecture 1.4. *If G is a connected graph with maximum degree Δ other than $K_{\Delta+1}$, $K_{\Delta,\Delta}$ and odd cycle, then $\chi^*_{eq}(G) \leq \Delta(G)$.*

Although Conjectures 1.3 and 1.4 were confirmed for many classes of graphs such as graphs with $\Delta \leq 3$ [4, 5] or $\Delta = 4$ [17], bipartite graphs [9], planar graphs with maximum degree at least 9 [11], series-parallel graphs [24] and outerplanar graphs [23], they are still much open. One can refer to the survey contributed by Lih [8] for more progresses concerning the research on equitable coloring of graphs.

A graph is *pseudo-outerplanar* if each block has an embedding on the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. This notion was firstly introduced by Zhang, Liu and Wu [26], where is proved that the class of outerplanar graphs is the intersection of the classes of pseudo-outerplanar graphs and series-parallel graphs and the edge coloring and the linear arboricity of pseudo-outerplanar were considered. Recently, Zhang [25] also proved that every pseudo-outerplanar graphs with maximum degree $\Delta \geq 5$ is totally $(\Delta + 1)$-choosable.

In this paper, we aim to confirm Conjecture 1.1 for pseudo-outerplanar graphs with maximum degree at least 5 by Theorem 2.5, Conjectures 1.2 for all pseudo-outerplanar graphs by Theorem 2.6, and Conjectures 1.3 and 1.4 for all pseudo-outerplanar graphs by Theorem 2.10.

2 Main results and their proofs

Lemma 2.1. [26] *Every pseudo-outerplanar graph with minimum degree at least 2 contains one of the first seventeen configurations in Figure* 1.

Lemma 2.2. *Every pseudo-outerplanar graph contains one of the configurations in Figure* 1*, where the degree of a solid vertex is exactly shown, and the degree of a hollow vertex is at least the number of edges incident to the hollow vertex in the figure, and moreover, hollow vertices may be not distinct while solid vertices are distinct.*

Proof. Let G be a pseudo-outerplanar graph. If $\delta(G) \geq 2$, then G contains one of the configurations among G_1–G_{17} by Lemma 2.1. We now assume that $\delta(G) = 1$ and G contains none of the above configurations. Due to the absence of the configuration G_{18}, G has only one vertex of degree 1, say v. Let u be the neighbor of v in G. If $\delta(G - v) = 1$, then $d(u) = 2$, which implies the appearance of the configuration G_{19} in G. Therefore, $\delta(G - v) \geq 2$ and then $G - v$ contains one of the configurations among G_1–G_{17} by Theorem 4.2 of [26], say G_i. If v is not adjacent

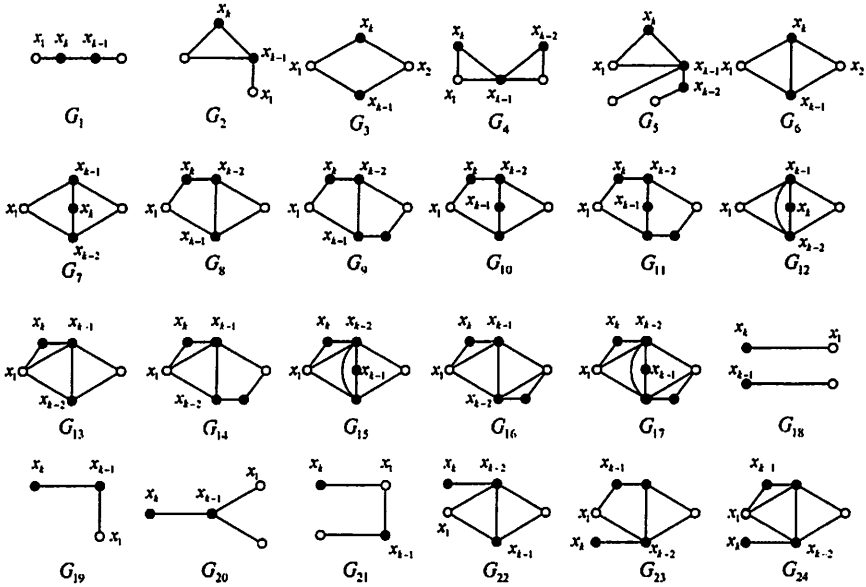


Figure 1: The unavoidable structures of pseudo-outerplanar graphs

to any solid vertex in G_i, then the configuration G_i occurs in G. If v is adjacent to some solid vertex in G_i, then one of the configurations among G_{19}–G_{23} appears in G. □

By Lemma 2.2, we immediately have the following corollary.

Corollary 2.3. *Every pseudo-outerplanar graph contains either a vertex of degree at most two or a configuration G_6 as shown in Figure* 1.

Lemma 2.4. [16] *Let G be the graph from Figure 2 and let L be an edge-list assignment for G such that $|L(e)| \geq 2$ if e is incident with x or z, and $|L(e)| \geq 4$ otherwise. Then G admits an edge L-coloring.*

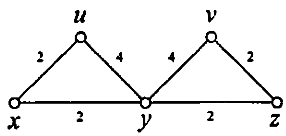
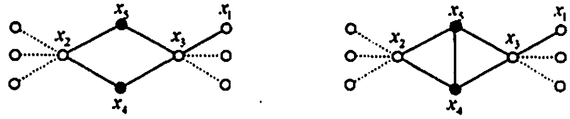


Figure 2: A special graph with the numbers of remaining colors



$G_3: 3 \leq d(x_3) \leq 5$ $G_6: 3 \leq d(x_3) \leq 5$

Figure 3: Label the vertices in S when G contains G_3 or G_6 and $k = \Delta = 5$

Theorem 2.5. *Let G be a pseudo-outerplanar graph with maximum degree Δ. If $\Delta \leq k$ and $k \geq 5$, then $\chi'_l(G) \leq k$.*

Proof. Suppose, to the contrary, that G is a minimum counterexample to this theorem. It is easy to see that $\delta(G) \geq 2$. If G contains G_3, then we delete the four edges in this configuration from G and denote the resulted graph by G'. Since G is the minimum counterexample to this theorem, G' has a list edge k-coloring ϕ. Since 4-cycles are edge 2-choosable, we can color the four deleted edges from their lists so that the extended coloring is still a list edge k-coloring. Hence G does not contain G_3. If G contains an edge uv with $d(u) + d(v) \leq k + 1$, then $G - uv$ has a list edge k-coloring ϕ since G is the minimum counterexample to this theorem. Since the edge uv is incident with at most $k - 1$ colored edges under ϕ, one can easily extend ϕ to a list edge k-coloring of G by coloring uv with a color from its list. Therefore, $d(u) + d(v) \geq k + 2$ for every edge $uv \in E(G)$. The above two facts along with Lemma 2.2 imply that $k = 5$ and G contains the configuration G_{17}: a 7-path $x'uxvywy'$ such that $d(u) = d(v) = d(w) = 2$, $d(x) = d(y) = 5$ and $xx', yy', xy, x'y, xy' \in E(G)$. Here we shall also assume that $d(x') = d(y') = 5$ since x' and y' are adjacent to 2-vertices in G. Let L be the list assigned to the edges of G with $|L(e)| = k$ for every edge $e \in E(G)$ and let $G' = G - \{u, v, w, x, y\}$. By the minimality of G, G' has a list edge k-coloring ϕ under the list L. Let $A_\phi(e)$ be the set of the available colors in $L(e)$ to color an edge $e \in \{ux', ux, vx, vy, wy, wy', xx', xy, yy', x'y, xy'\}$ so that the color received by e is different with the colors incident with e under ϕ. It is easy to see that $|A_\phi(xx')|, |A_\phi(yy')|, |A_\phi(x'y)|, |A_\phi(xy')| \geq 3$. Without loss of generality, assume that $|A_\phi(xx')| = |A_\phi(yy')| = |A_\phi(x'y)| = |A_\phi(xy')| = 3$. We now claim that one can color $x'y$ and xy' from their available lists so that the extended coloring θ satisfies $|A_\theta(xx')|, |A_\theta(yy')| \geq 2$ by discussing the following two cases. First, if $A_\phi(x'y) \cap A_\phi(xy') \neq \emptyset$, then we construct the above coloring θ by coloring $x'y$ and xy' with a same coloring from $A_\phi(x'y) \cap A_\phi(xy')$. Second, if $A_\phi(x'y) \cap A_\phi(xy') = \emptyset$, then there are two colors $\alpha \in A_\phi(x'y)$ and $\beta \in A_\phi(xy')$ so that $\{\alpha, \beta\} \not\subseteq A_\phi(xx')$ and $\{\alpha, \beta\} \not\subseteq A_\phi(yy')$, thus we can construct the above coloring θ by coloring $x'y$ and

xy' with α and β. One can also check that the extended partial coloring θ satisfies $|A_\theta(ux')|, |A_\theta(wy')| \geq 2$, $|A_\theta(xy)| \geq 3$ and $|A_\theta(ux)|, |A_\theta(vx)|, |A_\theta(vy)|, |A_\theta(wy)| \geq 4$. Without loss of generality, we assume that all of the above equalities hold (otherwise we meet easier cases and can deal with them much more easily). We claim that θ can be extended by coloring wy, wy' and yy' properly to another partial coloring λ of G which satisfies $|A_\lambda(ux')|, |A_\lambda(xx')|, |A_\lambda(xy)|, |A_\lambda(vy)| \geq 2$ and $|A_\lambda(ux)|, |A_\lambda(vx)| \geq 4$. First, if $A_\theta(yy') \not\subseteq A_\theta(xy)$, then we color yy' with $\lambda(yy') \in A_\theta(yy') \setminus A_\theta(xy) \neq \emptyset$, wy' with $\lambda(wy') \in A_\theta(wy') \setminus \{\lambda(yy')\}$ and wy with $\lambda(wy) \in A_\theta(wy) \setminus \{\lambda(wy'), \lambda(yy')\}$. Second, if $A_\theta(yy') \subseteq A_\theta(xy)$, then we color wy with $\lambda(wy) \in A_\theta(wy) \setminus A_\theta(xy) \neq \emptyset$, wy' with $\lambda(wy') \in A_\theta(wy') \setminus \{\lambda(wy)\}$ and yy' with $\lambda(yy') \in A_\theta(yy') \setminus \{\lambda(wy')\}$ (note that $\lambda(wy) \notin A_\theta(yy')$). In either case, one can confirm that the partial coloring λ of G satisfies the above required conditions. Therefore, λ can be extended to a final list edge k-coloring φ of G by Lemma 2.4. □

Theorem 2.6. *Every pseudo-outerplanar graph with maximum degree* Δ *is edge* $(\Delta + 1)$*-choosable.*

Proof. This is an immediate corollary from Theorem 2.5 and the fact that every graph with maximum degree $\Delta = 3$ [12] or $\Delta = 4$ [15] is edge $(\Delta + 1)$-choosable. □

Corollary 2.7. *LECC holds for pseudo-outerplanar graph with maximum degree at least 5.*

Corollary 2.8. *WLECC holds for all pseudo-outerplanar graph.*

Lemma 2.9. [27] *Let* $S = \{v_1, v_2, \ldots, v_k\}$ *where* $\{v_1, v_2, \ldots, v_k\}$ *are distinct vertices in graph* G. *If* $G - S$ *has an equitable* k*-coloring, and* $|N_G(v_i) - S| \leq k - i$ *with* $1 \leq i \leq k$, *then* G *has an equitable* k*-coloring.*

Theorem 2.10. *Every connected pseudo-outerplanar graph with maximum degree* Δ *has an equitable coloring with* k *colors for every* $k \geq \max\{\Delta, 5\}$.

Proof. We prove the theorem by induction on the order of G. If $\Delta \leq 4$, then the result holds by the Hajnal-zemerédi Theorem on equitable coloring, so we assume that $k \geq \Delta \geq 5$ in the following arguments. Since G is pseudo-outerplanar, G contains one of the 24 configurations by Lemma 2.2.

If G contains a configuration among G_4, G_5, G_7–G_{17} and G_{22}–G_{24}, then label the vertices v_1, v_{k-2}, v_{k-1} and v_k as they are in the Figure 2 and fill the remaining unspecified positions in S as described in Lemma 2.9 from highest to lowest indices by choosing at each step a vertex of degree at most 3 in the graph obtained from G by deleting the vertices thus far chosen for S. This can be done by using Corollary 2.3. Since $|N_G(v_i)-S| \leq k-i$ for all $1 \leq i \leq k$ and $G-S$ has an equitable k-coloring by the induction hypothesis, G has an equitable k-coloring by Lemma 2.9.

If G contains a configuration among G_1, G_2 and G_{18}–G_{21}, then we first label the vertices v_1, v_{k-1} and v_k as they are in the Figure 1. Let H_i be the graph derived from G by deleting the labeled vertices in the configuration G_i if G contains G_i. By Corollary 2.3, we consider two cases for $i \in \{1, 2, 18, 19, 20, 21\}$. If H_i contains a 2-vertex, then label this vertex by v_{k-2}. If H_i contains a pair of adjacent 3-vertices, then label these two vertices by v_2 and v_{k-2}. In either case, we fill the remaining unspecified positions in S as described in Lemma 2.9 from highest to lowest indices by choosing at each step a vertex of degree at most 3 in the graph obtained from G by deleting the vertices thus far chosen for S. Since $|N_G(v_i)-S| \leq k-i$ for all $1 \leq i \leq k$ and $G-S$ has an equitable k-coloring by the induction hypothesis, G has an equitable k-coloring by Lemma 2.9.

If G contains a configuration G_3 or G_6 and $k \geq 6$, then we first label the vertices v_1, v_2, v_{k-1} and v_k as they are in the Figure 1. If H_3 or H_6 contains a 2-vertex, then label this vertex by v_{k-2}. If H_3 or H_6 contains a pair of adjacent 3-vertices, then label these two vertices by v_3 and v_{k-2}. In either case, we fill the remaining unspecified positions in S as described in Lemma 2.9 from highest to lowest indices by choosing at each step a vertex of degree at most 3 in the graph obtained from G by deleting the vertices thus far chosen for S. Since $|N_G(v_i)-S| \leq k-i$ for all $1 \leq i \leq k$ and $G-S$ has an equitable k-coloring by the induction hypothesis, G has an equitable k-coloring by Lemma 2.9.

If G contains a configuration G_3 or G_6 and $k = \Delta = 5$, then one neighbor of the 2-vertices in this configuration is of degree at least 3, otherwise G is isomorphic to C_4, contradicting the fact that $\Delta = 5$. Under this condition, we label the vertices v_1, v_2, v_3, v_4 and v_5 as they are in the Figure 3. It is easy to see that $|N_G(v_i)-S| \leq k-i$ for all $1 \leq i \leq 5$. Since $G-S$ has an equitable k-coloring by the induction hypothesis, G has an equitable k-coloring by Lemma 2.9. □

Since ECC and EΔCC holds for graphs with maximum degree 3 [4, 5] and 4 [17], we immediately have the following corollaries.

Corollary 2.11. *ECC and EΔCC holds for all pseudo-outerplanar graphs.*

Corollary 2.12. *Every connected pseudo-outerplanar graph except K_4 with maximum degree Δ has an equitable coloring with k colors for every $k \geq \max\{\Delta, 3\}$.*

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