On incidence energy of some graphs *

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Abstract

Let G be a simple graph. The incidence energy (IE for short) of G is defined as the sum of the singular values of the incidence matrix. In this paper, a new lower bound for IE of graphs in terms of the maximum degree is given. Meanwhile, an upper bound and a lower bound for IE of the subdivision graph and the total graph of a regular graph G are obtained, respectively.

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1 Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. All graphs considered here are simple and undirected. Denote by $A(G)$ and $D(G)$ the adjacency matrix and the diagonal matrix with the vertex degrees of G on the diagonal, respectively. The matrix $L(G) = D(G) - A(G)$ (resp., $L^+(G) = D(G) + A(G)$) is called the Laplacian matrix (resp., signless Laplacian matrix [2, 3, 4, 5]) of G, for details on those matrices see [17, 18]. The multiset of eigenvalues of $A(G)$ (resp., $L(G)$, $L^+(G)$) are called the adjacency (resp., Laplacian, signless Laplacian) spectrum of G. Since $A(G)$, $L(G)$ and $L^+(G)$ are all real symmetric matrices, their eigenvalues are real numbers. So we can assume that

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$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ (resp., $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$, $\mu_1^+(G) \geq \mu_2^+(G) \geq \cdots \geq \mu_n^+(G)$) are the adjacency (resp., Laplacian, signless Laplacian) eigenvalues of G. It is well-known that all Laplacian (resp., signless Laplacian) eigenvalues of G are non-negative. If the graph G is connected, then $\mu_i(G) > 0$ for $i = 1, 2, \ldots, n-1$ and $\mu_n(G) = 0$ [17]. If G is a connected non-bipartite graph, then $\mu_i^+(G) > 0$ for $i = 1, 2, \ldots, n$ [2].

One of the most remarkable chemical applications of graph theory is based on the close correspondence between the graph eigenvalues and the molecular orbital energy levels of π-electrons in conjugated hydrocarbons. For the Hüchkel molecular orbital approximation, the total π-electron energy in conjugated hydrocarbons is given by the sum of absolute values of the eigenvalues corresponding to the molecular graph G in which the maximum degree is not more than four in general. In the 1970s I. Gutman [6] extended the concept of energy to all simple graphs G, and defined the energy of G as

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)|. \tag{1}$$

Research on graph energy is nowadays very active, as seen from the recent papers [7, 9, 12, 13, 15, 16, 20, 24] and the references quoted therein.

The singular values of a real matrix (not necessarily square) M are the square roots of the eigenvalues of the matrix MM^t, where M^t denotes the transpose of M. The energy $E(M)$ of the matrix M is then defined as the sum of its singular values [19]. Obviously, $E(G) = E(A(G))$.

Let $I(G)$ be the (vertex-edge) incidence matrix of the graph G. For a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{e_1, e_2, \ldots, e_m\}$, the (i, j)-entry of $I(G)$ is 0 if v_i is not incident with e_j and 1 if v_i is incident with e_j. Jooyandeh et al. [14] introduced the incidence energy IE of G, which is defined as the sum of the singular values of the incidence matrix of G. Gutmann et al. [10] shown that

$$IE = IE(G) = \sum_{i=1}^{n} \sqrt{\mu_i^+(G)}. \tag{2}$$

Some basic properties of IE were established in [10, 11, 14].

From the Equation (2), one can immediately get the incidence energy of a graph by computing the signless Laplacian eigenvalues of the graph. However, even for special graphs, it is still complicated to find the signless Laplacian eigenvalues of them. Hence it makes sense to establish lower and upper bounds to estimate the invariant for some classes of graphs. Zhou [23] obtained the upper bounds for the incidence energy using the first Zagreb index. Gutman et al. [11] gave several lower and upper bounds for

IE. In particular, an upper bound for IE of the line graph of a regular graph was established [11]. Similar to graph energy, it appears to be much more difficult to find lower bounds than upper bounds for IE.

The rest of this paper is organized as follows. In Section 2 we give a new lower bound for IE of graphs in terms of the maximum degree. An upper bound and a lower bound for IE of the subdivision graph and the total graph of a regular graph G are obtained in Section 3.

2 Lower bounds for incidence energy

In this section, we will give a lower bound of IE of a nonempty graph. Denote by K_n the complete graph with n vertices. If $G = K_1$, we have nothing to discuss. So we assume that $G \neq K_1$ throughout this paper. The following fundamental properties of the IE were established in [14].

Lemma 2.1 *[14] Let G be a graph with n vertices and m edges. Then*
(i) $IE(G) \geq 0$, and equality holds if and only if $m = 0$;
(ii) If the graph G has components $G_1, \ldots, G_p$, then $IE(G) = \sum_i^p IE(G_i)$.

From Lemma 2.1 (ii), when we study the incidence energy of a graph G, we may assume that G is connected.

The following Lemmas will be used later.

Lemma 2.2 *[22] Let G be a simple and connected graph with $n > 1$ vertices, then $\mu_1^+ \geq \Delta + 1$, equality holds if and only if G is a star with n vertices.*

Lemma 2.3 *[11] Let $c_1, c_2, \ldots, c_t$ be positive integers. Then*

$$\sum_{i=1}^{t} c_i \geq \sqrt{\frac{(\sum_{i=1}^{t} c_i^2)^3}{\sum_{i=1}^{t} c_i^4}}, \tag{3}$$

and the equality holds if and only if $c_1 = c_2 = \cdots = c_t$.

The first Zagreb index $Zg(G)$ of a graph G is defined as

$$Zg(G) = \sum_{u \in V(G)} d_u^2,$$

where d_u denotes the degree of vertex u in G. This quantity found many applications in chemistry [8].

Theorem 2.4 *Let G be a connected graph with n vertices, $m \geq 1$ edges and maximum degree Δ. Then*

$$IE(G) \geq 2m\sqrt{\frac{2m}{n\Delta^2 + 2m}}, \tag{4}$$

and the equality holds if and only if $G \cong K_2$.

Proof. Note that

$$Zg(G) = \sum_{u \in V(G)} d_u^2 \leq \Delta \sum_{u \in V(G)} d_u = \Delta(2m) \leq n\Delta^2, \tag{5}$$

and the equality holds if and only if either $d_u = \Delta$ for all $u \in V(G)$, i.e., G is Δ-regular, or $d_u = 0$ for all $u \in V(G)$. The latter case is impossible since $m \geq 1$. Hence, $Zg(G) = n\Delta^2$ if and only if G is Δ-regular.

Note that $\sum_{i=1}^n \mu_i^+ = 2m$ and $\sum_{i=1}^n (\mu_i^+)^2 = Zg(G) + 2m$. It follows from (2), (3) and (5) that

$$IE(G) = \sum_{i=1}^n \sqrt{\mu_i^+} \geq \sqrt{\frac{(2m)^3}{Zg(G) + 2m}} \geq 2m\sqrt{\frac{2m}{n\Delta^2 + 2m}}$$

and the equality holds if and only if all nonzero signless Laplacian eigenvalues are equal.

From the argument above, it is clear that the equality in (4) holds if and only if G is a regular and all nonzero signless Laplacian eigenvalues are equal. If G is a connected bipartite graph, then $\mu_1^+ = \mu_2^+ = \ldots = \mu_{n-1}^+$, and $\mu_n^+ = 0$. Therefore G is a $\Delta-$regular graph and $L^+(G)$ has two distinct eigenvalues $\mu_1^+, 0$. It follows that G is a regular graph with two distinct eigenvalues (of the adjacency matrix) $\mu_1^+ - \Delta$ and $-\Delta$ with multiplicities $n-1$ and 1, respectively. Notice that the sum of the eigenvalues of G is equal to zero. Then, by Lemma 2.2, $\Delta = (n-1)(\mu_1^+ - \Delta) \geq (n-1) \geq \Delta$. Thus $\mu_1^+ = \Delta + 1$. By Lemma 2.2 again, we have that G is a star. Since G is a regular graph, $G \cong K_2$. If G is a connected non-bipartite graph, then $\mu_i > 0, i = 1, 2, \ldots, n$. It follows that G is a $\Delta-$regular graph and $L^+(G)$ has n equal signless Laplacian eigenvalues. Therefore,

$$\Delta = \frac{2m}{n} = \frac{\sum_{i=1}^n \mu_i^+}{n} = \mu_1^+,$$

this contradicts with the Lemma 2.2.

Hence, we complete the proof of Theorem 2.4. □

Recall from [11] that a lower bound for IE was given as follows.

Lemma 2.5 *Let G be a graph with n vertices and m edges. Then*

$$IE(G) \geq \frac{2m}{\sqrt{n}}, \tag{6}$$

with equality if and only if $G \cong \overline{K_n}$ or $G \cong K_2$.

Remark 1 *For $m, n \geq 1$, the two lower bounds in (4) and (6) of IE are not comparable. Indeed, let G be the cycle with 5 vertices. Then $n = 5$, $m = 5$, $\Delta = 2$ and the bound in (4) is $\frac{10\sqrt{3}}{3}$, but the bound in (6) is $2\sqrt{5}$. Let G be the star with 5 vertices. Then $n = 5$, $m = 4$, $\Delta = 4$ and the bound in (4) is $\frac{8}{\sqrt{11}}$, but the bound in (6) is $\frac{8}{\sqrt{5}}$.*

Combine Theorem 2.4 with Lemma 2.5, we have

Proposition 2.6 *Let G be a graph with n vertices, $m \geq 1$ edges and maximum degree Δ. Then*

$$IE(G) \geq \max\left\{2m\sqrt{\frac{2m}{n\Delta^2 + 2m}}, \frac{2m}{\sqrt{n}}\right\} \tag{7}$$

and the equality holds if and only if $G \cong K_2$.

3 Incidence energy in subdivision and total graphs of a regular graph

In this section, we will explore the incidence energy of the subdivision graph and total graph of a regular graph. The following result is well known [2, 17, 18].

Lemma 3.1 *The spectra of $L(G)$ and $L^+(G)$ coincide if and only if the graph G is bipartite.*

We first consider the case for subdivision graphs. The subdivision graph of a graph G, denoted by $s(G)$, is the graph obtained by replacing every edge in G with a copy of P_2 ("subdividing" each edge).

Theorem 3.2 *Let G be a regular graph of n vertices and of degree d. Then*
(i) $IE(s(G)) > \frac{\sqrt{2}n(d-2)}{2} + n\sqrt{d+2}$;
(ii) $IE(s(G)) \leq (n-1)\sqrt{d} + \sqrt{d+2} + \frac{\sqrt{2}(nd-2)}{2}$, the equality holds if and only if $G \cong K_2$.

Proof. Note that the subdivision graph $s(G)$ of a simple graph is bipartite graph. It follows from Lemma 3.1 that $\mu_i(s(G)) = \mu_i^+(s(G))$, $i = 1, 2, \ldots, m+n$. By virtue of (2) and Theorem 3.4 [21] we have in fact already established the statements in the theorem. □

Next we consider the case of the total graphs. The total graph of a graph G, denoted by $t(G)$, is the graph whose vertices correspond to the union of the set of vertices and edges of G, with two vertices of $t(G)$ being adjacent if and only if the corresponding elements are adjacent or incident in G. Complete information about the spectrum of $t(G)$ is provided by D. Cvetković [1] in terms of the adjacency eigenvalues of G.

Lemma 3.3 *[1] If G is a regular graph of degree $d > 1$ with n vertices and m edges, then $t(G)$ has $m-n$ eigenvalues equal to -2 and the following $2n$ eigenvalues*

$$\frac{1}{2}(2\lambda_i(G) + d - 2 \pm \sqrt{4\lambda_i(G) + d^2 + 4}) \quad (i = 1, 2, \ldots, n). \tag{8}$$

If G is regular, then $t(G)$ is also regular. In particular, $t(G)$ is a regular graph of order $\frac{n(d+2)}{2}$ and of degree $2d$, where n and d are the order and degree of G, respectively. It follows from Lemma 3.3 and $L^+(t(G)) = D(t(G)) + A(t(G))$ that the signless Laplacian eigenvalues of $t(G)$ are $\mu_j^+(t(G)) = 2d + \lambda_j(t(G))$, $j = 1, \ldots, n, n+1, \ldots, m+n$, i.e., the signless Laplacian spectrum of $t(G)$ is

$$\begin{pmatrix} 2d-2 & \mu_1^+(t(G)) & \mu_1'^+(t(G)) & \cdots & \mu_n^+(t(G)) & \mu_n'^+(t(G)) \\ \frac{n(d-2)}{2} & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \tag{9}$$

where

$$\mu_i^+(t(G)) = \frac{5d - 2 + 2\lambda_i(G) + \sqrt{d^2 + 4 + 4\lambda_i(G)}}{2}$$

and

$$\mu_i'^+(t(G)) = \frac{5d - 2 + 2\lambda_i(G) - \sqrt{d^2 + 4 + 4\lambda_i(G)}}{2}$$

are the signless Laplacian eigenvalues of $t(G)$, $i = 1, 2, \ldots, n$.

The following result gives the bounds of the incidence energy of the total graph of a regular graph.

Theorem 3.4 *Let G be a regular graph of n vertices and degree d. Then*

(i) $IE(t(G)) \geq \frac{\sqrt{2(nd-2)}}{2}\sqrt{d-1} + (n+1)\sqrt{d} + \sqrt{3d-2}$, the equality holds if and only if $G \cong K_2$;

(ii) $IE(t(G)) < \frac{\sqrt{2n(d-2)}}{2}\sqrt{d-1} + 2n\sqrt{d} + n\sqrt{3d-2}$.

Proof. Note that the total graph of K_2 is K_3. If $d = 1$, then G is a disjoint union of copies of K_2 and hence $t(G)$ is a disjoint union of copies of K_3, i.e., if $G \cong \frac{n}{2}K_2$, then $t(G) \cong \frac{n}{2}K_3$, where n is even. It follows from Lemma 2.1 (*ii*) and $IE(K_3) = 4$ that

$$IE(t(G)) = 2n.$$

In this case, that is $d = 1$,

$$\frac{\sqrt{2}n(d-2)}{2}\sqrt{d-1} + 2n\sqrt{d} + n\sqrt{3d-2} = 3n > 2n$$

and

$$\frac{\sqrt{2}(nd-2)}{2}\sqrt{d-1} + (n+1)\sqrt{d} + \sqrt{3d-2} = n + 2 \leq 2n,$$

the equality holds if and only if $n = 2$, that is, $G \cong K_2$.

Assume that $d \geq 2$. It follows from (2) and (9) that

$$\begin{aligned} IE(t(G)) &= \frac{n(d-2)}{2}\sqrt{2(d-1)} + \sum_{i=1}^{n}\left(\sqrt{\mu_i^+(t(G))} + \sqrt{\mu_i'^+(t(G))}\right) \\ &= \frac{\sqrt{2}n(d-2)}{2}\sqrt{d-1} + 2\sqrt{d} + \sqrt{3d-2} \\ &\quad + \sum_{i=2}^{n}\left(\sqrt{\frac{5d-2+2\lambda_i+\sqrt{d^2+4+4\lambda_i}}{2}}\right. \\ &\quad \left. + \sqrt{\frac{5d-2+2\lambda_i-\sqrt{d^2+4+4\lambda_i}}{2}}\right). \end{aligned}$$

Note that $-d \leq \lambda_i(G) < d$, $i = 2, 3, \ldots, n$, by Perron-Frobenius theorem. Consider the function

$$g(t) = \sqrt{\frac{5d-2+2t+\sqrt{d^2+4+4t}}{2}} + \sqrt{\frac{5d-2+2t-\sqrt{d^2+4+4t}}{2}},$$

where $-d \leq t < d$. In what follows we shall show that $g(t)$ is increasing for $-d \leq t < d$. Let $f_1(t) = \sqrt{d^2+4+4t}$, $f_2(t) = 10d - 4 + 4t$. Then the derivative function of $g(t)$ is

$$g'(t) = \frac{(f_1(t)+1)\sqrt{f_2(t)-2f_1(t)} + (f_1(t)-1)\sqrt{f_2(t)+2f_1(t)}}{f_1(t)\sqrt{f_2(t)+2f_1(t)}\sqrt{f_2(t)-2f_1(t)}},$$

where $-d < t < d$. In order to prove that $g'(t) > 0$, we need only to show that

$$(f_1(t)+1)\sqrt{f_2(t)-2f_1(t)}+(f_1(t)-1)\sqrt{f_2(t)+2f_1(t)}>0,$$

i.e.,

$$\frac{f_1(t)\left(\sqrt{f_2(t)-2f_1(t)}+\sqrt{f_2(t)+2f_1(t)}\right)}{\sqrt{f_2(t)+2f_1(t)}-\sqrt{f_2(t)-2f_1(t)}}>1.$$

Further, we need only to indicate that

$$f_2(t)+\sqrt{(f_2(t)+2f_1(t))(f_2(t)-2f_1(t))}>2,$$

i.e.,

$$10d-4+4t+\sqrt{96d^2-80d+80td-48t+16t^2}>2.$$

Let $f(t) = 10d-4+4t+\sqrt{96d^2-80d+80td-48t+16t^2}$. Then the derivative function of $f(t)$ is

$$f^{'}(t)=4+\frac{80d-48+32t}{2\sqrt{96d^2-80d+80td-48t+16t^2}}. \tag{10}$$

It follows from (10) that $f(t)$ is increasing for $-d \le t < d, d > 1$. Therefore, the minimum of $f(t)$ on $[-d, d)$ is $f(-d) = 6d-4+\sqrt{32d^2-32d} > 2$.

Bearing in mind the above fact that $g(t)$ is increasing for $-d \le t < d$, we can deduce that

$$\begin{aligned} IE(t(G)) &\ge \frac{\sqrt{2}n(d-2)}{2}\sqrt{d-1}+2\sqrt{d}+\sqrt{3d-2}+\sum_{i=2}^{n}g(-d) \\ &= \frac{\sqrt{2}(nd-2)}{2}\sqrt{d-1}+(n+1)\sqrt{d}+\sqrt{3d-2} \end{aligned} \tag{11}$$

and

$$\begin{aligned} IE(t(G)) &< \frac{\sqrt{2}n(d-2)}{2}\sqrt{d-1}+2\sqrt{d}+\sqrt{3d-2}+\sum_{i=2}^{n}g(d) \\ &= \frac{\sqrt{2}n(d-2)}{2}\sqrt{d-1}+2n\sqrt{d}+n\sqrt{3d-2}. \end{aligned}$$

The equality in (11) holds if and only if G is a regular graph and $\lambda_1(G) = d$, $\lambda_2(G) = \cdots = \lambda_n(G) = -d$. It follows that G is 1-regular. This is impossible since $d \ge 2$.

Hence, we complete the proof of Theorem 3.4. □

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