Long Paths containing k-ordered vertices in Graphs[1]

Emlee W. Nicholson[a,b] and Bing Wei[a]
a. *Department of Mathematics, University of Mississippi University, MS 38677, USA*

b. *Winthrop University Department of Mathematics Rock Hill, SC 29733, USA*

email: nicholsone@winthrop.edu, bwei@olemiss.edu

Abstract

Let G be a graph on n vertices. If for any ordered set of vertices $S = \{v_1, v_2, \ldots, v_k\}$, that is, the vertices in S appear in order of the sequence $v_1, v_2, \ldots, v_k$, there exists a $v_1 - v_k$ (hamiltonian) path containing S in the given order, then G is k-ordered (hamiltonian) connected. In this paper we will show that if G is $(k+1)$-connected and k-ordered connected, then for any ordered set S, there exists a $v_1 - v_k$ path P containing S in the given order such that $|P| \geq \min\{n, \sigma_2(G) - 1\}$ where $\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G); uv \notin E(G)\}$ when G is not complete, otherwise set $\sigma_2(G) = \infty$. Our result generalizes several related results known before.

Keywords: long paths, k-ordered sets, k-ordered connected, k-ordered hamiltonian connected

1. Introduction

All graphs considered in this paper are finite, simple and undirected. For any graph G, we use $V(G)$ or just V to denote its vertex set and $E(G)$ or just E to denote its edge set. Let $|G|$ or $|V|$ denote the cardinality of V. Let H and S be subgraphs of G or vertex subsets of G and $v \in V(G)$. We denote the set of vertices in S that are adjacent to some vertices in H by $N_S(H)$. Likewise, $N_S(v)$ will denote the set of vertices in S that are adjacent to v. Hence the degree of v with respect to S is $|N_S(v)|$ and is denoted by $d_S(v)$. We denote the degree of v with respect to G by $d_G(v)$ or $d(v)$. Let $\delta(G) = \min\{d_G(v) : v \in V(G)\}$. For a graph G of order $n \geq 3$, we define $\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G); uv \notin E(G)\}$ when G is not complete, otherwise set $\sigma_2(G) = \infty$. A cycle (path) containing all vertices of G is called a hamiltonian cycle (path). A graph G is said to be hamiltonian if it possesses a hamiltonian cycle and to be hamiltonian connected if for any distinct vertices x, y in G there is a hamiltonian path connecting x and

[1]The work of B. Wei is supported by the Faculty Research Program of the University of Mississippi

y.

Various degree conditions have been studied for the hamiltonicity and the hamiltonian connectivity of graphs. Two well-known results regarding the hamiltonian graphs are due to Dirac and Ore.

Theorem 1 (Dirac (1952) [1]). Let G be a graph of order $n \geq 3$. If $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian.

Theorem 2 (Ore (1960) [6]). Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G is hamiltonian.

By increasing the lower bound for $\sigma_2(G)$ in Theorem 2 by just one, Ore proved the following:

Theorem 3 (Ore (1963) [7]). Let G be a graph on n vertices. If $\sigma_2(G) \geq n+1$, then G is hamiltonian connected.

Very recently, people became interested not only in finding long paths and cycles in graphs, but also in finding long paths and cycles containing any ordered set of k vertices. A vertex set $S = \{v_1, v_2, \ldots, v_k\}$ is said to be an ordered set if the vertices in S appear in the order of the sequence $v_1, v_2, \ldots, v_k$. A graph G is said to be k-ordered (hamiltonian) if for every ordered set of vertices S such that $|S| = k, (k \leq n)$, G contains a (hamiltonian) cycle C encountering S in the given order. A graph G is said to be k-ordered (hamiltonian) connected if for every ordered set of vertices $S = \{v_1, v_2, \ldots, v_k\}$ such that $|S| = k, (k \leq n)$, G contains a $v_1 - v_k$ (hamiltonian) path P encountering S in the given order. We define a requisite path as a $v_1 - v_k$ path containing S in the given order. Let $p(S)$ be the length of the longest requisite $v_1 - v_k$ path containing S. Set $p_o(G) = \min\{p(S) : \text{S is an ordered set with } |S| = k\}$. Of course, if a graph is hamiltonian (hamiltonian connected), then it is obviously k-ordered hamiltonian (k-ordered hamiltonian connected) when $2 \leq k \leq 3$. This observation led Ng and Schultz to investigate the degree conditions of k-ordered hamiltonian graphs and they obtained:

Theorem 4 (Ng and Schultz (1997) [5]). Let G be a graph of order $n \geq 3$ and let k be an integer with $3 \leq k \leq n$. If $\sigma_2(G) \geq n + 2k - 6$, then G is k-ordered hamiltonian.

Faudree et al. improved Theorem 4 as follows:

Theorem 5 (Faudree et al. (2003) [2]). Let k be an integer with $3 \leq k \leq \frac{n}{2}$ and let G be a graph of order n. If $\sigma_2(G) \geq n + \frac{(3k-9)}{2}$, then G is k-ordered hamiltonian.

In the same paper, Ng and Schultz considered the k-ordered hamiltonian connectedness of graphs under the same conditions as in Theorem 4 but for $k \geq 4$ and got the following:

Theorem 6 **(Ng and Schultz (1997) [5]).** Let G be a graph of order $n \geq 4$ and let k be an integer with $4 \leq k \leq n$. If $\sigma_2(G) \geq n + 2k - 6$, then G is k-ordered hamiltonian connected.

Corollary 1 **(Ng and Schultz (1997) [5]).** Let G be a graph of order $n \geq 4$ and let k be an integer such that $4 \leq k \leq n$. If $\delta(G) \geq \frac{n}{2} + k - 3$, then G is k-ordered hamiltonian connected.

By assuming that n is sufficiently large, Faudree et al. reduced the lower bound for $\sigma_2(G)$ in Theorem 6 and proved the following result:

Theorem 7 (Faudree et al. (2003) [3]). Let G be a graph of sufficiently large order n and $k \geq 3$. If $\sigma_2(G) \geq n + \frac{3k-6}{2}$, then G is k-ordered hamiltonian connected.

The purpose of this paper is to investigate the minimum ordered path length for any $(k+1)$-connected and k-ordered connected graph of order n. Our result is as follows:

Theorem 8. Let G be a $(k+1)$-connected $(k \geq 2)$ graph on n vertices. If G is k-ordered connected, then $p_o(G) \geq \min\{n-1, \sigma_2(G) - 2\}$.

Notice that Theorem 8 gives a lower bound for the length of any longest path connecting two distinct vertices and containing S ($|S| = k$) in the given order for a k-ordered connected graph. Theorem 8 generalizes Theorem 3 if we take $k = 2$. Since a graph G is k-ordered connected if G is k-ordered hamiltonian, by Theorem 5 and the fact that $\sigma_2(G) \geq n + \frac{3k-6}{2}$ implies G is $(k+1)$-connected for $k \geq 3$, one can easily check that Theorem 8 also generalizes Theorem 7.

We will first construct the following graphs to demonstrate the sharpness of the bounds of Theorem 8 and then show several lemmas in Section 2. The proof of Theorem 8 will be given in Section 3.

That the lower bound for $p_o(G)$ in Theorem 8 is best possible is shown by the following example. The graph G is composed of 3 subgraphs. The first subgraph is a clique on k vertices less the edges of a cycle on k vertices $K_k - E(C_k)$. The next subgraph is a clique K_{k-1}. The last subgraph is a set of independent vertices, M. Let $S = \{v_1, v_2, \ldots, v_k\}$ be the vertices of $K_k - E(C_k)$ and $E(C_k) = \{v_1v_2, v_2v_3, \cdots, v_{k-1}v_k, v_kv_1\}$. Let $V(K_{k-1}) = \{x_1, x_2, \ldots, x_{k-1}\}$ and $V(M) = \{u_1, u_2, \ldots, u_s\}$ where $s \geq k$. All possible edges exist between M and K_{k-1} and all possible edges exist between K_{k-1} and $K_k - E(C_k)$. The vertices of M are adjacent to only those vertices in $K_k - E(C_k)$ that possess an odd index. For $k \geq 7$, we have $\sigma_2(G) = d(u_i) + d(u_j) \leq 2((k-1) + \frac{k+1}{2}) = 3k - 1 = n - s + k$. Since $s \geq k$, $\sigma_2(G) \leq n$. Note that between every pair of consecutive vertices in S on P, there must be an element of K_{k-1}. Since P begins and ends in S and $|S| = k$, P must miss some $u_i \in M$ and $|P| \leq n-1$. In this case,

$p_o(G) = 3k - 3 = \min\{n-1, \sigma_2(G) - 2\}$.

Also consider the following example which indicates that the connectivity condition $k+1$ in Theorem 8 is best possible. Let G be a graph composed of 3 cliques: K_t, K_s, and K_k, and $K_k - E(C_k)$ as in the previous example. Let $k-2 \leq s < 2k-7$ and $k-2 \leq t < 2k-7$. All possible edges exist between the vertices of K_t and K_k, K_s and K_k and K_k and $K_k - E(C_k)$. Let $V(K_k) = \{x_1, x_2, \ldots, x_k\}$ and $V(K_t) = \{t_1, t_2, \ldots, t_t\}$. Let $S = \{v_1, v_2, \ldots, v_k\}$ be the vertices of $K_k - E(C_k)$. It is easy to check that G is k-connected. Since $t \geq k-2$ and $s \geq k-2$, we can choose two nonadjacent vertices in $K_k - E(C_k)$ to be those in the minimum degree sum. Then $\sigma_2(G) = 2(k-3) + 2k = 4k - 6$. Then a longest requisite path P must miss all the vertices of either K_t or K_s. Thus $p_o(G) \leq 2k + max\{s,t\} - 1 < 4k - 8 = min\{n-1, \sigma_2(G) - 2\}$. Actually, G can be generalized by adding more copies of K_s to a graph of arbitrarily large order.

2. Several Lemmas

For a path P with a given orientation and $u \in V(P)$, let u^+ denote the first successor of u on P and u^- denote the first predecessor of u on P. Also, if $v \in V(G)$ then $N_P^+(v)$ and $N_P^-(v)$ denote the set of vertices succeeding the neighbors of v on P and the set of vertices preceding the neighbors of v on P respectively. For $u, v \in V(P)$, $P[u,v]$ denotes the subpath of the path P from u to v in the given direction. For $P[u^+, v]$ we write $P(u,v]$. Similarly, for $P[u, v^-]$, we write $P[u,v)$. We use $\overline{P}[u,v]$ to denote the subpath of P from u to v in the reverse order.

In this section, we assume that $p_o(G) < n-1$ and let P be a longest requisite path for any ordered set $S = \{v_1, v_2, \ldots, v_k\}$ with $|P| < n$, then there exists some component $H \subseteq G - P$. Let $N_P(H) = \{x_1, x_2, \ldots, x_t\}$ in order along P and set $P_i = P(x_i, x_{i+1})$ where $1 \leq i \leq t-1$. We will first investigate some properties of P. For $u, v \in V(P)$, we call any $P(u,v)$ a good segment of P if $V(P(u,v)) \cap S = \emptyset$. Since G is $(k+1)$-connected, $t \geq k+1$. Since $|S| = k$ and the endpoints of P are by definition the first and last vertex in the ordered set S, by the pigeonhole principle, at least two of those segments of P defined by $N_P(H)$ are good. Let P_i for some $1 \leq i \leq t-1$, be a good segment. A vertex v in $V(P_i)$ is said to be insertible on P if v is adjacent to two consecutive vertices in $V(P - P_i)$. For any two vertices u, v in G and a subgraph H of G, we use uP_Hv to denote a longest path connecting u and v with all internal vertices in H. We will first present several lemmas regarding P.

Lemma 1. Let $u, v \in V(P)$. If $N_P^+(x) \cap N(x) \cap P[u,v] = \emptyset$, then $d_{P[u,v)}(x) \leq \frac{|P[u,v]|+1}{2}$.

Lemma 2. For every component $H \subseteq G - P$, $N_P^+(H) \cap N(H) = \emptyset$.

Proof: Suppose there exist $u, v \in V(H)$ such that without loss of generality $N_P^+(v) \cap N(u) \neq \emptyset$. Then by inserting uP_Hv, we get a requisite path that is longer than P; a contradiction. □

The following two lemmas give some structural properties for the vertices in a good segment.

Lemma 3. Let P_i $(1 \leq i \leq t)$ be a good segment and $x \in V(P_i)$.

(i) If for every $y \in V(P(x_i, x))$, y is insertible, then all vertices in $P(x_i, x)$ can be inserted into $V(P) - V(P_i)$.

(ii) If for every $y \in V(P(x, x_{i+1}))$, y is insertible, then all vertices in $P(x, x_{i+1})$ can be inserted into $V(P) - V(P_i)$.

Proof: We only prove (i) here and (ii) can be easily checked by a symmetric argument to that of the proof of (i).

The proof of (i) is by induction. If $|V(P(x_i, x))| = 1$, then the result holds by the definition of an insertible vertex. Suppose that $|V(P(x_i, x))| \geq 2$ and assume that the result holds for all integers p when $|V(P(x_i, x))| \leq p$. Now we consider $|V(P(x_i, x))| = p + 1$. Since x^- is insertible, there are two consecutive vertices say w and w^+ in $V(P - P_i)$ such that $x^-w \in E(G)$ and $x^-w^+ \in E(G)$. When $N(y) \cap \{w, w^+\} = \emptyset$ for any $y \in V(P(x_i, x^-))$, as $|V(P(x_i, x^-))| = p$ and x^- can be inserted using w and w^+, the result holds by the induction hypothesis. When $N(y) \cap \{w, w^+\} \neq \emptyset$ for some $y \in V(P(x_i, x^-))$, then choose the first such vertex, say y_1, in $V(P(x_i, x^-))$ and we can insert all vertices in $P[y_1, x^-]$ into $V(P) - V(P_i)$ using w and w^+. Since $|V(P(x_i, y_1))| < p$, by the induction hypothesis and the choice of y_1, all vertices in $P(x_i, y_1)$ can be inserted into $V(P) - V(P_i)$. Hence (i) holds. □

Lemma 4. Let P_i be any good segment and $u_i \neq u_i'$ be two vertices in $V(P_i)$ with $u_i' \in V(P[u_i, x_{i+1}))$ such that all vertices in $V(P(x_i, u_i)) \cup V(P(u_i', x_{i+1}))$ are insertible.

(i) If there is a vertex x in $N(y) \cap V(P - P[x_i, x_{i+1}])$ for some y in $V(P(x_i, u_i])$, then $N(y') \cap \{x^+, x^-\} = \emptyset$ for any $y' \in V(P[u_i', x_{i+1}))$.

(ii) All vertices in $V(P(x_i, u_i)) \cup V(P(u_i', x_{i+1}))$ can be inserted into a path containing $P[v_1, x_i] \cup P[x_{i+1}, v_k]$.

Proof: By contradiction, suppose there are some $x \in V(P - P[x_i, x_{i+1}])$, $y \in V(P(x_i, u_i])$ and $y' \in V(P[u_i', x_{i+1}))$ such that $\{x^+, x^-\} \cap N(y') \neq \emptyset$ and $xy \in E(G)$. Choose such y and y' with $|V(P(x_i, y))| + |V(P(y', x_{i+1}))|$ as small as possible (that is, $N(x) \cap V(P(x_i, y)) = \emptyset$ and $(N(x^+) \cup N(x^-)) \cap V(P(y', x_{i+1})) = \emptyset$). If $x \in V(P[v_1, x_i))$, then we can find a requisite path $P' = P[v_1, x)xyP(y, y')y'x^+P(x^+, x_i)x_iP_Hx_{i+1}P(x_{i+1}, v_k]$ when $x^+y' \in$

$E(G)$ (see Figure 1) and
$P' = P[v_1, x^-)x^-y'\overline{P}(y', y)yxP(x, x_i)x_iP_Hx_{i+1}P(x_{i+1}, v_k]$ when $x^-y' \in E(G)$ (see Figure 2).

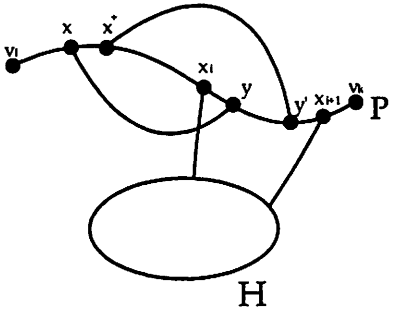

Figure 1

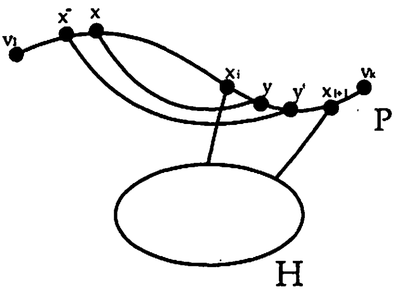

Figure 2

If $x \in V(P(x_{i+1}, v_k])$, then we can find a requisite path
$P' = P[v_1, x_i)x_iP_Hx_{i+1}P(x_{i+1}, x)xyP(y, y')y'x^+P(x^+, v_k]$, when $x^+y' \in E(G)$ (see Figure 3) and
$P' = P[v_1, x_i)x_iP_Hx_{i+1}P(x_{i+1}, x^-)x^-y'\overline{P}(y', y)yxP(x, v_k]$ when $x^-y' \in E(G)$ (see Figure 4).

In either case, by the choice of y, y' and Lemma 3, we can find a requisite path longer than P by inserting all vertices in $V(P(x_i, y)) \cup V(P(y', x_{i+1}))$ into P', contradicting the choice of P.

(ii) By Lemma 3 and Lemma 4(i), we can easily verify (ii). □

Next, we will use the parameter $D(G)$ as defined by Fraisse and Jung in [4]. For any 2-connected graph G, let $D(G)$ be the maximum integer m such that for any two distinct vertices u, v in G, there is a path of length at least m connecting u and v. For a complete graph K_n ($n \geq 2$), set $D(K_n) = n - 1$. If G has connectivity one, set $D(G) = \max\{D(G') :$

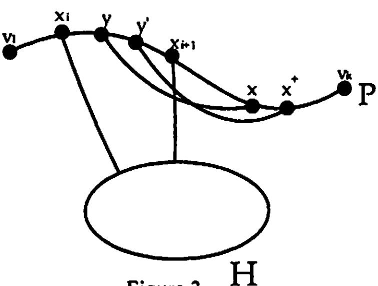

Figure 3

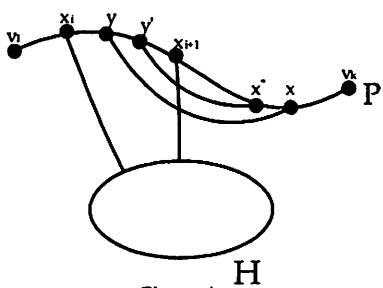

Figure 4

G' is an endblock of $G\}$. For an arbitrary graph, set $D(G) = \max\{D(G') :$ G' is a component of $G\}$.

Lemma 5 (Fraisse and Jung (1989) [4]). Let G be a noncomplete connected graph. Then there exist nonadjacent vertices v_1 and v_2 in G such that v_i is not a cut vertex of G and $D(G) \geq d(v_i)$ $(i = 1, 2)$.

Thus for a complete graph, $d(v) \leq D(G)$ for all $v \in V(G)$. If G is not 2-connected, let G_1 and G_2 be distinct endblocks of G. Then for any vertex $v_i \in V(G_i)$ that is not a cut vertex, there exists a path $P[v_i, v_j]$ such that $|P[v_i, v_j]| \geq D(G_1) + D(G_2) + 1$.

Let B_1 be an endblock of H with $D(B_1) = D(H)$. If H is 2-connected or $|H| \leq 2$, set $B_1 = H$. Let c_1 be the unique cut vertex of H in B_1 when H is not 2-connected and $|H| \geq 3$. Otherwise, let c_1 be an arbitrary vertex of H. Set $B = B_1 - \{c_1\}$.

Lemma 6. Let $P(u, v)$ be a good segment of P. Let $y_1 \neq y_2 \in V(H)$ with $\{y_1, y_2\} \cap V(B) \neq \emptyset$ and $uy_1 \in E$ and $vy_2 \in E$. Then there exists a good segment $P(u', v') \subseteq P(u, v)$ such that $V(P(u', v')) \cap N_P(H) = \emptyset$ and $|P(u', v')| \geq D(H) + 1$.

Proof: Without loss of generality let $y_1 \in V(B)$. Let u' be the last vertex in $V(P[u, v))$ such that $u'y_1 \in E(G)$. Since $vy_2 \in E$, choose v' to be the first vertex in $V(P(u', v])$ such that $v'y_2' \in E(G)$ for some $y_2' \in V(H) - \{y_1\}$. Since $P(u, v)$ is a good segment, $P(u', v')$ is a good segment such that $P(u', v') \subseteq P(u, v)$ and $V(P(u', v')) \cap N(H) = \emptyset$. Since P is maximal and $D(B_1) = D(H)$, this implies that $|P(u', v')| \geq D(B_1) + 1 = D(H) + 1$. □

Lemma 7. If G is $(k + 1)$-connected and $|P| < \sigma_2(G) - 1$, then for every component $H \subseteq G - P$ there exists a vertex $v \in H$ such that $\mathrm{d}_G(v) < \frac{\sigma_2(G)}{2}$.

Proof: Let $L = \frac{\sigma_2(G)}{2}$. Suppose that there exists a component $H \subseteq G - P$ such that for every vertex $v \in H$, $\mathrm{d}_G(v) \geq L$. Since $|P| < 2L - 1$ and by Lemma 2 we may assume $|B_1| \geq 2$. We first claim that $|B_1| \geq 3$.

Otherwise, assume $|B_1| = 2$. Then for any vertex z in B_1 we have $|N_P(z)| = d_G(z) - 1 \geq L - 1$. Hence, we have $|N_P(z)| = L - 1$ and $|N_P(H)| = |N_P(z)| \geq k + 1$ as G is $(k + 1)$-connected. By Lemma 2, we know for any i, $|P_i| \geq 1$ where $1 \leq i \leq t - 1$. Since $|S| = k$ and the longest requisite path P begins at v_1 and terminates at v_k where v_1 and v_k are the first and last vertices of the ordered set S respectively, there exists i_1 and i_2 such that $V(P_{i_1}) \cap S = \emptyset$ and $V(P_{i_2}) \cap S = \emptyset$. Thus $|P_{i_1}| \geq 2$ and $|P_{i_2}| \geq 2$. Since for any $i \neq i_1, i_2$ with $1 \leq i \leq L - 2$ we have $|P_i| \geq 1$. Thus $|P| \geq L - 1 + L - 2 + 2 = 2L - 1$; a contradiction. Hence $|B_1| \geq 3$.

Next we will show the following claim which is useful in our later proofs.

Claim 1: If $|P| < 2L-1$, then there exists $1 \le i_1 < i_2 < \ldots < i_{k+1} \le t$ such that

(i) $|\{x_{i_1}, \ldots, x_{i_{k+1}}\} \cap N_P(B)| \ge k$

(ii) For any $i_p \ne i_s$, there exist distinct vertices $v, v' \in V(H)$ such that $x_{i_p}v \in E(G)$, $x_{i_s}v' \in E(G)$, and $\{v, v'\} \cap V(B) \ne \emptyset$, that is $|vP_Hv'| \ge D(H)+1$.

In fact since $|B_1| \ge 3$, we have $|B| \ge 2$. Define $X = \{x_i : d_B(x_i) \ge 2, x_i \in V(P)\}$. Then we have $X \subseteq N_P(B)$ and $\sum_{y \in V(B)} d_{V(P)-X}(y) = |N_{V(P)-X}(B)| \le t - |X|$. Hence

$$\begin{aligned} |B|L &\le \sum_{y \in V(B)} d_G(y) \\ &\le \sum_{y \in V(B)} (d_{B_1}(y) + d_X(y)) + (t - |X|) \\ &\le |B|(|B| + |X|) + t - |X|. \end{aligned}$$

If $|X| \ge k$, since G is $(k+1)$-connected, Claim 1 holds by the definition of X. Assume $|X| \le k-1$ and set $r = k+1-|X|$ $(r \ge 2)$. Notice that by Lemma 2, $k+1 \le t < L$ which implies $t - L \le -1$. From the above inequality, we have

$$\begin{aligned} (|B|-1)L &\le |B|(|B|+|X|) + t - |X| - L \\ &= |B|(|B|+|X|+1) - |B| - |X| + t - L \\ &\le |B|(|B|+|X|+1) - (|B|+|X|+1) \\ &= (|B|-1)(|B|+|X|+1). \end{aligned}$$

Since $|B| \ge 2$, we obtain $(|B_1| + |X|) \ge L \ge k+2$ which implies that $|B_1| \ge k+2-|X| = r+1$. Since $|P| - |X| \ge 2t-1-|X| > t - |X| \ge k+1-|X| = r$ and $G - (X \cup \{c_1\})$ is $(r-1)$-connected, there exist distinct vertices $z_1, z_2, \ldots, z_{r-1}$ in $V(B)$ and $y_1, y_2, \ldots, y_{r-1}$ in $V(P) - X$ such that $z_iy_i \in E(G)$ $(1 \le i \le r-1)$. Set $Y = \{y_1, y_2, \ldots, y_{r-1}\}$ and $Z = \{z_1, z_2, \ldots, z_{r-1}\}$. Notice that $|X \cup Y| = k+1-r+r-1 = k$. Similarly, $|X \cup Z| = k$. Because G is $(k+1)$-connected, there exists some $z_r \in V(H) - Z$ and $y_r \in V(P) - (X \cup Y)$ such that $z_ry_r \in E(G)$. By the definition of X, we can easily check that $X \cup Y \cup \{y_r\}$ is a set that satisfies (i) and (ii).

Now we turn to prove Lemma 7. By Lemma 5, there exists $v \in V(H)$ such that $d_H(v) \le D(H)$. Since G is $(k+1)$-connected, $|S| = k$, and P is a $v_1 - v_k$ path which has at least two good segments. Thus, by

Claim 1 and Lemmas 5 and 6, we have $|P| \geq 2|N_P(v)| - 1 + 2(D(H)) \geq 2d_P(v) - 1 + 2d_H(v) \geq 2L - 1$; a contradiction. □

Lemma 8. If $|P| < \sigma_2(G) - 1$, there is only one component H of $G - P$.

Proof: Suppose there are at least two components $H, H' \subseteq G - P$. Then by Lemma 7, we can take $v \in H$ such that $\mathrm{d}(v) < \frac{\sigma_2(G)}{2}$ and $v' \in H'$ such that $\mathrm{d}(v') < \frac{\sigma_2(G)}{2}$. Since v and v' are nonadjacent, this implies $\sigma_2(G) \leq \mathrm{d}(v) + \mathrm{d}(v') < \sigma_2(G)$, a contradiction. □

So for every longest requisite path P, if $|P| < \sigma_2(G) - 1$, then $G - P$ has only one component, H. Recall $N(H) = \{x_1, x_2, \ldots, x_t\}$ in order along P and $P_i = P(x_i, x_{i+1})$ where $1 \leq i \leq t - 1$. The following corollary is a consequence of Lemmas 7 and 8 and the definition of $\sigma_2(G)$.

Corollary 2. For any $v \in V(P) - N_P(H)$, $d_{G-P}(v) = 0$ and $d_G(v) > \frac{\sigma_2(G)}{2}$.

Lemma 9. If $|P| < \sigma_2(G) - 1$, then for every good segment P_i where $1 \leq i \leq t - 1$, there exist at least two noninsertible vertices in P_i.

Proof: Suppose there is a good segment P_i with no noninsertible vertices, then we can find a requisite path $P' = P[v_1, x_i)x_i P_H x_{i+1} P(x_{i+1}, v_k]$. Since every vertex in P_i can be inserted into P' by Lemma 3, we can find a requisite path that is longer than P, contradicting the choice of P.

Suppose there exists some good segment, P_i with only one noninsertible vertex v. Then applying Lemmas 3 and 4 we can insert all vertices in $V(P_i) - \{v\}$ into $V(P - P_i)$ to get a requisite path P' including all vertices in $V(P) - \{v\}$ and at least one vertex in H. Thus $|P'| \geq |P|$. Since v is the only noninsertible vertex in P_i and by Corollary 2, $d_{G-P}(v) = 0$, v is itself a component of $G - P'$. And since $d_G(v) > \frac{\sigma_2(G)}{2}$, this component contains no vertex y such that $d(y) < \frac{\sigma_2(G)}{2}$ which contradicts Lemma 7. □

3. Proof of Theorem 8

By contradiction, assume that $p_o(G) < \min\{n - 1, \sigma_2(G) - 2\}$. Let P be a longest requisite path for some ordered set $S = \{v_1, v_2, \ldots, v_k\}$ with $|P| < \min\{n, \sigma_2(G) - 1\}$. By Lemma 8 there exists exactly one component $H \subseteq G - P$. Let $N_P(H) = \{x_1, x_2, \ldots, x_t\}$ in order along P. Since G is $(k+1)$-connected we know that P contains at least two good segments P_i and P_j with $i < j$ such that there exist distinct vertices $w_q, w'_q \in V(H)$ $(q = i, j)$ with $x_q w_q \in E(G)$, $x_{q+1} w'_q \in E(G)$ and $|w_q P_H w'_q| \geq D(H) + 1$ by Claim 1, and $V(P_q) \cap N_P(H) = \emptyset$ and $|P_q| \geq D(H) + 1$ by Lemma 6 . Then by Corollary 2, for any vertex $x \in V(P_i) \cup V(P_j)$, we have that $d_{G-P}(x) =$

0. By Lemma 9, we know that there are at least two noninsertible vertices in P_i and P_j respectively. Choose u_q and u'_q ($q = i, j$) to be the first and last noninsertible vertices in P_q, respectively. Notice that a path containing the vertex set $V(P) - (V(P_i) \cup V(P_j))$ in the same order as that in P is a requisite path. We consider the following two cases:

Case 1: $N(u_i) \cap V(P_j) = \emptyset$ or $N(u'_i) \cap V(P_j) = \emptyset$.

Since u_i is noninsertible and $N(u_i) \subseteq V(P)$, by applying Lemma 1 to $P - (P_i \cup P_j)$, we have $d_{P-(P_i \cup P_j)}(u_i) \leq \frac{|P-(P_i \cup P_j)|+3}{2}$ and $d_{P_i}(u_i) \leq |P_i| - 1$. As $d_{P_j}(u_i) = 0$, $d(u_i) = d_{P-(P_i \cup P_j)}(u_i) + d_{P_i}(u_i)$. Choose $h \in V(H)$ such that $d_H(h) \leq D(H)$. Then by Lemmas 1 and 2, $d_P(h) \leq \frac{|P-(P_i \cup P_j)|+3}{2}$. Now

$$\begin{aligned} \sigma_2(G) &\leq d(u_i) + d(h) \\ &\leq \frac{|P-(P_i \cup P_j)|+3}{2} + |P_i| - 1 + \frac{|P-(P_i \cup P_j)|+3}{2} + D(H) \\ &= |P - (P_i \cup P_j)| + |P_i| + D(H) + 2. \end{aligned}$$

Since $|P_j| \geq D(H) + 1$, $\sigma_2(G) \leq |P| + 1$; a contradiction. Thus $N(u_i) \cap V(P_j) \neq \emptyset$. Symmetrically, $N(u'_i) \cap V(P_j) \neq \emptyset$.

Case 2: $N(u_i) \cap V(P_j) \neq \emptyset$ and $N(u'_i) \cap V(P_j) \neq \emptyset$.

Case 2.1 There are some $z \neq w$ in $V(P_j)$ such that $N(z) \cap V(P(x_i, u_i]) \neq \emptyset$ and $N(w) \cap V(P[u'_i, x_{i+1})) \neq \emptyset$.

Choose such z and w with $|P(z, w)|$ as small as possible and, subject to that, choose $b_i \in N_P(z) \cap P(x_i, u_i]$ and $b'_i \in N_P(w) \cap P[u'_i, x_{i+1})$ such that $|P(x_i, b_i]| + |P[b'_i, x_{i+1})|$ is as small as possible.

Define $P' = P[v_1, x_i)x_iP_Hx_{i+1}P(x_{i+1}, z)zb_iP(b_i, b'_i)b'_iwP(w, v_k]$ when $z \in P(x_j, w)$ or $P' = P[v_1, x_i)x_iP_Hx_{i+1}P(x_{i+1}, w)wb'_i\overline{P}(b'_i, b_i)b_izP(z, v_k]$ when $w \in P(x_j, z)$, see Figure 5.

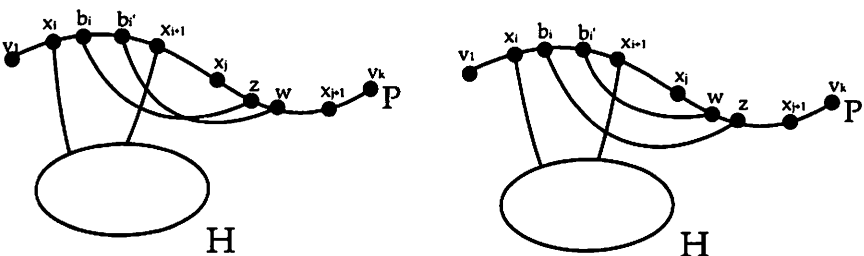

Figure 5

It is easy to check that P' is a requisite path. By the choices of w, z, b_i, b'_i and Lemmas 3 and 4, $N(u_i) \cap V(P(w, z)) = \emptyset$ and all vertices in $V(P(x_i, b_i)) \cup V(P(b'_i, x_{i+1}))$ can be inserted into P'. Thus by the maximality of P, we can conclude $|P(w, z)| \geq D(H) + 1$ (or $|P(z, w)| \geq D(H) + 1$).

As in Case 1, we choose $h \in V(H)$ such that $d_H(h) \le D(H)$. Then

$$\begin{aligned}\sigma_2(G) &\le d(u_i) + d(h)\\ &\le \tfrac{|P-(P_i \cup P(w,z)|+3}{2} + |P_i| - 1 + \tfrac{|P-(P_i \cup P(w,z))|+3}{2} + D(H)\\ &= |P - (P_i \cup P(w,z))| + |P_i| + D(H) + 2.\end{aligned}$$

Thus, $\sigma_2(G) \le |P| + 1$; a contradiction.

Case 2.2 There is only one vertex, say z, in $V(P_j)$ such that $zu_i \in E(G)$, $zu'_i \in E(G)$ and $N(y) \cap (V(P_j) - \{z\}) = \emptyset$ for every $y \in V(P(x_i, u_i)) \cup V(P(u'_i, x_{i+1}))$.

Symmetrically, we may assume that there is only one vertex, say m, in $V(P_i)$ such that $mu_j \in E(G)$, $mu'_j \in E(G)$ and $N(y') \cap (V(P_i) - \{m\}) = \emptyset$ for every $y' \in V(P(x_j, u_j)) \cup V(P(u'_j, x_{j+1}))$.

If $|P_j| \ge D(H) + 2$, then, ignoring z, we still have $D(H) + 1$ available vertices all of which are not adjacent to both v_i and u'_i. By using the same method as that in Case 1, we can get $|P| \ge \sigma_2(G) - 1$, a contradiction.

If $|P_j| = D(H) + 1$, symmetrically, we may assume that $|P_i| = D(H) + 1$. This implies that u_i and u'_i are the first and last vertices in P_i, respectively. Similarly, u_j and u'_j are the first and last vertices in P_j, respectively. Consider $P'_1 = P[v_1, u_i)u_i z P(z, u'_j)u'_j m P(m, x_j)x_j P_H x_{j+1} P(x_{j+1}, v_k]$ and $P'_2 = P[v_1, m)mu_j P(u_j, z) z u'_i P(u'_i, x_j) x_j P_H x_{j+1} P(x_{j+1}, v_k]$. See Figure 6. Clearly, P'_1 and P'_2 are both requisite paths. From P'_1, we see that $|P(u_i, m)| + |P[u_j, z)| \ge D(H) + 1$, otherwise $|P'_1| > |P|$. From P'_2, we see that $|P(m, u'_i)| + |P(z, u'_j]| \ge D(H) + 1$, otherwise $|P'_2| > |P|$. Including m and z in our calculation, we have that $|P_i| + |P_j| \ge 2D(H) + 2 + 2 = 2D(H) + 4$ which contradicts the fact that $|P_i| = |P_j| = D(H) + 1$. □

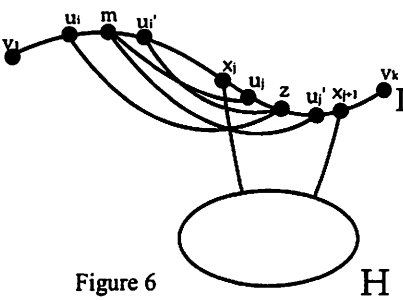

Figure 6

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