On the Oriented Line Graph

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Abstract

Kotani and Sunada introduced the oriented line graph as a tool in the study of the Ihara zeta function of a finite graph. The spectral properties of the adjacency operator on the oriented line graph can be linked to the Ramanujan condition of the graph. Here, we present a partial characterization of oriented line graphs in terms of forbidden subgraphs. We also give a Whitney-type result, as a special case of a result by Balof and Storm, establishing that if two graphs have the same oriented line graph, they are isomorphic.

1 Introduction

This work treats some of the structural properties of oriented line graphs. Oriented line graphs were first introduced by Kotani and Sunada [7] as a tool to show the rationality of the Ihara zeta function. Storm attached an oriented line graph to a hypergraph [10] for the same reasons. Also, Scott and Storm were able to describe the coefficients of the Ihara zeta function in terms of graph structure by exploiting the connection with oriented line graphs [9].

In addition to being useful for studying the rationality of the Ihara zeta function and the structure of the associated graph, there is a direct connection between oriented line graphs and Ramanujan graphs. The connection comes through the graph "Riemann hypothesis" and can be expressed as a spectral condition on the adjacency operator of the oriented line graph. We refer the interested reader to a paper by Murty [8] which highlights this connection. Details can also be found in the paper by Kotani and Sunada [7]. More recently, Angel, Friedman, and Hoory made an investigation of non-backtracking walks in graphs [1], a closely related subject, and added further motivation for a graph theoretical Riemann hypothesis, which describes a Ramanujan type condition, proposed by Horton, Stark, and Terras [6]. The aim of this work is to increase our understanding of this important intermediary structure, which we hope will increase our understanding of both the Ihara zeta function and of Ramanujan conditions on graphs.

The goal of this work is to provide a partial characterization of oriented line graphs in terms of forbidden subgraphs. All graphs and digraphs treated here will be finite. We begin by establishing our definitions and notation. We refer the reader to the books by Harary, and Chartrand and Lesniak [5, 3] for a good overview of graphs and digraphs.

A *graph* $X = (V, E)$ is a finite nonempty set V of *vertices* and a finite multiset E of unordered pairs of vertices, called *edges*. If $\{u, v\} \in E$, we say that u is *adjacent* to v and write $u \sim v$. A graph X is *simple* if there are no edges of the form $\{v, v\}$ and if there are no repeated edges. We denote by $d(x)$, the *degree* of vertex x, which is the number of vertices to which x is adjacent. Finally, a graph is *minimum degree 2*, denoted "md2", if every vertex has degree at least 2. We will restrict our focus to md2 graphs.

A *directed graph* or *digraph* $D = (V, E)$ is a finite nonempty set V of *vertices* and a finite multiset E of ordered pairs of vertices called *arcs*. For an arc $e = (u, w)$, we define the *origin* of e to be $o(e) = u$ and the *terminus* of e to be $t(e) = w$. The *inverse arc* of e, written $\overline{e}$, is the arc formed by switching the origin and terminus of e: $\overline{e} = (w, u)$. In general, the inverse arc of an arc need not be present in the arc set of a digraph.

A digraph D is called *symmetric* if, whenever (u, w) is an arc of D, its inverse arc (w, u) is as well. There is a natural one-to-one correspondence between the set of symmetric digraphs and the set of graphs, given by identifying an edge of the graph to an arc and its inverse arc on the same vertices. We denote by $D(X)$ the symmetric digraph associated with the graph X. We give an example in Figure 1.

Crucial to our discussion of these structures are graph isomorphisms and edge-isomorphisms. A *graph isomorphism* $\varphi : X_1 \to X_2$ from a graph X_1 to a graph X_2 is a one-to-one map from $V(X_1)$ to $V(X_2)$ which preserves edge adjacencies. A seemingly weaker notion is that of edge-isomorphism: ϕ is an *edge-isomorphism* of X_1 and X_2 if ϕ is a one-to-one map from $E(X_1)$

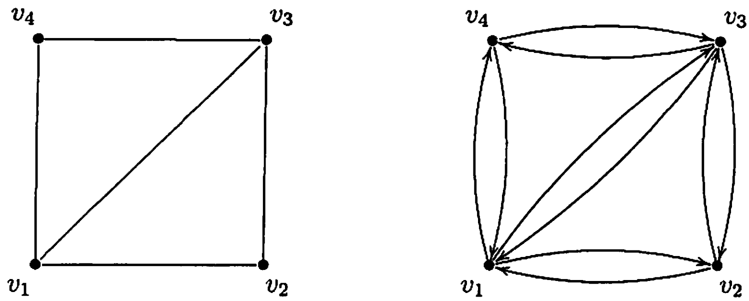

Figure 1: The complete graph K_4 minus an edge and its symmetric digraph

to $E(X_2)$ which preserves adjacencies. We leave it to the reader to fill in the appropriate definitions for isomorphisms of digraphs. We are primarily concerned with graph isomorphisms; however, edge-isomorphisms are useful as a tool in Section 2.

We now give the construction for the oriented line graph as first proposed by Kotani and Sunada [7]. The idea behind the construction is to begin with a graph and produce a new structure, a digraph, where any walk in the digraph corresponds to a non-backtracking walk in the original graph. A *backtrack* in a walk is an immediate reuse of an edge, which would return you to the vertex you just left. To this end, we will construct the oriented line graph by first changing from X to the symmetric digraph $D(X)$ and then using the arcs in $D(X)$ as the vertices of the oriented line graph. Arcs in the oriented line graph arise whenever an arc in $D(X)$ "feeds into" another arc, which is not the inverse arc of the first one (backtracking!). Thus the oriented line graph is a digraph whose cycles correspond to cycles in the original graph that do not have a backtrack. This structure is the focus of the rest of the paper.

Construction 1.1 (Kotani and Sunada). *We begin with a graph X and form its symmetric digraph $D(X)$. Hence $D(X)$ has $2|E(X)|$ arcs. We construct the* oriented line graph $L^oX = (V_L, E_L^o)$ *by*

$$V_L = E(D(X)),$$
$$E_L^o = \{(e_i, e_j) \in E(D(X)) \times E(D(X));\ \bar{e}_i \neq e_j,\ t(e_i) = o(e_j)\}.$$

Definition 1.2. *We say that a digraph D is an* oriented line graph *if it arises as a result of this construction.*

We give an example of the construction in Figure 2.

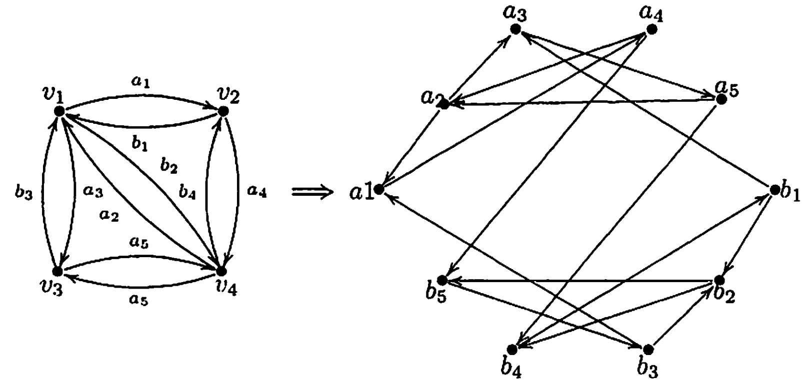

Figure 2: Construction of an oriented line graph of K_4 minus an edge.

In general, each vertex in L^oX is induced by an arc in $D(X)$. At times, we may refer to a vertex in L^oX as (u, v) if it is the vertex induced by the arc (u, v) in $D(X)$. Context will make it clear when we mean the vertex in L^oX and when we mean the arc in $D(X)$.

Remark 1.3. Often we are interested in the adjacency operator of the oriented line graph. This operator, sometimes referred to as Hashimoto's T-matrix, has received much attention in the literature. It also sometimes appears at the "edge-routing" matrix. By studying oriented line graphs, we are studying the structure of these matrices.

In Section 2 we detail a Whitney result, establishing that if two md2 graphs have the same oriented line graph, they are isomorphic. Then in Sections 3 and 4, we provide a list of forbidden subgraphs that an oriented line graph cannot contain. We do this by first giving an algorithm that takes a directed graph and, in the event that the directed graph is an oriented line graph, produces the "parent" graph. By analyzing this algorithm, we provide infinite families of forbidden subgraphs. We also detail some structural properties that oriented line graphs must satisfy. We conjecture, at the end, that the families we have provided actually are a complete characterization.

2 A Whitney result

In 1932, Whitney addressed the following question "Suppose we have two line graphs L_X and L_Y, what can we conclude about the graphs X and Y

which induced them?" He answered this question as follows: If $L_X \cong L_Y$, then $X \cong Y$ with the exception of a finite number of situations (reproduced in Figure 3) [11]. In this section, we establish a similar theorem for oriented line graphs and graphs.

The goal of this section is to provide a proof of Theorem 2.1. This theorem was first presented by Balof and Storm [2] in the more general setting of oriented line graphs of hypergraphs. We present here just the case of oriented line graphs of graphs since that is the setting in which we find ourselves.

Theorem 2.1 (Balof and Storm). *Suppose X and Y are md2 graphs. If $L^oX \cong L^oY$, then $X \cong Y$.*

We first consider the case in which every vertex has exactly degree 2; then we consider the more general case in which at least one vertex is of degree 3 or higher. In the event that every vertex is degree 2, the original graph is a disjoint union of cycles. The oriented line graph is a disjoint union of twice as many directed cycles of the same lengths. Given the construction of the oriented line graph, it is clear that if we begin with a disjoint union of directed cycles, we can deduce what graph it came from.

To show the more general case, we look more closely at the vertices in the oriented line graph. In particular, given a vertex induced by an arc e in the symmetric digraph, we are able to identify the vertex induced by the inverse arc $\bar{e}$. Pairing each vertex with the vertex induced by its inverse arc will be very helpful. This idea first appears in [4] as part of an algorithm to reconstruct the graph that induced an oriented line graph.

We establish some notation and then see how to do this. The *outdegree* of v, $d^+(v)$, is the number of vertices that are at the terminus of an arc beginning at v. Similarly, the *indegree* of v, $d^-(v)$, is the number of vertices that are at the origin of an arc terminating at v.

Theorem 2.2 (Cooper). *Given an oriented line graph D which was constructed from a connected md2 graph X with at least one vertex of degree 3 or greater, we can match each vertex induced by an arc with the vertex induced by the inverse arc.*

Proof. We begin by considering a vertex $v \in V(D)$ which satisfies $d^+(v) \geq 2$. By the construction of the oriented line graph, this suggests that v corresponds to an arc in the symmetric digraph which "feeds into" at least 2 other arcs, which we will call x and y, without backtracking. Now let us look at the inverse arc associated with x. This arc will also feed into y and any other arcs which v feeds into with the exception of the arc x. However, this arc will feed into exactly one arc which v does not feed into; namely, the inverse arc of v. Making use of this fact allows us to identify $\bar{v}$ so that we can identify it with v in the oriented line graph.

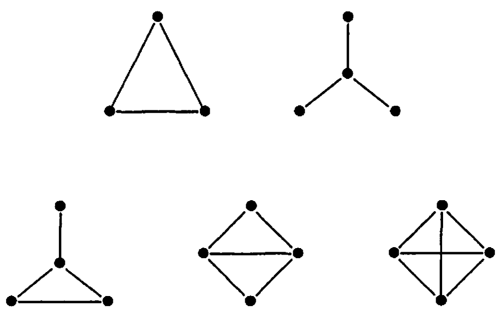

Figure 3: Exceptional graphs in Whitney's theorem.

We now consider a vertex $v \in V(D)$ which satisfies $d^+(v) = 1$. Since X is connected and md2, there exists a backtrackless path in X which begins with the directed arc associated with v and terminates at z, where z is any vertex of X satisfying $d(z) \geq 3$ (such a vertex exists by assumption placed on X). We argue by induction on the length of this path.

Our base case is if the path is length 1. If the length is 1, the directed arc associated with v in X terminates at z, a vertex of degree 3 or higher. This implies, necessarily, that this directed arc feeds into more than one other directed arc, forcing $d^+(v) > 1$, a contradiction. Then we can use the arguments from the first paragraph to identify its inverse arc $\bar{v}$.

Now we suppose that the length is n. In that case, we look at the vertex to which v points. We denote it by w. Then w is a vertex which is length $n-1$ away from z, so by induction we can identify the vertex which came from $\bar{w}$. Then $\bar{w}$ satisfies $d^+(\bar{w}) = 1$, and the sole vertex it points at is $\bar{v}$, so we can identify $\bar{v}$ as desired. □

Theorem 2.2 will let us take an oriented line graph and establish a correspondence between vertices induced by inverse arcs. An isomorphism of oriented line graphs will then induce an edge-isomorphism of the parent graphs because of this correspondence. We make use of a theorem, due to Whitney [11], regarding edge-isomorphisms as found in Chartrand and Lesniak [3].

Theorem 2.3 (Whitney). *Let ϕ be an edge-isomorphism, a one-to-one map from the edge set of one graph to another which preserves adjacencies, from a connected graph X_1 to a connected graph X_2, where X_1 is different from the graphs G_1, G_2, G_3, G_4, and G_5 shown in Figure 3. Then ϕ is induced by an isomorphism from X_1 to X_2.*

We use this theorem to prove Theorem 2.1.

Proof of Theorem 2.1. We suppose that we have two graphs X and Y which are md2 and whose oriented line graphs are isomorphic; ie, $L^oX \cong L^oY$. In the event that L^oX is a disjoint union of direct cycles, we see directly that X and Y are disjoint unions of cycles and must be isomorphic.

We suppose that X has a vertex of degree at least 3. In that case, we use Theorem 2.2 and identify each vertex induced from an arc in L^oX with the vertex induced from the inverse arc. We do the same for L^oY. Then any isomorphism $\phi : L^oX \to L^oY$ will preserve this identification since it was forced upon us. Moreover, by now identifying the pairs as having come from the same undirected edge in the original graph, we see that ϕ actually provides us with an edge-isomorphism from X to Y. We now invoke Theorem 2.3 to see that $X \cong Y$.

The exceptional graphs in Figure 3 don't pose a problem as we can resolve them on a case-by-case basis. □

This gives us the Whitney-type result that we desired. We now turn our attention to forbidden subgraphs in oriented line graphs in an effort to provide a structural characterization.

3 Forbidden subgraphs

We begin our investigation of forbidden subgraphs by giving an algorithm which takes a directed graph D and gives a graph X. In the event that D is an oriented line graph, the graph X will have the following property: D and L^oX are isomorphic as oriented line graphs.

Remark 3.1. The algorithm is technical. Here is an intuitive idea of what happens. Recall that each vertex of L^oX corresponds to an arc in a directed graph C. Let's focus for a moment on digraphs. We will mentally break each arc into two parts: the beginning which is leaving a vertex v and the end which is arriving at a vertex w. Then we identify a vertex by looking at all of the arcs which leave or arrive at that vertex. Thus our goal will be to identify which vertices in L^oX correspond to arcs which originate from the same vertex or which arrive at the same vertex. Finally, we will construct the adjacencies in C by gluing the arcs back together so that we can tie our vertices together.

Construction 3.2. *To do this, we first label the vertices of D as $v_1, \ldots, v_m$ and the arcs as $e_1, \ldots, e_k$. We construct a digraph $\mathcal{H}$ as follows. The vertex set of $\mathcal{H}$ will be a partition of the set $S = \{1, 2, \ldots, 2m\}$ where m was the number of vertices in D. We generate the particular partition algorithmically.*

For $i \in S$, let $S_i(t)$ be the subset of S containing i at time $0 \leq t \leq k$. We begin with $S_i(0) = \{i\}$ for each i. At time j, we examine the arc

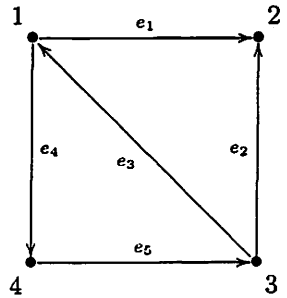

Figure 4: A digraph D upon which we run the algorithm

$e_j = (v_{j_1}, v_{j_2})$. We let $S_{j_2}(j) = S_{j_1+m}(j) = S_{j_2}(j-1) \cup S_{j_1+m}(j-1)$. In this way, we are identifying that the terminus of v_{j_1} (represented by S_{j_1+m}) is the same as the origin of v_{j_2} (represented by S_{j_2}). We note that $S_{j_2}(t) = S_{j_1+m}(t)$ will remain identified with each other for the rest of the process so that if one set grows, the other grows with it. To this end, for any $\ell \in S_{j_2}(j)$, we update $S_\ell(j) = S_{j_2}(j)$. For $\ell \notin S_{j_2}(j)$, we update $S_\ell(j) = S_\ell(j-1)$. We complete this process after we have considered each edge.

To give the digraph $\mathcal{H}$, we let $S_i = S_i(k)$; then the vertex set of $\mathcal{H}$ is given by removing any duplication among the S_i's and taking that resulting partition. The edge set of $\mathcal{H}$ is given by $E(\mathcal{H}) = \{(S_i, S_{i+m}) | 1 \leq i \leq m\}$. We note that $\mathcal{H}$ is a well-defined multi-digraph; however, it may not be a simple digraph. It is possible that, at this point, we have loops and multiple edges.

The graph X is formed as follows. For each pair of arcs in $\mathcal{H}$ which are inverses, we combine them into a single undirected edge on the same vertices. After this, we forget the directions in $\mathcal{H}$ and take any directed edges as undirected edges on the same vertices. This leaves us with a multigraph.

In Theorem 3.6 we show that if D is an oriented line graph satisfying certain conditions and if X is the graph generated through Construction 3.2, then D is isomorphic to L^oX. In addition, at the end of the section in Theorem 3.14, we show that if D is an arbitrary digraph that does not contain certain forbidden subdigraphs, then D is a subdigraph of L^oX.

Example 3.3. We illustrate the algorithm on the digraph in Figure 4. The digraph D has 4 vertices which we have labeled 1 through 4. We now create eight sets $S_1, S_2, \cdots, S_8$ by setting $S_i(0) = \{i\}$.

The first arc that we consider is e_1, which originates at vertex 1 and terminates at vertex 2. Thus we let $S_5(1) = S_2(1) = S_5(0) \cup S_2(0) = \{2, 5\}$.

Set	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$S_1(t)$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1,2,5,7\}$	$\{1,2,4,5,7\}$	$\{1,2,4,5,7\}$
$S_2(t)$	$\{2\}$	$\{2,5\}$	$\{2,5,7\}$	$\{1,2,5,7\}$	$\{1,2,4,5,7\}$	$\{1,2,4,5,7\}$
$S_3(t)$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3,8\}$
$S_4(t)$	$\{4\}$	$\{4\}$	$\{4\}$	$\{4\}$	$\{1,2,4,5,7\}$	$\{1,2,4,5,7\}$
$S_5(t)$	$\{5\}$	$\{2,5\}$	$\{2,5,7\}$	$\{1,2,5,7\}$	$\{1,2,4,5,7\}$	$\{1,2,4,5,7\}$
$S_6(t)$	$\{6\}$	$\{6\}$	$\{6\}$	$\{6\}$	$\{6\}$	$\{6\}$
$S_7(t)$	$\{7\}$	$\{7\}$	$\{2,5,7\}$	$\{1,2,5,7\}$	$\{1,2,4,5,7\}$	$\{1,2,4,5,7\}$
$S_8(t)$	$\{8\}$	$\{8\}$	$\{8\}$	$\{8\}$	$\{8\}$	$\{3,8\}$

Figure 5: We record the updates to the sets S_i for each arc considered.

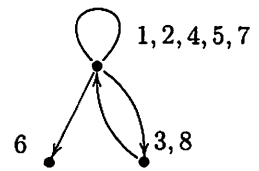

Figure 6: The digraph that results from our algorithm

We also update $S_i(1) = \{i\}$ for $i \neq 2,5$. We now identify S_2 and S_5 so that if one is updated later, the other will be as well. We have finished with e_1 and now consider $e_2 = (3,2)$. This asks us to update $S_7(2) = S_2(2) = \{2,5,\} \cup \{7\} = \{2,5,7\}$. In addition, since S_2 and S_5 are identified from before, we update $S_5(2) = \{2,5,7\}$. The remaining sets S_i for $i \neq 2,5,7$ remain with $S_i(2) = \{i\}$. We continue in this manner for the arcs e_3, e_4, and e_5. A complete list of the updates which happen for each arc appears in Figure 5.

Now that we have completed building the sets S_i, we see that $S_1 = S_2 = S_4 = S_5 = S_7 = \{1,2,4,5,7\}$, that $S_3 = S_8 = \{3,8\}$, and that $S_6 = \{6\}$. This gives us the partition of $\{1,2,\cdots,8\}$ that we claimed. We build the digraph $\mathcal{H}$ from the sets S_i.

The partition that we found above is given by $\{\{1,2,4,5,7\},\{3,8\},\{6\}\}$. Then the adjacencies we get in forming the digraph $\mathcal{H}$ are as follows: $\{1,2,4,5,7\}$ is adjacent to itself, to $\{6\}$, and to $\{3,8\}$ since $1+4=5$, $2+4=6$, and $4+4=8$. Also, $\{3,8\}$ is adjacent to $\{1,2,4,5,7\}$. The resulting digraph is shown in Figure 6.

We note that this digraph is a bit "messy"; however, this is in large part because the original digraph we started with is not an oriented line graph. This conclude the example.

Before moving on, we describe how to tell when the sets S_i and S_j, constructed in the above algorithm, are identified with each other. We first give a useful technical definition.

Definition 3.4 (Alternating paths). *In a digraph D, an* alternating path *is a sequence of distinct arcs $\{f_1, f_2, \cdots, f_\ell\}$ for some $\ell \in \mathbb{Z}$ such that the path formed by replacing f_{2j} with the inverse arc $\bar{f}_{2j}$ forms a path in the digraph, either starting at the origin of f_1 and going forwards or starting at the origin of f_ℓ and going backwards in our sequence. In other words, by reversing the direction of the evenly positioned arcs, we see what we expect for a path.*

We can now fully describe when $S_i = S_j$.

Lemma 3.5. *Let D be a digraph with m vertices and $\mathcal{H}$ the digraph formed from D by the algorithm above. Then $S_i = S_j$ for $i \neq j$ if and only if one of the three conditions below is satisfied:*

1. *For $1 \leq i < j \leq m$, there is an alternating path $\{f_1, f_2, \cdots, f_\ell\}$, with ℓ even, and $f_1 = (x, v_i)$ and $f_\ell = (y, v_j)$ for some vertices $x, y \in V(D)$.*
2. *For $1 \leq i \leq m < j \leq 2m$, there is an alternating path $\{f_1, f_2, \cdots, f_\ell\}$, with ℓ odd, and $f_1 = (x, v_i)$ and $f_\ell = (v_{j-m}, y)$ for some vertices $x, y \in V(D)$.*
3. *For $m < i < j \leq 2m$, there is an alternating path $\{f_1, f_2, \cdots, f_\ell\}$, with ℓ even, and $f_1 = (v_{i-m}, x)$ and $f_\ell = (v_{j-m}, y)$ for some vertices $x, y \in V(D)$.*

Proof. If such an alternating path exists, application of the algorithm leads directly to $S_i = S_j$. To show the existance of an alternating path when the sets are equal we proceed by induction on the number of arcs in the graph D. If D has no arcs, the constructed digraph $\mathcal{H}$ is a collection of m independent edges and no two sets are equal, so the condition holds vacuously. For a general digraph D, we label the arcs consecutively and inspect each arc in turn during the construction of $\mathcal{H}$. In applying the algorithm to $D - e_k$, we may use the same labels on the arcs. Denote the digraph so obtained by $\mathcal{H}'$. There are two cases.

Case 1. $S_i = S_j$ before the inspection of e_k. In this case, $S_i = S_j$ for $\mathcal{H}'$, a smaller graph, so the alternating path condition holds by the induction hypothesis.

Case 2. $S_i = S_j$ only after the inspection of e_k. Using the notation of the construction, $e_k = (v_{k_1}, v_{k_2})$ and inspection of the arc yields the identification $S_{k_2} = S_{k_1+m}$. Thus, prior to this inspection, $S_i = S_{k_2}$ and $S_j = S_{k_1+m}$ (or vice versa) so that after the inspection the sets are the same. However, these conditions are also true for $\mathcal{H}'$ so we have alternating paths $\{f_1, f_2, \ldots, f_l\}$ between v_i or v_{i-m} and v_{k_1} with $f_l = (v_{k_1}, x)$ and $\{g_1, g_2, \ldots, g_{l'}\}$ between v_{k_2} and v_j or v_{j-m} with $g_1 = (y, v_{k_2})$. However, $\{f_1, f_2, \ldots, f_l, e_k, g_1, g_2, \ldots, g_{l'}\}$ is an alternating path between v_i or v_{i-m} and v_j or v_{j-m}. The parity condition follows from inspection of this path and the conditions on i and j. □

We immediately make use of Lemma 3.5 to justify the construction given at the beginning of this section.

Theorem 3.6. *Let D be the oriented line graph of a graph X such that every vertex in X has degree at least 3. Then the digraph $\mathcal{H}$ constructed from D by our algorithm is exactly the symmetric digraph $D(X)$ associated to X, making the corresponding graph X.*

Proof. We begin with some numerical considerations. If m is the number of edges in X, then the number of vertices in D is exactly $2m$. By construction, there is one arc in $\mathcal{H}$ for each vertex in D, so the number of arcs in $\mathcal{H}$ is also $2m$, the same as the number of arcs in $D(X)$. There are two considerations to conclude the proof: that each arc in $\mathcal{H}$ has an inverse arc and that adjacent arcs in $\mathcal{H}$ correspond to adjacent edges in X.

In Theorem 2.2 we argued that it was possible to identify the vertex induced by the inverse arc of a vertex v in the oriented line graph if $d^+(v) \geq 2$. Since X is md3, each arc feeds into at least two other arcs, so every vertex in D has out degree at least 2. Thus for each vertex $v \in D$, we can identify the vertex $\bar{v}$ induced by the associated inverse arc. In the proof of this result, a length 3 alternating path was constructed, as $\{(v, x), (y, x), (y, \bar{v})\}$. If we identify $v = v_i$ and $\bar{v} = v_j$, this alternating path falls into case 2 of Lemma 3.5 so that $S_{i+m} = S_j$. However, if instead we start with v_j, the same considerations lead to an alternating path $\{(\bar{v}, a), (b, a), (b, v)\}$ so that $S_{j+m} = S_i$ and the arcs (S_i, S_{i+m}) and (S_j, S_{j+m}) are inverses, as desired.

We conclude by showing that adjacent arcs in $\mathcal{H}$ correspond to adjacent edges in X. Consider an alternating path $\{(v_1, v_2), (v_3, v_2), (v_3, v_4), (v_5, v_4), (v_5, v_6)\}$ in D. Since we know that D is an oriented line graph, if there is no arc (v_1, v_4), then the above considerations identify $v_4 = \bar{v}_1$. Similarly, if there is no arc (v_3, v_6), then $v_6 = \bar{v}_3$. However, since the arc corresponding to v_3 in $D(X)$ feeds into the arc corresponding to $v_4 = \bar{v}_1$, v_1 must feed into $\bar{v}_3$, so the arc (v_1, v_6) must exist. Thus, the shortest odd alternating path between two vertices has length at most 3. Since every arc has an inverse, the only adjacencies to consider in $\mathcal{H}$ are those of the form (S_i, S_{i+m}) adjacent to (S_j, S_{j+m}) (so $S_{i+m} = S_j$). Since this only happens when there is an alternating path from v_i to v_j, and the shortest alternating path is of length at most 3, either these are inverse arcs, or they are adjacent in D, so v_i feeds into v_j in $D(X)$. This concludes the proof. □

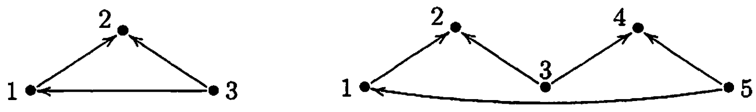

Figure 7: The digraphs $\mathcal{F}_1$ and $\mathcal{F}_2$.

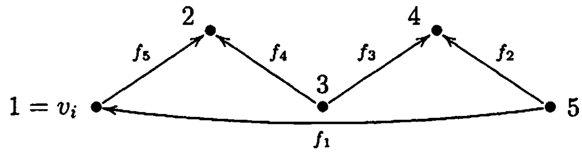

Figure 8: Illustration for the proof of proposition 1 in the case $n = 2$

Lemma 3.5 also gives us a concrete way to detect certain structures which are forbidden in oriented line graphs of simple graphs. We first focus on the structures in D that lead to loops in X. From the definition of the edge set of $\mathcal{H}$, we get a loop in $\mathcal{H}$ (and thus in X) if $\mathcal{S}_i = \mathcal{S}_{i+m}$. We define the following class of digraphs.

Definition 3.7 (Forbidden subdigraphs: $\mathcal{F}_n$). *Let $\mathcal{F}_n$ be a digraph with $2n+1$ vertices, written $\{1, 2, \cdots, 2n+1\}$. The arcs in $\mathcal{F}_n$ are given by $(i-1, i)$ and $(i+1, i)$ for each even i satisfying $1 < i < 2n+1$. In addition, we add the arc $(2n+1, 1)$.*

We illustrate $\mathcal{F}_1$ and $\mathcal{F}_2$ in Figure 7.

Proposition 3.8 (Forbidden subdigraphs: $\mathcal{F}_n$). *Suppose D contains no subdigraphs isomorphic to $\mathcal{F}_n$ for each $n \in \mathbb{N}$, then the digraph $\mathcal{H}$, and consequently the graph X, formed from D by our algorithm contains no loops.*

Proof. A loop is formed if an arc is given from a vertex to itself. From the construction of $\mathcal{H}$, the arcs are $(\mathcal{S}_i, \mathcal{S}_{i+m})$. Thus, loops may only be formed if $\mathcal{S}_i = \mathcal{S}_{i+m}$. Since $1 \leq i \leq m < i+m \leq 2m$, this falls into case 2 of Lemma 3.5 and there is an odd alternating path from v_i to itself in D. With $v_i = 1$, $f_1 = (2n+1, 1)$, $f_2 = (2n+1, 2n)$, $f_3 = (2n-1, 2n)$, and so on until $f_{2n+1} = (1, 2)$, this is a subdigraph isomorphic to the graph $\mathcal{F}_n$. This isomorphism is illustrated in Figure 8. □

Thus the graphs $\mathcal{F}_n$ arise in an oriented line graph if the parent graph has a loop.

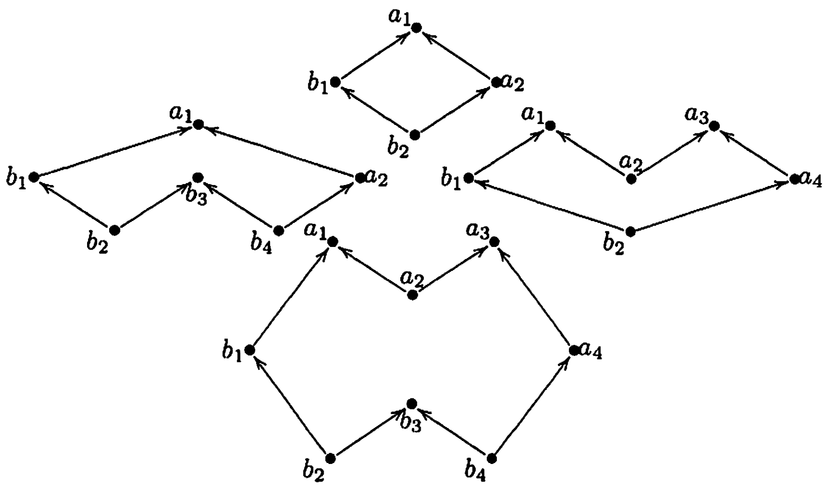

Figure 9: The digraphs $\mathcal{G}_{1,1}$, $\mathcal{G}_{1,2}$, $\mathcal{G}_{2,1}$, and $\mathcal{G}_{2,2}$.

Example 3.9. We have already seen an example where $\mathcal{F}_1$ induces a loop. We refer to the digraph in Figure 4. Then the subdigraph induced by the vertices $\{1, 2, 3\}$ is isomorphic to $\mathcal{F}_1$. Indeed, when we run our algorithm, we see how a loop is produced in the output, as shown in Figure 6.

We also identify the structures in D that give rise to double edges in X. From the definition of the edge set of $\mathcal{H}$, we get a double edge in $\mathcal{H}$ (and thus in X) if $\mathcal{S}_i = \mathcal{S}_j$ and $\mathcal{S}_{i+m} = \mathcal{S}_{j+m}$ for some pair $i \neq j$. We define the following class of digraphs.

Definition 3.10 (Forbidden subdigraphs: $\mathcal{G}_{n,k}$). *Let $\mathcal{G}_{n,k}$ be a digraph with $2(n+k)$ vertices labeled $\{a_1, a_2, \cdots, a_{2n}, b_1, b_2, \cdots, b_{2k}\}$. The arcs are given by (a_i, a_{i+1}), (a_i, a_{i-1}) for odd i and (b_i, b_{i+1}), (b_i, b_{i-1}) for even i. In addition, we have the arcs (b_1, a_1) and (b_{2k}, a_{2n}).*

We illustrate $\mathcal{G}_{1,1}$, $\mathcal{G}_{1,2}$, $\mathcal{G}_{2,1}$, and $\mathcal{G}_{2,2}$ in Figure 9.

Proposition 3.11 (Forbidden subgraphs: $\mathcal{G}_{n,k}$). *Suppose D contains no subdigraphs isomorphic to $\mathcal{G}_{n,k}$ for each pair $(n, k) \in \mathbb{N} \times \mathbb{N}$, then the digraph $\mathcal{H}$ from our algorithm, does not contain a double arc with both arcs directed in the same way. In particular, the resulting graph X will not have a double edge.*

Proof. This proof follows the same general idea as the proof of 3.8: we first identify when double arcs appear and then identify an isomorphism between

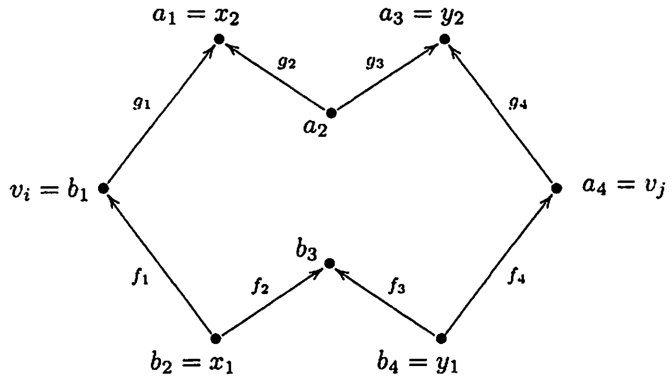

Figure 10: Illustration for the proof of proposition 2 in the case $n = k = 2$

the required edges and one of the digraphs $\mathcal{G}_{n,k}$. Specifically, double arcs in D result from arcs $(\mathcal{S}_i, \mathcal{S}_{i+m})$ and $(\mathcal{S}_j, \mathcal{S}_{j+m})$ when $\mathcal{S}_i = \mathcal{S}_j$ and $\mathcal{S}_{i+m} = \mathcal{S}_{j+m}$. If $\mathcal{S}_i = \mathcal{S}_j$, then this is an instance of case 1 of Lemma 3.5. As such, there is an alternating path $\{f_1, f_2, \cdots, f_{2k}\}$ with $f_1 = (x_1, v_i)$ and $f_{2k} = (y_1, v_j)$. If $\mathcal{S}_{i+m} = \mathcal{S}_{j+m}$, then this is an instance of case 3 of Lemma 3.5. Thus, there is an alternating path $\{g_1, g_2, \cdots, g_{2n}\}$ with $g_1 = (v_i, x_2)$ and $g_{2n} = (v_j, y_2)$. We identify $v_i = b_1$, $v_j = a_{2n}$, $x_1 = b_2$, $y_1 = b_{2k}$, $x_2 = a_1$, and $y_2 = a_{2n-1}$. In addition, $g_2 = (a_3, a_2)$, $g_3 = (a_3, a_4)$, etc., until $g_{2n-1} = (a_{2n-2}, a_{2n-a}$, $f_2 = (b_2, b_3)$, $f_3 = (b_4, b_3)$, and so on until $f_{2k-1} = (b_{2k}, b_{2k-1})$. This is an isomorphism to $\mathcal{G}_{n,k}$ and is illustrated in Figure 10. □

The graphs $\mathcal{G}_{n,k}$ arise if the parent graph has a double edge, which shows up in $\mathcal{H}$ as two arcs directed in the same direction. There is a second case of a double edge in X which arises from two arcs in $\mathcal{H}$ which are directed in opposite directions. We must take care to allow this when the arcs are playing the role of "inverse arcs" and disallow it when they were induced by different edges in X. To this end, we define another class of digraphs.

Definition 3.12. *We define $\tilde{\mathcal{G}}_{n,k}$ by starting with $\mathcal{G}_{n,k}$ and removing the arc (b_1, a_1). We then replace that arc with the arc (a_1, b_1).*

Proposition 3.13 (Forbidden digraphs: $\tilde{\mathcal{G}}_{1,k}$). *Suppose D contains no subdigraphs isomorphic to $\tilde{\mathcal{G}}_{n,k}$ for each pair $(n, k) \in \mathbb{N}\times\mathbb{N}$, then the digraph $\mathcal{H}$ contains no double arcs directed in opposing directions. In particular, if D contains no subdigraphs isomorphic to $\tilde{\mathcal{G}}_{1,k}$ for each $k \in \mathbb{N}$, then the digraph $\mathcal{H}$ contains no arcs which would arise from a pair of inverse arcs. In this case, the graph X formed from $\mathcal{H}$ will not have a double edge resulting from failing to identify a pair of inverse arcs.*

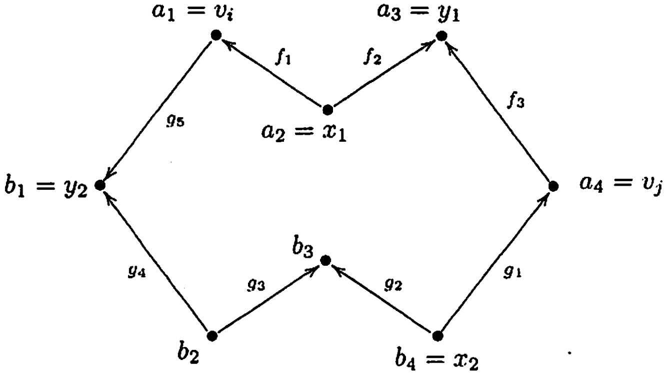

Figure 11: Illustration for the proof of proposition 3 in the case $n = k = 2$

Proof. Arcs occur in opposite directions in $\mathcal{H}$ when arcs (S_i, S_{i+m}) and (S_j, S_{j+m}) have $S_i = S_{j+m}$ and $S_j = S_{i+m}$. Both of these fall into case 2 of Lemma 3.5, so there are alternating paths $\{f_1, f_2, \cdots, f_{2n-1}\}$ and $\{g_1, g_2, \cdots, g_{2k+1}\}$ with $f_1 = (x_1, v_i)$, $f_{2n-1} = (v_j, y_1)$, $g_1 = (x_2, v_j)$ and $g_{2k+1} = (v_i, y_2)$. With the identification that $a_1 = v_i$, $a_2 = x_1$, $a_{2n-1} = y_1$, $a_{2n} = v_j$, $b_1 = y_2$, and $b_{2k} = x_2$, this gives an isomorphism to the graph $\mathcal{G}_{n,k}$. However, we must take care to allow inverse arcs, so this cuts down the family of disallowed subgraphs. An arc in $\mathcal{H}$ would not be adjacent to its inverse arc in the oriented line graph, so the vertices corresponding to inverse arcs cannot be adjacent in D. The first part of this theorem completely describes when one arc and its inverse both arise from D. The vertices in $\tilde{\mathcal{G}}_{n,k}$ which correspond to the inverse arcs are a_1 and a_{2n}. There is no value of k to make these adjacent, but they are adjacent when $n = 1$, so the graphs $\tilde{\mathcal{G}}_{1,k}$ are not allowed. □

Now that we have identified the families of digraphs $\mathcal{F}_n$, $\mathcal{G}_{n,k}$, and $\tilde{\mathcal{G}}_{1,k}$, we more precisely state the relationship between D, $\mathcal{H}$, and X when D is an arbitrary digraph.

Theorem 3.14. *Let D be a digraph and X the graph resulting from the above algorithm. Suppose D contains no subdigraph isomorphic to an instance of $\mathcal{F}_n$, $\mathcal{G}_{n,k}$, or $\tilde{\mathcal{G}}_{1,k}$, then $\mathcal{H}$ is a simple digraph with no loops and D is a subdigraph of L^oX, the oriented line graph of X.*

Proof. The statement that $\mathcal{H}$ is simple follows from Propositions 3.8 and 3.11. For the remaining statement, that D is a subdigraph of L^oX, we present an explicit injective mapping of both the vertices and the arcs in D to the vertices and arcs in L^oX. This injection is illustrated in Figure

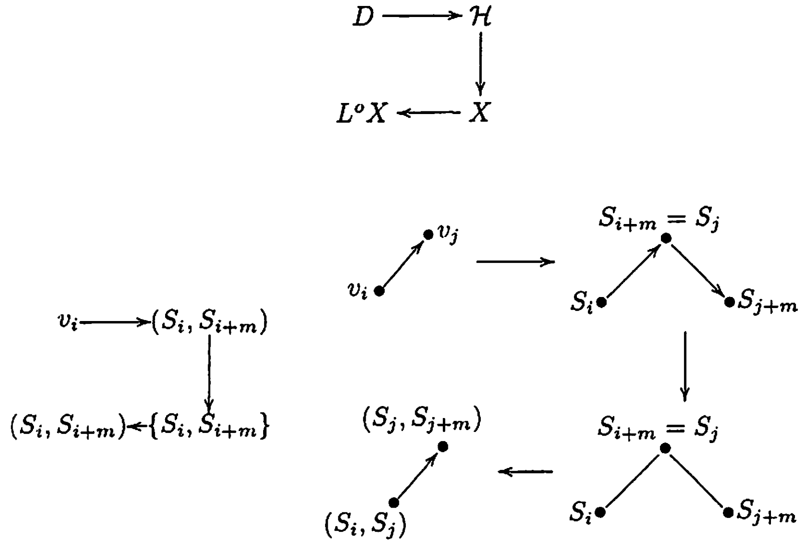

Figure 12: Our mapping.

12. To begin with the vertices, a vertex $v_i \in V(D)$ corresponds in the algorithm to the arc $(S_i, S_{i+m}) \in E(\mathcal{H})$. This mapping is bijective by construction. Since $\mathcal{H}$ is simple, and we form X by combining inverse arcs ignoring direction, there is at most one other arc in $E(\mathcal{H})$ corresponding to the edge $\{S_i, S_{i+m}\} \in E(X)$. Recalling that $V(L^oX) = E(D(X))$, the edge $\{S_i, S_{i+m}\}$ forms two vertices in $V(L^oX)$, specifically, (S_i, S_{i+m}) and (S_{i+m}, S_i). As such, we introduce the correspondence that $v_i \mapsto (S_i, S_{i+m})$ and note that this mapping is injective.

It remains only to show that if (v_i, v_j) is an arc in D then the corresponding vertices (S_i, S_{i+m}) and (S_j, S_{j+m}) in $V(L^oX)$ are adjacent. However, the existence of the edge (v_i, v_j) causes $S_{i+m} = S_j$ from our algorithm. Thus, in $\mathcal{H}$, the arc (S_i, S_{i+m}) and the arc (S_j, S_{j+m}) share the vertex $S_{i+m} = S_j$. By Proposition 3.13, the vertices S_i and S_{j+m} are distinct, so the arcs (S_i, S_{i+m}) and (S_j, S_{j+m}) in $E(D(X))$ are not inverses. Since the terminus of the first arc is the origin of the second arc, the arc $((S_i, S_{i+m}), (S_j, S_{j+m}))$ is in $E(L^oX)$. The injectivity of the mapping $(v_i, v_j) \mapsto ((S_i, S_{i+m}), (S_j, S_{j+m}))$ follows since each of the aforementioned arcs is unique. □

We note that the conclusion that D is a subdigraph of L^oX is important. There are several ways for D to not be isomorphic to L^oX. It is possible that the intermediate graph $\mathcal{H}$ is not isomorphic to $D(X)$. This can happen particularly when $\mathcal{H}$ has some arcs without corresponding inverse arcs. In this case, X will have more edges than half the number of vertices of D. A second possibility is that the number of edges in X is correct, but L^oX has more arcs than D.

4 Structural characterizations

The discussion in the previous section led to forbidden subgraphs upon considering what structures in an oriented line graph would be induced by a loop or double edge in X. In this section, we take a more rigorous look at the structure of the oriented line graph and provide some characterizations based upon these observations.

We fix a graph X, its symmetric digraph $D(X)$, and the resulting oriented line graph L^oX. We begin by looking at the arcs in $D(X)$ that an arc e_1 can legally feed into. We first establish some notation. For a vertex v in a digraph $\mathcal{D}$, we let $\Gamma_+(v) = \{w \in V(\mathcal{D}) | \{v, w\} \in E(\mathcal{D})\}$. Then $\Gamma_+(v)$ is exactly those neighbors of v for which there is an arc beginning at v and terminating at w. Similarly, $\Gamma_-(v) = \{w \in V(\mathcal{D}) | \{w, v\} \in E(\mathcal{D})\}$ describes those vertices for which there is an arc beginning at w and terminating at v. We also have $d^+(v) = |\Gamma_+(v)|$ and $d^-(v) = |\Gamma_-(v)|$.

We now focus on the indegrees and outdegrees of vertices in an oriented line graph.

Proposition 4.1 (Characterization 1: Degree considerations). *Suppose L^oX is the oriented line graph of X. Then the following are true:*

1. *Suppose $v \in V(L^oX)$ and $d^+(v) > 1$. Then for vertices $w_1, w_2 \in \Gamma_+(v)$, we have $d^-(w_1) = d^-(w_2)$. Moreover, the symmetric difference of $\Gamma_-(w_1)$ and $\Gamma_-(w_2)$ contains exactly 2 elements: $\bar{w}_1 \in \Gamma_-(w_2) \setminus \Gamma_-(w_1)$ and $\bar{w}_2 \in \Gamma_-(w_1) \setminus \Gamma_-(w_2)$.*

2. *Suppose $v \in V(L^oX)$ and $d^-(v) > 1$. Then for vertices $w_1, w_2 \in \Gamma_-(v)$, we have $d^+(w_1) = d^+(w_2)$. Moreover, the symmetric difference of $\Gamma_+(w_1)$ and $\Gamma_+(w_2)$ contains exactly 2 elements: $\bar{w}_1 \in \Gamma_+(w_2) \setminus \Gamma_+(w_1)$ and $\bar{w}_2 \in \Gamma_+(w_1) \setminus \Gamma_+(w_2)$.*

Proof. We consider the first case. Fix a vertex v with $d^+(v) > 1$ and two vertices w_1 and $w_2 \in \Gamma_+(v)$. This means that v feeds into both w_1 and w_2. If we consider what is happening in the symmetric digraph $D(X)$ used in the construction of L^oX, we see that v corresponds to an arc which points at the tail of the arcs described by w_1 and w_2. Hence, w_1 and w_2 have the same origin in $D(X)$. That they satisfy $d^-(w_1) = d^-(w_2)$ now follows.

The claim regarding the symmetric difference is immediate from our discussion of Theorem 2.1 in Section 2.

The second claim follows by symmetry with the first. □

We will now focus on cycles in the oriented line graph and on paths where each vertex has indegree 1 and outdegree 1. Before giving our next characterization, we give a definition from Cooper [4].

Definition 4.2 (Cooper). *In a digraph D, a* 2-path *is a sequence of vertices*
$\{d_1, d_2, \cdots, d_k\}$ such that $\Gamma_+(d_i) = \{d_{i+1}\}$ for $1 \le i < k$. A 2-path containing a vertex v is maximal *if there is no 2-path with more arcs which contains v.*

In the oriented line graph L^oX, we can get 2-paths whenever we have vertices of degree 2 in X since 2-paths consist exactly of those paths where each vertex has indegree 1 and outdegree 1. We describe a structural property of L^oX relating to 2-paths and cycles.

Proposition 4.3 (Characterization 2: 2-paths and cycles). *Suppose L^oX is the oriented line graph of X. Then the following are true:*

1. *If $\{d_1, \cdots, d_k\}$ is a maximal 2-path in L^oX, there there is a maximal 2-path $\{a_1, \cdots, a_k\}$ satisfying $t(a_k) = o(d_1)$ and $t(d_k) = o(a_1)$ in X.*
2. *If $\{e_1, \cdots, e_n\}$ is a cycle in L^oX such that $e_i = e_j$ if and only if $i = j$, then there is another cycle $\{b_1, \cdots, b_n\}$ in L^oX which is disjoint from the first and satisfies $b_i = b_j$ if and only if $i = j$.*

Proof. Both statements will follow by considering the oriented line graph construction. We consider the first statement. If $\{d_1, \cdots, d_k\}$ is a maximal 2-path in L^oX, then there is a corresponding path of k edges in X where each of the interior vertices has degree 2 and the vertices at either end have larger degree. Traversing this path in one direction will give rise to $\{d_1, \cdots, d_k\}$. By traversing it in the other direction, we will produce another maximal 2-path $\{a_1, \cdots, a_k\}$ which ends where the first started, and starts where the first ends. This is exactly what we have claimed.

The second statement is similar. Given a cycle in X which does not repeat edges, we induce two cycles in L^oX depending upon our direction of travel around the cycle. □

We conclude by conjecturing that between the forbidden subgraphs of the previous section and the structural properties described in this section, we have described oriented line graphs.

Conjecture 4.4 (Description of oriented line graphs). *Suppose a digraph D has no subgraphs isomorphic to an instance of $\mathcal{F}_n$, $\mathcal{G}_{n,k}$, or $\tilde{\mathcal{G}}_{1,k}$, and suppose D satisfies the properties given in Proposition 4.1 and Proposition 4.3. Then let X be the undirected graph obtained by following the algorithm in the previous section and combining arcs which point in opposing directions into a single undirected edge, then D is the oriented line graph of X.*

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