The optimal pebbling number of square of paths and cycles *

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Abstract

A pebbling move is taking two pebbles off one vertex and then placing one on an adjacent vertex. The optimal pebbling number of G, denoted by $f_{opt}(G)$, is the least positive integer n such that n pebbles are placed suitably on vertices of G and for any specified vertex v of G, we can move one pebble to v by a sequence of pebbling moves. In this paper, we determine the optimal pebbling number of the square of paths and cycles.

AMS Classification: 05C38
Keywords: Paths; Square of cycles; Optimal pebbling number

1 Introduction

Pebbling in graphs was first introduced by Chung [1]. Consider a connected graph with a fixed number of pebbles distributed on its vertices. A pebbling move consists of the removal of two pebbles from a vertex and the placement of one pebble on an adjacent vertex.

A distribution of pebbles to the vertices of a graph is said to be solvable when a pebble may be moved to any specified vertex using a sequence of admissible pebbling moves. (This includes sequences of length 0, i.e., when the initial distribution places at least one pebble on the vertex in question).

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*Supported by National Natural Science Foundation of China (No. 10971248) and Anhui Provincial Natural Science Foundation(No. KJ2013Z279, KJ2011B152).

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The pebbling number of a graph G, denoted by $f(G)$, is the least number such that any distribution of $f(G)$ pebbles to the vertices of G is solvable. And the optimal pebbling number of a graph G, denoted by $f_{opt}(G)$, is the least number such that there exists a solvable distribution of $f_{opt}(G)$ pebbles on G. Clearly, $f_{opt}(G) \leq f(G)$ for any graph G. There are some known results about $f_{opt}(G)$ for some specific graphs(see[2-5]).

Let G be a connected graph. For $u, v \in V(G)$, we denote by $d_G(u, v)$ the distance between u and v in G. The kth power of G, denoted by G^k, is the graph obtained from G by adding an edge uv to G whenever $2 \leqslant d_G(u, v) \leqslant k$ in G.

Throughout this paper, unless stated otherwise, G will denote a simple connected graph on n vertices. Given a distribution D of pebbles on G, we use $|D|$ to indicate the size of D, i.e. the number of pebbles in D. Denote by $\langle v_1, v_2, \ldots, v_n \rangle$(respectively, $[v_1, v_2, \ldots, v_n]$) the path (respectively, cycle) with vertices $v_1, v_2, \ldots, v_n$ in order. For a path $\langle v_i, v_{i+1}, \cdots, v_{i+j} \rangle$, we write $D(\langle v_i, v_{i+1}, \cdots, v_{i+j} \rangle) = [p_i, p_{i+1}, \cdots, p_{i+j}]$ to indicate that D places p_k pebbles on vertex v_k for each $k = i, i+1, \cdots, i+j$. Similarly, $D(v) = p$ means that D places p pebbles on the vertex v. We then say that a vertex v is occupied if $D(v) \geq 1$ and unoccupied when $D(v) = 0$. Moreover, denote by $\widetilde{D}(v)$ the number of pebbles on v after some sequence of pebbling moves. Denote Z^+ the set of positive integers.

The optimal pebbling number was first considered by Pachter, Snevily, and Voxman. They obtained the optimal number of paths and the pebbling number of P_n^2 (see [3]). In [4], T. Friedman and C. Wyels gave the optimal pebbling number of cycles. Y. Ye et al. gave that the pebbling number of C_n^2 [6, 7]. Motivated by this, we obtain the optimal pebbling numbers of squares of paths and cycles in this paper.

2 Optimal pebbling number of P_n^2

This section studies the optimal pebbling number of P_n^2. Let $P_n = \langle v_1, v_2, \ldots, v_n \rangle$. First, we give some useful lemmas.

Lemma 2.1 ([4])$f_{opt}(P_{3t+r}) = f_{opt}(C_{3t+r}) = 2t + r$, *where* $t \in Z^+$ *and* $r = 0, 1, 2$.

Lemma 2.2 *Let s and q be two positive integers with $s \leq q$. If P_n^2 admits a solvable distribution D of size q such that $D(v_n) = s$, then P_n^2 also admits* *a solvable distribution D^* of size q such that $D^*(v_n) = 0$ for $s = 1$ and $D^*(v_n) = s - 2$ for $s \geq 2$.*

Proof. For $s = 1$, there exist a sequence of pebbling moves from $\langle v_1, v_2, \cdots, v_{n-2}\rangle$ to v_{n-1} with $\widetilde{D}(v_{n-1}) \geq 1$. Now we can define $D^*(v_{n-1}) = D(v_{n-1}) + 1$, $D^*(v_n) = 0$ and $D^*(v_j) = D(v_j)$ for each of the left vertices v_j. Clearly, D^* is also a solvable distribution.

Next let $s \geq 2$. We can define $D^*(v_{n-1}) = D(v_{n-1}) + 2$, $D^*(v_n) = s - 2$ and $D^*(v_j) = D(v_j)$ for each of the left vertices v_j. Clearly, D^* is also a solvable distribution. □

Lemma 2.3 *Let s and q be two positive integers with $s \leq q$. If P_n^2 admits a solvable distribution D of size q such that $D(\langle v_{n-1}, v_n\rangle) = [s, 0]$, then P_n^2 also admits a solvable distribution D^* of size q such that $D^*(\langle v_{n-1}, v_n\rangle) = [0, 0]$ for $s = 1$ and $D^*(\langle v_{n-1}, v_n\rangle) = [s - 2, 0]$ for $s \geq 2$.*

Proof. For $s = 1$, it is clear that there exist a sequence of pebbling moves such that $\widetilde{D}(v_{n-2}) \geq 2$ or $\widetilde{D}(v_{n-3}) \geq 2$. Suppose that $D^*(v_{n-2}) = D(v_{n-2}) + 1$, $D^*(v_{n-1}) = 0$ and $D^*(v_j) = D(v_j)$ for each of the left vertices v_j. Then D^* is a solvable distribution.

Next let $s \geq 2$. Similarly, suppose that $D^*(v_{n-2}) = D(v_{n-2}) + 2$, $D^*(v_{n-1}) = s - 2$ and $D^*(v_j) = D(v_j)$ for each of the left vertices v_j. Hence D^* is also a solvable distribution. □

By Lemma 2.2 and Lemma 2.3, we easily give the following result.

Corollary 2.4 *For P_n^2, if D is a solvable distribution of size q, then there is some solvable distribution D^* of size q such that $D^*(v_n) = D^*(v_{n-1}) = 0$.*

A solvable distribution of P_n^2 is said to be good, if $\widetilde{D}(v_{n-1}) \geq 2$ or $\widetilde{D}(v_n) \geq 2$ after appropriate pebbling moves.

Lemma 2.5 *For $n \geq 3$, let P_n^2 have a good distribution of size q. Then P_n^2 admits a good distribution D of size q such that $D(v_n) \neq 0$ or $D(v_{n-1}) \neq 0$.*

Proof. Let k be the maximum positive integer, obtain $D(v_k) \neq 0$, such that P_n^2 admits a good distribution of size q. Using proof by contradiction, we can assume that $k \leq n - 2$, i.e., $D(v_{k+1}) = D(v_{k+2}) = \ldots = D(v_n) = 0$.

First, we consider the case that $D(v_k)$ is odd or $k = 1, 2$. We can define two new distributions D'_j $(j = 1, 2)$ such that $D'_j(v_k) = D(v_k) - 1$, $D'_j(v_{k+j}) = D(v_{k+j}) + 1$ and $D'_j(v_i) = D(v_i)$ for else i. Note that any pebble can go through one of v_{k+1} and v_{k+2} to reach v_{n-1} and v_n. So at least one of D'_1 and D'_2 is a good distribution. Hence it is a contradiction in the maximal of k.

Second, it remains the case that $D(v_k)$ is even and $k \geq 3$. Next, we consider two cases according to the parity of $D(v_{k-1})$. If $D(v_{k-1})$ is odd, then we can define two new distributions D''_j $(j = 1, 2)$ such that $D''_j(v_k) = D(v_k) - 2$, $D''_j(v_{k-1}) = D(v_{k-1}) + 1$, $D''_j(v_{k+j}) = D(v_{k+j}) + 1$ and $D''_j(v_i) = D(v_i)$ for else i. Clearly, either D''_1 or D''_2 is a good distribution. Otherwise, $D(v_{k-1})$ is even. We can define two new distributions D'''_j $(j = 1, 2)$ such that $D'''_j(v_k) = D(v_k) - 2$, $D'''_j(v_{k-2}) = D(v_{k-2}) + 1$, $D'''_j(v_{k+j}) = D(v_{k+j}) + 1$ and $D'''_j(v_i) = D(v_i)$ for else i. Then one of D'''_1 and D'''_2 is a good distribution. A contradiction. □

The result of $f_{opt}(P_n^2)$ depends on the value of n *mod* 5. Thus we write P_n as $P_n = P_{5t+r} = \langle v_1, v_2, \ldots, v_{5t+r} \rangle$, where $r \in \{-2, -1, 0, 1, 2\}$ and t is a positive integer.

Theorem 2.6 *Let* $n = 5t + r$. *Then*

$$f_{opt}(P_n^2) = f_{opt}(P_{5t+r}^2) = \begin{cases} 2t & if \quad r = -2, -1, 0; \\ 2t + 1 & if \quad r = 1, 2. \end{cases}$$

Proof. Mark the right of the equation with A_r. We give a distribution D of P_{5t+r}^2, which places 2 pebbles at $v_3, v_{5+3}, \ldots, v_{5(t-1)+3}$ when $r = -2, -1, 0$, places an additional pebble at v_{5t} when $r = 1$ *or* 2. Clearly, D is solvable. Then $f_{opt}(P_n^2) \leq A_r$. Next, we use mathematical induction on n to prove $f_{opt}(P_n^2) = A_r$.

First, let $n = 3$. Note that $P_3^2 \cong C_3$. By Lemma 2.1, $f_{opt}(P_3^2) = 2$.

Second, assume that the theorem is true for each $n > 3$. We need show that the theorem is true for $n + 1$. If $n = 5t - 2$, then by induction hypothesis, $f_{opt}(P_{5t-2}^2) = 2t$. Since P_{5t-2}^2 is a subgraph of P_{5t-1}^2, we have

$$f_{opt}(P_{n+1}^2) = f_{opt}(P_{5t-1}^2) \geq f_{opt}(P_{5t-2}^2) = 2t = A_{-1}.$$

Similarly, if $n = 5t - 1$, then

$$f_{opt}(P_{n+1}^2) = f_{opt}(P_{5t}^2) \geq f_{opt}(P_{5t-1}^2) = 2t = A_0;$$

if $n = 5t + 1$, then

$$f_{opt}(P_{n+1}^2) = f_{opt}(P_{5t+2}^2) \geq f_{opt}(P_{5t+1}^2) = 2t + 1 = A_2.$$

Next, it remains the cases $n = 5t$ and $n = 5t + 2$.

For $n = 5t$, we have to show $f_{opt}(P_{n+1}^2) = f_{opt}(P_{5t+1}^2) \geq 2t + 1 = A_1$. By induction hypothesis, $f_{opt}(P_{5t+1}^2) \geq f_{opt}(P_{5t}^2) = 2t = A_0$. Now, assume that D is a solvable distribution of P_{5t+1}^2 of size $2t$. If $D(v_{5t+1}) = 1$ or $D(\langle v_{5t}, v_{5t+1}\rangle) = [0, 2]$, then after any pebbling moves,

$$\sum_{i=1}^{5t-1} \widetilde{D}(v_i) \leq 2t - 1 \quad \text{or} \quad \sum_{i=1}^{5t} \widetilde{D}(v_i) \leq 2t - 1.$$

Note that by induction hypothesis, $f_{opt}(P_{5t-1}^2) = f_{opt}(P_{5t}^2) = 2t$. It is a contradiction in the solvability of D. If $D(v_{5t+1}) = 2$ and $D(v_{5t})$ is odd, then we can define a distribution D' such that $D'(v_{5t+1}) = 1$, $D'(v_{5t}) = D(v_{5t})+1$ and $D'(v_i) = D(v_i)$ for else i. If $D(v_{5t+1}) = 2$ and $D(v_{5t})$ is even and nonzero, then we can define a distribution D' such that $D'(v_{5t+1}) = 1$, $D'(v_{5t-1}) = D(v_{5t-1}) + 1$ and $D'(v_i) = D(v_i)$ for else i. Clearly, in both cases, D' is also solvable for P_{5t+1}^2 with $D'(v_{5t+1}) = 1$. A contradiction to the above. If $D(v_{5t+1}) \geq 3$, then by Lemma 2.2, P_{5t+1}^2 admits a solvable distribution D'' with $D''(v_{5t+1}) = 1$ or 2. Similarly, a contradiction. Hence $D(v_{5t+1}) = 0$, that is, $\sum_{i=1}^{5t} D(v_i) = 2t$. We give the following

Claim D is not a good distribution for P_{5t}^2.

The proof of the claim will be given in the last section.

Since D is not a good distribution for P_{5t}^2, the vertex v_{5t+1} can not get a pebble after any pebbling moves, it is a contradiction in the solvability of D. So D is not solvable for P_{5t+1}^2. Then $f_{opt}(P_{5t+1}^2) = 2t + 1$. Now, we give the proof of the claim.

By the induction hypothesis that $f_{opt}(P_{5t}^2) = 2t$, let D be a solvable distribution of size $2t$ in P_{5t}^2. Similar to the preceding proof, we have $D(v_{5t}) = 0$. Next we prove that $D(v_{5t-1}) = 0$. If $D(v_{5t-1}) = 1$, then $\sum_{i=1}^{5t-2} D(v_i) = 2t - 1$. By the induction hypothesis that $f_{opt}(P_{5t-1}^2) = 2t$, it is a contradiction. If $D(\langle v_{5t-3}, v_{5t-2}, v_{5t-1}\rangle) = [0, 0, 2]$, then after any pebbling moves, $\sum_{i=1}^{5t-4} \widetilde{D}(v_i) = 2t - 2$. Since $f_{opt}(P_{5t-4}^2) = 2t - 1$ by induction hypothesis, it is a contradiction in the solvability of D. Suppose that at least one of the values of $D(v_{5t-3})$ and $D(v_{5t-2})$ is nonzero, say $D(v_{5t-3}) \neq 0$. If $D(v_{5t-3})$ is odd, then we can define a distribution D^* such that $D^*(v_{5t-1}) = 1$, $D^*(v_{5t-3}) = D(v_{5t-3}) + 1$ and $D^*(v_i) = D(v_i)$ for else i. Otherwise, $D(v_{5t-3})$ is even. We can define a distribution D^* such that $D^*(v_{5t-1}) = 1$, $D^*(v_{5t-4}) = D(v_{5t-4}) + 1$ and $D^*(v_i) = D(v_i)$ for else i. Therefore, in both cases, D^* is also solvable for P_{5t}^2 with $D^*(v_{5t-1}) = 1$. A contradiction to the above. If $D(v_{5t-1}) \geq 3$, then, by Lemma 2.3, we can conclude that P_{5t}^2 admits a solvable distribution D^{**} with $D^{**}(v_{5t-1}) = 1$ or 2. A contradiction. Then $D(v_{5t-1}) = 0$. By Lemma 2.5, the solvable distribution D of P_{5t}^2 is not a good distribution.

For $n = 5t + 2$, we can similarly prove that $f_{opt}(P_{n+1}^2) = 2t + 2$. □

3 Optimal pebbling number of C_n^2

We write C_n as $C_{5t+r} = [v_1 v_2 \cdots v_{5t+r}]$ where $r \in \{-2, -1, 0, 1, 2\}$ and t is positive integer. For the convenience of our proofs, if we remove a vertex v_i of C_n^2, then it is understood that the edge incident to v_i is also removed, after adding an edge between v_{i-1} and v_{i+2}, v_{i-2} and v_{i+1}, we obtain the graph C_{n-1}^2. If we remove two vertices v_i and v_{i+1} then adding an edge between v_{i-1} and v_{i+2}, v_{i-1} and v_{i+3}, v_{i-2} and v_{i+2}, respectively. We create the graph C_{n-2}^2. Since the graph C_n^2 may obtain by adding an edge between v_1 and v_n, v_1 and v_{n-1}, v_2 and v_n in the graph P_n^2, we have a corollary as follows:

Corollary 3.1 $f_{opt}(P_n^2) \geq f_{opt}(C_n^2)$.

Theorem 3.2 *Let $n = 5t + r$, where $t \in Z^+$ and $r = -2, -1, 0, 1, 2$.*

$$f_{opt}(C_n^2) = f_{opt}(C_{5t+r}^2) = \begin{cases} 2t & if \quad r = -2, -1, 0; \\ 2t + 1 & if \quad r = 1, 2. \end{cases}$$

Proof. Mark the right of the equation with A_r. By Corollary 3.1,

$$f_{opt}(C_{5t+r}^2) \leq A_r.$$

Now we show $f_{opt}(C_{5t+r}^2) \geq A_r$. Thus $f_{opt}(C_{5t+r}^2) = A_r$.

First, we note that $C_3^2 \cong K_3$, $C_4^2 \cong K_4$ and $C_5^2 \cong K_5$. Obviously,

$$f_{opt}(C_3^2) = f_{opt}(C_4^2) = f_{opt}(C_5^2) = 2.$$

Second, assume that the theorem is true for $n > 5$. We need show that the theorem is true for $n + 1$. If $n = 5t - 2, 5t - 1, 5t + 1$, then Theorem 3.2 holds for $n + 1$, which the proof is similar to the proof of Theorem 2.6.

Next, it remains the cases $n = 5t$ and $n = 5t + 2$. We only need prove that if $f_{opt}(C_{5t}^2) = 2t$ then $f_{opt}(C_{5t+1}^2) \geq 2t + 1$. The proof of the case $n = 5t + 2$ is similar.

Using proof by contradiction, suppose that $f_{opt}(C_{5t+1}^2) < 2t + 1$, and $5t + 1$ is the smallest index such that the theorem does not hold. That is $f_{opt}(C_{5t+1}^2) \leq 2t$. Since C_{5t}^2 is a subgraph of C_{5t+1}^2, we have $f_{opt}(C_{5t+1}^2) \geq f_{opt}(C_{5t}^2) = 2t$. Thus $f_{opt}(C_{5t+1}^2) = 2t$. Suppose that D is a solvable distribution of size $2t$ in C_{5t+1}^2. We will modify D to create D^*, a solvable distribution on a smaller square of cycles with fewer than the number of pebbles, thus producing our desired contradiction.

Case 1: D places exactly 2 pebbles on each occupied vertex of C_{5t+1}^2.

We consider the corresponding sequence of the number of pebbles on the vertices of C_{5t+1}^2. Since D was assumed to be solvable, we may assume that there are at most four consecutive unoccupied vertices in D. Also, there is a sequence of vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$ with $D([v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}])$ $= [2,0,0,0,2]$ or $[2,0,0,2,0]$ or $[2,0,2,0,0]$ or $[2,2,0,0,0]$. Otherwise, there would be exactly four unoccupied vertices between every pair of occupied vertices. Thus we have $r = 0$. This is a contradiction.

Suppose that $D([v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}]) = [2,0,0,0,2]$ or $[2,0,0,2,0]$ or $[2,0,2,0,0]$, we can modify D and C_{5t+1}^2 by removing vertices v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4} and their associated pebbles. Since v_{i+5} and v_{i+6} are either occupied or can be pebbled by v_i, and other vertices are unaffected, D^* is a solvable distribution. Note that D^* is a distribution on C_{5t-3}^2 with $|D^*| = 2t-2$. We have reached a contradiction since $f_{opt}(C_{5t-3}^2) = 2t-1$. In addition, if $D([v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}]) = [2,2,0,0,0]$, then we can modify D and C_{5t+1}^2 by removing vertices v_{i+1}, v_{i+2} and two pebbles on v_{i+1}, and put one pebble to v_{i-1}. Observe that D^* is a solvable distribution. Then D^* is a distribution on C_{5t-1}^2 with $|D^*| = 2t - 1$. By the hypothesis $f_{opt}(C_{5t-1}^2) = 2t$, it is a contradiction.

Next, we consider that C_{5t+1}^2 contains the vertex v_1(after relabeling if necessary) with $D(v_1) = 1$ or $D(v_1) \geq 3$. Without loss of generality, we may assume that v_j $(j > 1)$ need to obtain pebbles from $\{v_{5t}, v_{5t+1}, v_1\}$ by a sequence of pebbling moves. Thus the pebbling moves must be involved with v_2 and v_3. We assume that the number of pebbles which can be moved from v_{5t} to others is at most m_1, while the number of pebbles which can be moved simultaneously from v_{5t+1} is at most m_2. Let $D(v_1) = q$, $D(v_2) = d$.

Case 2 $q = 1$.

If $m_2 = 0$, then we modify D and C_{5t+1}^2 by removing v_1, v_{5t+1} and the pebble on them. Thus creating a distribution D^* of C_{5t-1}^2 with at most $2t-1$ pebbles. We claim that all the remaining vertices may still be pebbled, in other words, that D^* is solvable. Using distribution D, $\lfloor \frac{m_1+1}{2} \rfloor$ pebbles could be moved from $\{v_{5t}, v_1\}$ to v_2, $\lfloor \frac{m_1+1}{2} \rfloor + \lfloor \frac{d}{2} \rfloor$ pebbles can be moved to v_3. Using D^*, m_1 pebbles may be moved from v_{5t} to v_2. $m_1 + \lfloor \frac{d}{2} \rfloor$ pebbles pebbles can be moved to v_3. Since $m_1 \geq \lfloor \frac{m_1+1}{2} \rfloor$ for all $m_1 \geq 0$, all vertices initially that are reachable from D are still reachable from D^*. Then D^* is solvable. Then contradicting the fact that $f_{opt}(C_{5t-1}^2) = 2t$.

If $m_2 \neq 0$ and $d = 0$, then we modify D and C_{5t+1}^2 by removing v_1, v_2 and the pebble on them. Thus creating a distribution D^* of C_{5t-1}^2 with $2t-1$ pebbles. We claim that all the remaining vertices may still be pebbled, in other words, that D^* is solvable. Using distribution D, $\lfloor \frac{m_1+m_2+1}{2} \rfloor$ pebbles could be moved from $\{v_{5t}, v_{5t+1}, v_1\}$ to v_3. Using D^*, $m_1 + m_2$ pebbles may be moved from $\{v_{5t}, v_{5t+1}\}$ to v_3. Since $m_1 + m_2 \geq \lfloor \frac{m_1+1+m_2}{2} \rfloor$ for all $m_1 \geq 0$, all vertices initially that are reachable from D are still reachable from D^*. Then D^* is solvable. Then contradicting the fact that $f_{opt}(C_{5t-1}^2) = 2t$.

If $m_2 \neq 0$ and $d \neq 0$, then we modify D and C_{5t+1}^2 by removing v_1 and the pebble on it. Thus creating a distribution D^* of C_{5t}^2 with $2t-1$ pebbles. We claim that D^* is solvable. Using distribution D, $\lfloor \frac{m_1+1}{2} \rfloor + m_2$ pebbles could be moved from $\{v_{5t}, v_{5t+1}, v_1\}$ to v_2, at most $\lfloor \frac{m_1+m_2+1+d}{2} \rfloor$ pebbles could be moved to v_3. Using D^*, $m_1 + m_2$ pebbles may be moved to v_2, v_3 can get $\lfloor \frac{m_1+d}{2} \rfloor + m_2$ pebbles. Since $m_1 \geq \lfloor \frac{m_1+1}{2} \rfloor$ and $\lfloor \frac{m_1+d}{2} \rfloor + m_2 \geq \lfloor \frac{m_1+m_2+1+d}{2} \rfloor$ for all $m_1 \geq 0$, all vertices initially that are reachable from D are still reachable from D^*. Then D^* is solvable. Then contradicting the fact that $f_{opt}(C_{5t}^2) = 2t$.

Case 3 $q \geq 3$.

We may assume that each occupied vertex of C_{5t+1}^2 has at least two pebbles. Otherwise, it can return to Case 2. When at least one of v_{5t+1} and v_2 is unoccupied, say v_2, we modify D and C_{5t+1}^2 by removing v_1, v_2 and pebbles on v_1, place 2 pebbles on v_{5t} and $q-3$ pebbles on v_3. Thus creating a distribution D^* of $2t-1$ pebbles on C_{5t-1}^2. We claim that D^* is solvable. For D, we can move $\lfloor \frac{q+m_1+m_2}{2} \rfloor$ pebbles to v_3. For D^*, we can move $m_1 + m_2 + q - 2$ pebbles to v_3. Since $m_1 + m_2 + q - 2 \geq \lfloor \frac{m_1+m_2+q}{2} \rfloor$, all vertices initially that are reachable from D are still reachable from D^*. Then D^* is solvable. Then contradicting the fact that $f_{opt}(C_{5t-1}^2) = 2t$. When $D(v_2) \neq 0$ and $D(v_{5t+1}) \neq 0$, i.e., $d \geq 2$ and $D(v_{5t+1}) \geq 2$, we modify D and C_{5t+1}^2 by removing v_1 and pebbles on it, place 2 pebbles on

v_{5t+1} and $q-3$ pebbles on v_2. Thus creating a distribution D^* of $2t-1$ pebbles on C_{5t}^2. Using distribution D, $\lfloor\frac{m_1+q}{2}\rfloor+m_2$ pebbles could be moved to v_2 and at most $\lfloor\frac{m_1+m_2+q+d}{2}\rfloor$ pebbles could be moved to v_3. Using D^*, m_1+m_2+q-2 pebbles could be moved to v_2 and $\lfloor\frac{m_1+q+d-3}{2}\rfloor+m_2+1$ pebbles could be moved to v_3. Since $m_1+q-2 \geq \lfloor\frac{m_1+q}{2}\rfloor$ and $\lfloor\frac{m_1+q+d-3}{2}\rfloor+m_2+1 \geq \lfloor\frac{m_1+m_2+q+d}{2}\rfloor$, all vertices initially that are reachable from D are still reachable from D^*. Then D^* is solvable. Thus contradicting the fact that $f_{opt}(C_{5t}^2)=2t$.

Combining with three cases, we conclude that our initial assumption is false, and the theorem holds for all positive integers. □

Acknowledgment Thanks to the anonymous referees for many useful comments and suggestions.

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