Some Commutativity Conditions in Prime MA-Semirings

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Abstract

In this paper, we investigate some commutativity conditions and extend a remarkable result of Ram Awtar, when Lie ideal U becomes the part of the centre of MA-semiring R.
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1 Introduction and Preliminaries

Throughout the paper, R will denote an MA-semiring with center $Z(R)$. Recall that R is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$ and semiprime if $aRa = 0$ implies that $a = 0$. MA-semiring R is said to be 2 torsion free if $2x = 0$ implies that $x = 0$ $\forall x \in R$. An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$, holds for all $x, y \in R$ and said to be an inner derivation if $d([t,y]) = [d(t), y] + [t, d(y)]$ indeed, $d([t,y]) = d(ty + yt') = d(t)y + td(y) + d(y)t' + y'd(t) = [d(t), y] + [t, d(y)]$. First, we introduce the Lie ideal as follows: Let U be an additive subsemigroup of R, then (i) U is a right Lie ideal of R if $[U, R] \subset U$, (ii) U is a left Lie ideal of R if $[R, U] \subset U$, (iii) if U is both a right Lie ideal and left Lie ideal of R, then U is a Lie ideal of R. In this paper, we investigate some commutativity conditions with the help of Lie ideals and derivations in MA-semirings. A celebrated result of Ram Awtar [1] is extended in MA-semiring as follows: Let R be a 2-torsion free prime MA-semiring, d be a non-zero derivation of R and U be a Lie ideal of R such that $[x, d(x)] = 0$ $\forall x \in U$, then $U \subseteq Z(R)$. A remarkable result of Herstein [8] is also extended in the setting of MA-semirings (see(Theorem 2.9)). Let R be a 2-torsion free MA-semiring, d be a non zero derivation of R. If $a \in R$ and $[d(R), a] = 0$, then $a \in Z(R)$.

By semiring we mean a non empty set R with two binary operations '+' and '.' such that $(R,+)$ and $(R,.)$ are semigroups, where + is ' **commutative** with absorbing (0) such that $a+0=0+a=a$ and $a0=0a=0$ for all $a\in R$ (see [7]) and $a.(b+c)=a.b+a.c$, $(b+c).a=b.a+c.a$ hold for all $a,b,c\in R$. In [2], present authors referred the class of additively commutative inverse semiring satisfying the condition (A_2) i.e. $a+a'\in Z(R)$ for all $a\in R$, where a' is the additive pseudo inverse (see [6]), as MA-**semirings** and introduce the notion of commutators and further they used commutators to develop the notion of dependent elements (see[4]). For any $x,y\in R$, we write $(x\circ y)=xy+yx$ and $[x,y]=xy+yx'$ and proved the Jacobian Theorem as: $[x,[y,z]]+[y,[z,x]]=[[x,y],z]$ $\forall x,y,z\in R$ and following identities also hold $(i)[x,yz]=[x,y]z+y[x,z]$, $(ii)[xy,z]=x[y,z]+[x,z]y$, $(iii)([x,y])'=[y,x]=[x,y']=[x',y]$ (see [2]). Examples of non commutative MA-semirings can also be found in [2] (see also [3, 4]). The aim of the present paper is to investigate some commutativity conditions in prime MA-semirings.

2 Some Commutativity Conditions on Prime MA-Semirings

In this section, we investigate some commutativity conditions for prime MA-semirings with the help of derivations and commutators. We recall the followings:

Lemma 2.1. *[4, Lemma 2.3] Let R be a semiprime MA-semiring and an element $a\in R$ such that*

(i) $[x,a]=0$, $\forall\ x\in R$, then $a\in Z(R)$.
(ii) $[x,a]a=0$ or $a[x,a]=0$ $\forall\ x\in R$, then $a\in Z(R)$.
(iii) $x=0$ if and only if $x'=0$.
(iv) $axb=0$ if and only if $ax'b=0$ or $a'xb=0$ or $axb'=0$.

Theorem 2.2. *[2, Theorem 3.5] Let R be an MA-semiring, then $[x,[y,z]]+[y,[z,x]]=[[x,y],z]$ holds for all $x,y,z\in R$.*

Theorem 2.3. *[2, Theorem 3.2] If R is an MA-semiring, then for all $x,y,z\in R$, the following identities are valid.*

$(i)[x,yz]=[x,y]z+y[x,z]$ *,(Jacobian Identity)*
$(ii)[xy,z]=x[y,z]+[x,z]y$, *(Jacobian Identity)*
$(iii)[x+y,z]=[x,z]+[y,z]$
$(iv)[x,0]=[0,x]=0$
$(v)([x,y])'=[y,x]=[x,y']=[x',y]$
$(vi)[[x,y],z]=[x,y]z+z[y,x]$
$(vii)[nx,y]=n[x,y]$, *for any positive integer n.*

Lemma 2.4. *Let R be a 2-torsion free semiprime MA-semiring. If $a \in R$ such that $[a,[a,z]] = 0 \ \forall \ z \in R$, then $a \in Z(R)$.*

Proof. By hypothesis

$$[a,[a,z]] = 0 \ \forall z \in R. \tag{1}$$

Replacing z by zy in (1), we get
$0 = [a, z[a,y] + [a,z]y] = z[a,[a,y]] + [a,z][a,y] + [a,z][a,y] + [a,[a,z]]y$.
Using (1) in the last equation, we get $2[a,z][a,y] = 0$. Since R is 2-torsion free so

$$[a,z][a,y] = 0. \tag{2}$$

Replacing y by yt in (2), we get $0 = [a,z][a,yt] = [a,z]y[a,t] + [a,z][a,y]t$. In view of (2), the last equation yields $[a,z]y[a,t] = 0$. Primeness of R implies that either $[a,z] = 0$ or $[a,t] = 0$ for all $z,t \in R$. Hence by Lemma 2.1(i) $a \in Z(R)$. □

The following Jordan identity is very useful in the development of the sequel:

Lemma 2.5. *Let R be an MA-semiring. Let $x,y,z \in R$, then the following identity holds:*

$$(xy \circ z) = x[y,z] + (x \circ z)y.$$

Proof. By definition of commutators in MA-semirings the right hand side of the last expression becomes:
$x[y,z] + (x \circ z)y = x(yz + zy') + (xz + zx)y = xyz + xzy' + xzy + zxy$
$= xyz + xz(y' + y) + zxy = xyz + x(y' + y)z + zxy = x(y + y' + y)z + zxy =$
$xyz + zxy = (xy \circ z)$. □

Lemma 2.6. *Let d be a derivation of a prime MA-semiring R and a be an element of R. If $ad(x) = 0 \ \forall \ x \in R$, then either $a = 0$ or d is zero.*

Proof. Let $ad(x) = 0 \ \forall \ x \in R$, replace x by xy. Then
$0 = ad(x) = a(d(x)y + xd(y)) = ad(x)y + axd(y) = axd(y) \ \forall \ x,y \in R$. As R is a prime MA-semiring, therefore either $d(y) = 0$ or $a = 0$, if $d(y) \neq 0$ for some $y \in R$, then $a = 0$. □

Theorem 2.7. *Let R be a 2-torsion free prime MA-semiring, d be a non zero derivation of R. If $a \in R$ and $[d(R),a] = 0$, then $a \in Z(R)$.*

Proof. Assume that $a \notin Z(R)$. By hypothesis, we have

$$[d(x),a] = 0 \ \forall \ x \in R. \tag{3}$$

Replacing x by xa in (3), we get
$0 = [d(x)a + xd(a), a] = [d(x)a, a] + [xd(a), a] = (d(x)a + a'd(x))a + [xd(a), a]$

$= [d(x),a]a + x[d(a),a] + [x,a]d(a)$. By using (3) in the last equation, we get

$$[x,a]d(a) = 0 \;\; \forall\, x \in R. \tag{4}$$

Replacing x by xy, $y \in R$ in (4), we get $0 = [xy,a]d(a) = x[y,a]d(a) + [x,a]yd(a) = 0$ and using (4), we get $[x,a]yd(a) = 0 \;\forall\; x,y \in R$. Primeness of R implies that $[x,a] = 0$ or $d(a) = 0$. As $a \notin Z(R)$, therefore $[x,a] \neq 0$ for some $x \in R$ by Lemma 2.1(i), so we obtain $d(a) = 0$. Now consider the following mapping on $R : D(x) = [x,a]$, where D is a non-zero inner derivation admitted by R (see [4]). By hypothesis $[d(x),a] = 0 \;\forall\; x \in R$, that is

$$Dd(x) = 0. \tag{5}$$

Since D is an inner derivation, so $D(xy) = D(x)y + xD(y) \;\forall\; x,y \in R$.
Now take, $dD(x) = d([x,a]) = d(xa + a'x) = d(xa) + d(a'x) = d(x)a + xd(a) + d(a)x' + a'd(x) = d(x)a + a'd(x) + xd(a) + d(a)x' = [d(x),a] + [x,d(a)]$. That is $dD(x) = [d(x),a] + [x,d(a)]$. From the last equation using (3) and the fact that $d(a) = 0$, we get

$$dD(x) = 0 \;\; \forall x \in R. \tag{6}$$

Replacing x by $xD(y)$ in (5) and using (6) and (5) again, we get $0 = Dd(xD(y)) = D(d(x)D(y) + xdD(y)) = D(d(x)D(y)) = Dd(x)D(y) + d(x)DD(y) = d(x)DD(y)$. Hence we get $d(x)DD(y) = 0 \;\forall\; x,y \in R$. As $d(x) \neq 0$ for some $x \in R$, therefore by Lemma 2.6, we have

$$DD(y) = 0 \;\; \forall y \in R. \tag{7}$$

Replacing y by xy in (7) and using it again, we get
$0 = DD(xy) = D(D(x)y + xD(y)) = DD(x)y + D(x)D(y) + D(x)D(y) + xDD(y) = 2D(x)D(y)$. Since R is a 2-torsion free, we have $D(x)D(y) = 0$. That is, $[x,a]D(y) = 0$, $\forall\; x,y \in R$. Suppose for some $x_\circ \in R$, $[x_\circ,a] \neq 0$, $[x_\circ,a]D(y) = 0$. Using Lemma 2.6, we obtain $D(y) = 0 \;\forall\; y \in R$, that is $[y,a] = 0, y \in R$ then by Lemma 2.1 $a \in Z(R)$. □

Corollary 2.8. *Let R be a 2-torsion free prime MA-semiring, d be a nonzero derivation of R. If $[d(R),a] = 0 \;\forall\; a \in R$, then R is commutative.*

Proof. Let $a \in R$, then by hypothesis $[d(R),a] = 0$. Since R be a 2-torsion free prime MA-semiring, d be a nonzero derivation of R, therefore by Theorem 2.7 $a \in Z(R)$ and hence $R \subseteq Z(R)$. Thus R is commutative. □

In the following theorem, we extend the celebrated result of Herstein [8] in the setting of MA-semirings.

Theorem 2.9. *Let R be a 2-torsion free prime MA-semiring, d be a nonzero derivation of R and $a \in R$. If $d([R,a]) = 0$, then $a \in Z(R)$.*

Proof. By hypothesis

$$d([r,a]) = 0 \ \forall\, r \in R. \tag{8}$$

Replacing r by ar in (8), we get
$0 = d([ar,a]) = d(ara + aar') = d(a(ra + ar')) = d(a[r,a]) = d(a)[r,a] + ad([r,a])$. Using (8) in the last equation, we get,

$$d(a)[r,a] = 0, \ \forall\, r \in R. \tag{9}$$

Replacing r by rs, $s \in R$ in (9) and using Theorem 2.3, we have
$0 = d(a)[rs,a] = d(a)(r[s,a] + [r,a]s) = d(a)r[s,a] + d(a)[r,a]s \ \forall\, r,s \in R$. Using (9), we get $d(a)r[s,a] = 0 \ \forall\, r,s \in R$. Since R is a prime MA-semiring, so $d(a) = 0$ or $[s,a] = 0 \ \forall\, s \in R$.
Case 1: If $d(a) \neq 0$, then $[s,a] = 0 \ \forall\, s \in R$ and this by Lemma 2.1(i) implies $a \in Z(R)$.
Case 2: Suppose $[s,a] \neq 0$ for some $s \in R$, then $d(a) = 0$. Now, from our hypothesis, $\forall\, r \in R$, we have: $0 = d([r,a]) = d(ra + ar') = d(ra) + d(ar') = d(r)a + rd(a) + d(a)r' + a'd(r)$.
As $d(a) = 0$, therefore, $d(r)a + a'd(r) = 0$. That is $[d(r),a] = 0, \ \forall\, r \in R$, then by Theorem 2.7, $a \in Z(R)$. □

Corollary 2.10. *Let R be a 2-torsion free prime MA-semiring, d be a nonzero derivation of R and $a \in R$. If $d([R,a]) = 0 \forall a \in R$, then R is commutative.*

Proof. Let $a \in R$, then by hypothesis $d([R,a]) = 0$. Since R be a 2-torsion free prime MA-semiring, d be a nonzero derivation of R, therefore by Theorem 2.9 $a \in Z(R)$ and hence $R \subseteq Z(R)$. Thus R is commutative. □

Theorem 2.11. *Let R be a 2-torsion free prime MA-semiring, I a non-zero ideal of R and $a \in R$. If $([x,b] \circ a) = 0, \ \forall x \in R, b \in I$, then $a \in Z(R)$.*

Proof. As I is a non-zero ideal, take $b \in I$ such that $b \neq 0$
Case 1: If $[x,b] = 0$, then in particular $[a,b] = 0$ for some $a \in R$, so

$$[a,b] = 0 \ \forall\, b \in I. \tag{10}$$

As I is ideal, so replacing b by xb in (10), we get $[a,xb] = 0 \ \forall b \in I$ *i.e.* $x[a,b] + [a,x]b = 0$. Using (10) in the last equation, we get

$$[a,x]b = 0 \ \ \forall b \in I, x \in R. \tag{11}$$

Replacing x by xy, in (11), we have
$0 = [a,xy]b = x[a,y]b + [a,x]yb \ \forall\, x,y \in R$. Using (11) in the last expression we get $[a,x]yb = 0 \ \forall\, x,y \in R$. As $b \neq 0$, therefore, by primeness of R,

$[a, x] = 0 \; \forall \; x \in R$ and by Lemma 2.1(i), we have $a \in Z(R)$.
Case 2: If $[x, b] \neq 0$, then consider the following mappings on R:

$$(i) \quad f(x) = [x, b], \qquad\qquad (ii) \quad g(x) = (x \circ a). \tag{12}$$

Then for any $x \in R$, we have $gf(x) = g([x, b]) = ([x, b] \circ a) = 0$. That is

$$gf(x) = 0, \forall x \in R. \tag{13}$$

Replacing x by xb in (12)(i), we get $f(xb) = [xb, b] = xbb + b'xb = \left(xb + b'x\right) b$ $= [x, b]b = f(x)b$, and so we get,

$$f(xb) = f(x)b, \quad \forall x \in R. \tag{14}$$

Replacing x by xb in (13), we get $0 = gf(xb) = g(f(x)b) = f(x)b \circ a = f(x)[b, a] + (f(x) \circ a)b$ (by Lemma 2.5). That is $f(x)[b, a] + ([x, b] \circ a)b = 0$. Using hypothesis in the last expression, we have

$$f(x)[b, a] = 0 \quad \forall x \in R. \tag{15}$$

Replacing x by xs, $s \in R$ in (15), we obtain $0 = f(xs)[b, a] = [xs, b]\,[b, a] = (x\,[s, b] + [x, b]\,s)\,[b, a] = x\,[s, b]\,[b, a] + [x, b]\,s[b, a] = xf(s)[b, a] + [x, b]s[b, a]$ and using (15), we get $[x, b]s[b, a] = 0$, $\forall \; x, s \in R$. By primeness of R either $[x, b] = 0$ or $[b, a]) = 0$. If $[x, b] = 0$ then in particular $[a, b]) = 0$ and hence $[b, a]) = 0$ for some $a \in R$. Then by case 1, $a \in Z(R)$. □

Corollary 2.12. *Let R be a 2-torsion free prime MA-semiring, I a non-zero ideal of R and $a \in R$. If $([R, I] \circ a) = 0 \; \forall \; a \in R$, then R is commutative.*

Proof. Let $a \in R$, then by hypothesis $([R, I] \circ a) = 0$. Since R be a 2-torsion free prime MA-semiring, I be a nonzero ideal of R, therefore by Theorem 2.11 $a \in Z(R)$ and hence $R \subseteq Z(R)$. Thus R is commutative. □

3 Lie ideals and Commutativity Conditions on Prime MA-Semirings

In this section we investigate commutativity conditions with the help of Lie ideals and derivations in MA-semirings. A remarkable result of Ram awtar [1] is also extended in the setting of MA-semirings. First we introduce the notion of Lie ideals in MA-semirings as a canonical extension of Lie ideals of rings.

Definition 3.1. Let U be an additive subsemigroup of R, then U is a Lie ideal of R, if $[U, R] \subset U$, and also $[R, U] \subset U$.

Proposition 3.2. *If R is a 2-torsion free prime MA-semiring and U is a Lie ideal of R such that $\forall x \in U$, $[[x,d(x)],r]=0$ $\forall r \in R$, and $x^2 \in U$, then $[x,d(x)]=0$ $\forall x \in U$.*

Proof. By supposition

$$[[x,d(x)],r]=0 \quad \forall x \in U, r \in R. \tag{16}$$

Replacing x by $x+x^2$ in the last relation and using it again, we get

$$\begin{aligned}
& [[x+x^2,d(x+x^2)],r]=0 \\
\Rightarrow \quad & [[x,d(x)],r]+[[x,d(x^2)],r]+[[x^2,d(x)],r]+[[x^2,d(x^2)],r]=0.
\end{aligned}$$

Using (16) in the last equation, we get

$$\begin{aligned}
& [[x,d(x^2)],r]+[[x^2,d(x)],r]=0 \\
\Rightarrow \quad & [[x,xd(x)],r]+[[x,d(x)x],r]+[x[x,d(x)]+[x,d(x)]x,r]=0 \\
\Rightarrow \quad & \left[x\left(xd(x)+d(x)x'\right),r\right]+\left[\left(xd(x)+d(x)x'\right)x,r\right]+[x[x,d(x)],r] \\
& +[[x,d(x)]x,r]=0 \\
\Rightarrow \quad & [x[x,d(x)],r]+[[x,d(x)]x,r]+[x[x,d(x)],r]+[[x,d(x)]x,r]=0.
\end{aligned}$$

In view of Lemma 2.1(i), (16) gives: $[x,d(x)] \in Z(R)$. So from the last equation, we get $4[[x,d(x)]x,r]=0$. Since R is 2-torsion free, we get: $0=[[x,d(x)]x,r]=[[x,d(x)],r]x+[x,d(x)][x,r]$ $\forall$ $x \in U$, $r \in R$. Using (16) from the last equation, we get

$$[x,d(x)][x,r]=0 \quad \forall x \in U,\ r \in R. \tag{17}$$

Replacing r by sr in the last equation then $\forall x \in U$, $r,s \in R$, we get $0=[x,d(x)][x,sr]=[x,d(x)]\,(s[x,r]+[x,s]r)=[x,d(x)]s[x,r]$ $+[x,d(x)]\,[x,s]\,r$. By(17) the last equation reduces to $[x,d(x)]s[x,r]=0$. By primness of R either $[x,d(x)]=0$ or $[x,r]=0$ $\forall x \in U$, $r,s \in R$.
Case 1: If $[x,r]=0$ $\forall$ $x \in U$, $r \in R$, in particular $[x,d(x)]=0$.
Case 2: If $[x,r_\circ] \neq 0$ for some $r_\circ \in R$, then $[x,d(x)]=0$ $\forall$ $x \in U$. □

Proposition 3.3. *Let R be a prime MA-semiring and U be a Lie ideal of R. Suppose that $[[x,d(x)],r]=0$ $\forall x \in U$, $r \in R$, then $[[d(r),x],x] \in Z(R)$ $\forall$ $x \in U, r \in R$.*

Proof. Let $x \in U$ and $r \in R$, and as U is a Lie ideal, therefore $[x,r] \in U$, so that $x+[x,r] \in U$. By assumption

$$[[x,d(x)],s]=0 \quad \forall x \in U,\ s \in R. \tag{18}$$

Replacing x by $x+[x,r]$ in (18) then $\forall\, x \in U,\ r,s \in R$, we get $[[x,d(x)],s]+[[x,d[x,r]],s]+[[[x,r],d[x,r]],s]+[[[x,r],d(x)],s]=0$. In view of (18) and using the fact that d is a derivation from the last equation, we get

$$[[x,[d(x),r]]+[x,[x,d(r)]]+[[x,r],d(x)],s]=0. \tag{19}$$

By Theorem 2.2 $[d(x),[r,x]]+[x,[d(x),r]]=[[x,d(x)],r]$. Using Theorem 2.3(v), from the last equation, we get $[[x,r],d(x)]+[x,[d(x),r]]=[r,[d(x),x]]$. Using it in (19), we get $[[r,[d(x),x]]+[x,[x,d(r)]],s]=0$. In view of (18), the last equation becomes: $0=[[x,[x,d(r)]],s]$ $=\left[[[x,d(r)],x]',s\right]=\left[\left[[d(r),x]',x\right]',s\right]=[[[d(r),x],x],s]$ (see (Theorem 2.3(v))). Then by Lemma 2.1(i) $[[d(r),x],x]\in Z(R)$. □

The following Lemma 3.4 can be established by using the steps of proof of Proposition 3.3.

Lemma 3.4. *Let R be a prime MA-semiring and U be a Lie ideal of R. Suppose that $[x,d(x)]=0\ \forall\ x\in U$, then $[[d(r),x],x]=0\ \forall x\in U, r\in R$.*

Theorem 3.5. *Let R be a 2-torsion free prime MA-semiring. Let d be a non-zero derivation of R and U be a Lie ideal of R such that $[x,d(x)]=0$ $\forall x\in U$. Then $U\subseteq Z(R)$.*

Proof. By Lemma 3.4

$$[[d(r),x],x]=0\ \ \forall x\in U, r\in R. \tag{20}$$

By linearization of (20) and using (20), we get

$$[[d(r),x],y]+[[d(r),y],x]=0. \tag{21}$$

Let $y\in U, z=[t,y]$, then $[yt,y]=yty+yyt'=y(ty+yt')=y[t,y]=yz$. This implies that $yz\in U$.
Now, replacing y by yz in (21) and expanding, we get
$y([[d(r),x],z]+[[d(r),z],x])+([[d(r),x],y]+[[d(r),y],x])z$ $+[d(r),y][z,x]+[y,x][d(r),z]=0$. In view of (21) the last equation reduces to

$$[d(r),y][z,x]+[y,x][d(r),z]=0\ \ \forall x,z,y\in U, r\in R. \tag{22}$$

Replacing x by y in (22), $\forall\ x,y,z\in U, r\in R$, we get

$$\left(d(r)y,y'd(r)\right)\left(zy+y'z\right)+\left(yy+yy'\right)\left(d(r)z+z'd(r)\right)=0.$$

or $0=d(r)yzy+y'd(r)zy+yd(r)yz+d(r)\left(yy+yy'\right)z+d(r)\left(yyz'+yyz\right)+d(r)yy'z=d(r)yzy+y'd(r)zy+yd(r)yz+d(r)\left(yy+yy'+yy'\right)z=d(r)yzy+$

$y'd(r)zy + yd(r)yz + d(r)y'yz = (d(r)y + y'd(r))zy + (y'd(r) + d(r)y)'yz$ $= [d(r), y][z, y]$. Replacing z by $[t, y]$ in the last expression, we get

$$[d(r), y][[t, y], y] = 0 \ \forall r, t \in R, y \in U. \quad (23)$$

Replacing t by $td(a)$ in (23), $\forall\ r, t \in R, y \in U$, yields on expansion

$$[d(r), y]\{t[[d(a), y], y] + 2[t, y][d(a), y] + [[t, y], y]d(a)\} = 0.$$

In view of (20) and (23) and using the fact that R is 2-torsion free the last equation reduces to

$$[d(r), y][t, y][d(a), y] = 0 \ \forall r, t, a \in R, y \in U. \quad (24)$$

Now, replacing x by z in (22), $\forall\ x, z, y \in U, r \in R$ and expanding, we get $0 = (d(r)y + y'd(r))(zz + zz') + (yz + zy')(d(r)z + z'd(r)) = d(r)yzz + d(r)yzz' + y'd(r)zz + y'd(r)zz' + yzd(r)z + yzz'd(r) + zy'd(r)z + zy'z'd(r) = yzz(d(r) + d'(r) + d'(r)) + yz(d'(r) + d(r) + d(r))z + zy'd(r)z + zy'z'd(r) = yzzd'(r) + yzd(r)z + zy'd(r)z + zy'z'd(r) = (yz + zy')z'd(r) + (yz + zy')d(r)z = (yz + zy')(z'd(r) + d(r)z) = [y, z][d(r), z]$. Replacing z by $[t, y]$ in the last relation, we get $0 = [y, [t, y]][d(r), [t, y]]\ \forall\ y \in U, r, t \in R$. In view of Theorem 2.3(v), we get

$$[[t, y], y][[t, y], d(r)] = 0 \ \forall y \in U, r, t \in R. \quad (25)$$

Now, replacing t by $t + d(a)$ and on expansion, we have

$$([[t, y], y] + [[d(a), y], y])([[t, y], d(r)] + [[d(a), y], d(r)]) = 0.$$

Using (20) and (25) in the last expression, we get

$$[[t, y], y][[d(a), y], d(r)] = 0 \ \forall\ y \in U,\ r, t, a \in R \quad (26)$$

Replacing t by $d(t)p, p \in R$ in (26) and expanding, we get

$$\{d(t)[[p, y], y] + 2[d(t), y][p, y] + [[d(t), y], y]p\}[[d(a), y], d(r)] = 0.$$

In view of (20) and (26) and the fact that R is 2-torsion free, the last equation becomes: $[d(t), y][p, y][[d(a), y], d(r)] = 0\ \forall\ y \in U, r, t, a, p \in R$. Using (24) in the last equation, we get $[d(t), y][p, y]d'(r)[d(a), y] = 0$. By Lemma (2.1)(iv), the last equation yields

$$[d(t), y][p, y]d(r)[d(a), y] = 0 \ \forall\ y \in U,\ r, t, a, p \in R. \quad (27)$$

Replacing t by $td(p)$, in (24) and expanding and then using equation (27), we get $[d(r), y]t[d(p), y][d(a), y] = 0$. Primeness of R implies that

$[d(r),y]=0$ or $[d(p),y]\,[d(a),y]=0$.

Case 1: If $[d(p),y]\,[d(a),y]\neq 0$, then $[d(r),y]=0\ \forall\ y\in U, r\in R$, then by Theorem 2.7 $y\in Z(R)\ \forall\ y\in U$.

Case 2: If $[d(r),y]\neq 0$ for some $y\in U, r\in R$, then

$$[d(p),y]\,[d(a),y]=0\ \ \forall\ y\in U,\ a,p\in R. \tag{28}$$

Replacing a by bc in (28), on expansion, we get $[d(p),y]\,d(b)\,[c,y] + [d(p),y]\,[d(b),y]\,c + [d(p),y]\,b\,[d(c),y] + [d(p),y]\,[b,y]\,d(c) = 0$. In view of (28), the last equation reduces to

$$[d(p),y]\,d(b)\,[c,y] + [d(p),y]\,b\,[d(c),y] + [d(p),y]\,[b,y]\,d(c) = 0.$$

Replacing b by $[t,y]$ in (28), we get $[d(p),y]\,d([t,y])\,[c,y] + [d(p),y]\,[t,y]\,[d(c),y] + [d(p),y]\,[[t,y],y]\,d(c) = 0$. In view of (23) and (24) the last equation becomes: $0 = [d(p),y]\,d([t,y])\,[c,y] = [d(p),y]\,([d(t),y] + [t,d(y)])\,[c,y] = [d(p),y]\,[d(t),y]\,[c,y] + [d(p),y]\,[t,d(y)]\,[c,y] = 0$. In view of (28), the last equation yields

$$[d(p),y]\,[t,d(y)]\,[c,y]=0\ \ \forall p,c,t\in R,\ y\in U \tag{29}$$

Replacing c by sc, in (29), we get: $[d(p),y]\,[t,d(y)]\,s\,[c,y] + [d(p),y]\,[t,d(y)]\,[s,y]\,c = 0$. In view of (29) the last equation reduces to: $[d(p),y]\,[t,d(y)]\,s\,[c,y] = 0\ \forall p,s,c,t\in R, y\in U$. Primeness of R implies , $[d(p),y]\,[t,d(y)]=0$ or $[c,y]=0$.

Case 2(a): If $[d(p),y]\,[t,d(y)]\neq 0$ then $[c,y]=0$ implies that $y\in Z(R)$.

Case 2(b): If $[c,y]\neq 0\ \forall y\in U$, then

$$[d(p),y]\,[t,d(y)]=0\ \ \forall\ p,t\in R, y\in U. \tag{30}$$

Replacing t by st, in (30) $\forall\ p,s,c,t\in R, y\in U$, we have: $[d(p),y]\,s\,[t,d(y)] + [d(p),y]\,[s,d(y)]\,t = 0$. In view of (30) from the last equation, we get $[d(p),y]\,s\,[t,d(y)]=0$. Primeness of R implies that either $[d(p),y]=0\ \forall p\in R$ or $[t,d(y)]=0$. If $[d(p),y]=0$, then by case 1, $y\in Z(R)$, if $[d(p),y]\neq 0$ for some $p\in R$, then

$$[t,d(y)]=0\ \ \forall t\in R, y\in U. \tag{31}$$

By Lemma 2.1(i) $d(y)\in Z(R), \forall y\in U$. This implies $d(U)\subseteq Z(R)$. This completes the proof of (i).

(ii):

Now, replacing y by $[a,y]$, the last equation becomes: $[t,d([a,y])]=0$. This implies that $[t,[d(a),y]+[a,d(y)]]=0$. In view of

(31), last equation becomes: $[t, [d(a), y]] = 0 \ \forall\ a, t \in R, y \in U$. In view of Lemma 2.1(iii) and Theorem 2.3, the last equation can be rewritten as:

$$[[d(a), y], t] = 0. \tag{32}$$

In particular $[[d(ay), y], t] = 0$. Consider $[d(ay), y] = [d(a)y + ad(y), y] = (d(a)yy + y'd(a))y + a[d(y), y] + [a, y]d(y) = [d(a), y]y + a[d(y), y] + [a, y]d(y)$.

By hypothesis $[d(y), y] = 0$, so the last expression becomes: $[d(ay), y] = [d(a), y]y + [a, y]d(y)$. Using last expression in (32), we get

$$[([d(a), y]y + [a, y]d(y)), t] = 0 \ \ \forall\ a, t \in R,\ y \in U. \tag{33}$$

In particular, $0 = [[d(a), y]y + [a, y]d(y), y] = [d(a), y]yy + y[d(a), y]'y + [[a, y]d(y), y] = \left([d(a), y]y + y[d(a), y]'\right)y + [[a, y]d(y), y] = [[d(a), y], y]y + [[a, y], y]d(y) + [a, y][d(y), y]$. Using the fact $[x, d(x)] = 0$ and (32), the last expression becomes: $[[a, y], y]d(y) = 0$. As d is a non zero derivation and by (i) $d(y) \in Z(R)$ therefore, $[[a, y], y]td(y) = 0 \ \forall t \in R$. Then primeness of R implies that $[[a, y], y] = 0 \ \forall\ a \in R, y \in U$.

In view of Theorem 2.3, the last equation reduces to $[y, [y, a]] = 0 \ \forall a \in R, y \in U$. Now by Lemma 2.4, $y \in Z(R) \ \forall y \in U$. If $[y, [y, a]] \neq 0$ for some $y \in U$, then $d(y) = 0$. Using the fact $d(y) = 0$, the expression (33) becomes: $[[d(a), y]y, t] = 0$. In particular $[[d(a), y]y, b] = 0$. This implies that $[d(a), y][y, b] + [[d(a), y], b]y = 0 \ \forall a, b \in R, y \in U$. In view of (32), the last equation becomes:

$$[d(a), y][y, b] = 0 \ \ \forall a, b \in R, y \in U. \tag{34}$$

Replacing b by $cb, c \in R$ in the last equation, we get $[d(a), y][y, cb] = 0$. This gives $[d(a), y]c[y, b] + [d(a), y][y, c]b = 0$. In view of (34), the last equation yields $[d(a), y]c[y, b] = 0$. Since R is prime, so either $[d(a), y] = 0$ or $[y, b] = 0$. $\forall\ a, b, c \in R, y \in U$. By hypothesis of case 2 $[d(a), y] \neq 0$, then $[y, b] = 0$ implies $y \in Z(R)$ (c.f Lemma 2.1(i)). This implies that $y \in Z(R)$ $\forall\ y \in U$. Thus $U \subseteq Z(R)$. This completes the proof. □

Corollary 3.6. *Let R be a 2-torsion free prime MA-semiring, d be a nonzero derivation of R. If $[x, d(x)] = 0 \ \forall x \in R$, then R is commutative.*

Proof. As R be itself a Lie ideal and by hypothesis $[x, d(x)] = 0 \ \forall x \in R$, therefore, by Theorem 3.5 $R \subseteq Z(R)$. Thus R is commutative. □

If d be taken as inner derivation then the last corollary can be written as follows:

Corollary 3.7. *Let R be a 2-torsion free prime MA-semiring, $[y, a] \neq 0$ for some $y \in R$. If $[x, [x, a]] = 0 \ \forall x \in R$, then R is commutative.*

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References

[1] Awtar Ram, *Lie and Jordan Structure in Prime Rings with Derivations*, Pro. Amer. Math. Soc. (1973) 67-74.

[2] M. A. Javed, M. Aslam and M. Hussain, *On Condition (A_2) of Bandlet and petrich for inverse semirings*, International Mathematical Forum, vol 7(2012), no.59, 2903-2914.

[3] M. A. Javed and M. Aslam, *Commuting mappings of semiprime MA-semirings.* To be appeared in WASJ.

[4] M. A. Javed, M. Aslam and K. P. Shum, *On Dependent Elements in MA-semirings.* To be appeared in ARS Combinatoria.

[5] N. Aydin and M. SoytÄurk, (σ, τ)-Lie ideals in prime rings with derivations, Doga Tr. J. of Math., 19(1993), 239-244.

[6] H.J. Bandlet, M. Petrich, *Subdirect products of rings and distrbutive lattics.* Proc. Edin Math. Soc. 25 (1982), 135-171.

[7] J. S. Golan, *The theory of semirings with applications in mathematics and theoretical computer science.* John Wiley & Sons. Inc., New York, 1992.

[8] I. N. Herstein, *A note on derivations II*, Canad. Math. Bull., 22(4)(1979), 509-511.

[9] E. C. Posner, *Derivation of prime rings*, Proc. Amer. Math. Soc., 8 (1957), 1093-1100.

[10] A. B. Thaheem and M. S. Samman, *A note on derivation on semiprime rings,* Demonstratio Math., 34 (2001), 783-788.