Self-Complementary Magic Squares

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Abstract

A *magic square of order* n is an $n \times n$ array of integers from $1, 2, \ldots, n^2$ so that the sum of the integers in each row, column and the diagonal is the same number. Two magic squares are said to be *equivalent* if one can be obtained from the other by rotation or reflection. If every entry a of a magic square M of order n is replaced by n^2+1-a, then we obtain the *complement* of M (which is also a magic square of order n). A magic square is said to be *self-complementary* if it is equivalent to its complement. In this paper, we prove a structural theorem which characterizes self-complementary magic squares. Further, we present a method of construction for self-complementary magic squares of even order. This construction, together with the structural theorem and some known results on magic squares imply the existence of self-complementary magic squares of order n for every $n \geq 3$.

1 Introduction

A *magic square of order* n is an $n \times n$ array of integers from $1, 2, \ldots, n^2$ so that the sum of the integers in each row, column and the diagonal is the same number. The common sum is called the *magic sum*. It is easy to see that the magic sum of a magic square of order n is $S_n = \frac{n(n^2+1)}{2}$. Magic squares are old mathematical objects and because of their enticing nature, not only they have been the subject of much discussion in recreational mathematics, but they have also been explored by the professional mathematicians.

Two magic squares M_1 and M_2 are said to be *equivalent* if one can be obtained from the other by rotation or reflection. This is equivalent to saying that $\sigma(M_1) = M_2$ for some $\sigma \in D_4$ where D_4 denotes the dihedral group of order 8.

Let M be a magic square of order n. If every entry a of M is replaced by n^2+1-a, then the resulting square is denoted by $\overline{M}$ and is called the *complement* of M. It turns out that $\overline{M}$ is also a magic square of order n. This is easy to see since the sum $\sum(n^2+1-a)$, when taken over all entries a in a row or column, or diagonal is equal to the magic sum S_n.

1	6	12	15
11	16	2	5
8	3	13	10
14	9	7	4

(a)

16	11	5	2
6	1	15	12
9	14	4	7
3	8	10	13

(b)

Figure 1: A magic square of order 4 and its complement

Figure 1(a) demonstrates a magic square of order 4 whose complement is depicted in Figure 1(b). Note that, for any magic square of order n, $M+\overline{M}$ is the matrix $(n^2+1)J_n$ where J_n denotes the $n \times n$ matrix in which all the entries are equal to 1.

A magic square M is said to be *self-complementary* if M is equivalent to its complement. Hence, one can say that M is a self-complementary magic square of order n if there exists a mapping $\sigma \in D_4$ such that $M+\sigma(M) = (n^2+1)J_n$. In this case, $\sigma(M)$ is the complement of M. For example, the magic square of Figure 1(a) is not self-complementary whereas the magic square M_1 of Figure 2(a) is self-complementary since it is equivalent to its complement (which is shown in Figure 2(b)).

1	2	15	16
12	14	3	5
13	7	10	4
8	11	6	9

(a)

16	15	2	1
5	3	14	12
4	10	7	13
9	6	11	8

(b)

Figure 2: A self-complementary magic square M_1 of order 4

The magic square M_2 depicted in Figure 3 is another example of a self-complementary magic square. Note that, for the magic square M_1, we have $\rho(M_1) = \overline{M_1}$ where ρ denotes the central vertical reflection of M_1, whereas for the magic square M_2, we have $\pi(M_2) = \overline{M_2}$ where π denotes a 180-degree rotation.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Figure 3: A self-complementary magic square M_2 of order 5

In Section 2, we shall show that all self-complementary magic squares have these properties (see Theorem 2). That is, if M is a self-complementary magic square of order n, then either $\overline{M}$ is the reflected copy (vertically or horizontally) of M or else $\overline{M}$ is a 180-degree rotation of M.

In Section 3, we shall present a construction for self-complementary magic squares of order n where n is even. This construction, together with the characterization theorem of Section 2 (Theorem 2) and some known results on magic squares imply the existence of self-complementary magic square of order n for every $n \geq 3$.

2 Characterization

We begin, in this section, by describing two types of magic squares of order n and show that any self-complementary magic square of order n must belong to one of these two types.

Let $M = (a_{i,j})$ be a magic square of order n where $a_{i,j}$ denotes the integer at the (i, j) position (or the $(i, j) -$ *cell*).

Under a 180-degree rotation π, the position (i, j) of M is mapped to the $(n - i + 1, n - j + 1)$ position. We shall call $(n - i + 1, n - j + 1)$ the *π-rotation position* of (i, j). Then M is said to be *ro-symmetrical* if whenever two entries are in their π-rotation positions, then their sum is equal to $n^2 + 1$. That is, $a_{i,j} + a_{n-i+1,n-j+1} = n^2 + 1$ for all $1 \leq i, j \leq n$. As such, for a ro-symmetrical magic square M of order n where n is odd, the integer $\frac{n^2+1}{2}$ must be at the central cell of M. Moreover we see that

$M + \pi(M) = (n^2 + 1)J_n$ and this means that $\pi(M)$ is the complement of M which in turn implies that M is self-complementary.

Under a central vertical reflection ρ, the position (i, j) of M is mapped to the $(i, n - j + 1)$ position. We shall call $(i, n - j + 1)$ the *ρ-reflection position* of (i, j). Then M is said to be *ref-symmetrical* if whenever two entries are in their ρ-reflection positions, then their sum is equal to $n^2 + 1$. That is, $a_{i,j} + a_{i,n-j+1} = n^2 + 1$ for all $1 \leq i, j \leq n$.

Note that in the above definition, we could use a central horizontal reflection instead. In this case, the horizontal reflection position of (i, j) is the position $(n - i + 1, j)$.

Since M is a ref-symmetrical magic square of order n and $M + \rho(M) = (n^2 + 1)J_n$, it follows that $\rho(M)$ is the complement of M and M is self-complementary.

The preceding observations lead to the following theorem.

Theorem 1 *Let M be a magic square of order n. If M is ro-symmetrical or ref-symmetrical, then M is self-complementary.*

The rest of this section is to prove that the converse of Theorem 1 is also true.

Theorem 2 *Let M be a magic square of order n.*

(i) Suppose n is odd. Then M is self-complementary if and only if M is ro-symmetrical.

(ii) Suppose n is even. Then M is self-complementary if and only if either M is ref-symmetrical, or else M is ro-symmetrical in which case $n \equiv 0 (\text{mod } 4)$.

PROOF: Let $M(k \times k)$ denote the central $k \times k$ subsquare of M.

Let φ be a mapping defined by

$$\varphi(x) = \overline{x}$$

for all $x \in \{1, 2, \ldots, n^2\}$, where $\overline{x} = n^2 + 1 - x$. Also, let $\varphi(M) = \overline{M}$.

Since M is self-complementary, there exists $\sigma \in D_4$ such that $\sigma(M) = \varphi(M)$.

(i) Suppose n is odd.

Then k is odd. As such, since $\varphi(\frac{n^2+1}{2}) = \frac{n^2+1}{2}$, we see that the integer $\frac{n^2+1}{2}$ must be at the $(2,2)$-cell of $M(3 \times 3)$ for any $\sigma \in D_4$.

Next, we shall look at the entries in the corner cells of $M(3 \times 3)$.

Suppose a is the entry at the $(1,1)$-cell of $M(3 \times 3)$. Since $\sigma(M) = \varphi(M)$, $\overline{a}$ must be at the corner cell of $M(3 \times 3)$. There are only two cases to be considered.

Case (1) $\overline{a}$ is at the $(1,3)$-cell or the $(3,1)$-cell of $M(3 \times 3)$.

Without loss of generality, assume that $\overline{a}$ is at the $(1,3)$-cell of $M(3\times3)$. In this case, if b is the entry at the $(3,1)$-cell of $M(3 \times 3)$, then $\overline{b}$ can only be at the $(3,3)$-cell of $M(3 \times 3)$. But this implies that σ has to be the reflection along the central vertical line of M.

Now suppose x is the entry at the $(1,2)$-cell of $M(3 \times 3)$. Since $\overline{x}$ can neither be at the $(2,1)$-cell nor at the $(2,3)$-cell of $M(3 \times 3)$, it follows that $\overline{x}$ can only be at the $(3,2)$-cell of $M(3 \times 3)$. But then $\sigma(M) \neq \varphi(M)$, a contradiction.

Case (2) $\overline{a}$ is at the $(3,3)$-cell of $M(3 \times 3)$.

In this case, if b is the entry at the $(1,3)$-cell of $M(3 \times 3)$, then $\overline{b}$ can only be at the $(3,1)$-cell of $M(3 \times 3)$. But this implies that σ has to be a 180-degree rotation on M.

It follows that if x and y are the entries at the $(1,2)$-cell and $(2,1)$-cell of $M(3 \times 3)$ respectively, then $\overline{x}$ and $\overline{y}$ could only be at the $(3,2)$-cell and $(2,3)$-cell of $M(3 \times 3)$ respectively. Hence $M(3 \times 3)$ takes the following form:

a	x	b
y	$\frac{n^2+1}{2}$	$\overline{y}$
$\overline{b}$	$\overline{x}$	$\overline{a}$

Now, if w is the entry at the (i,j)-cell of M, then $\overline{w}$ must be at the $(n-i+1, n-j+1)$-cell of M because σ is a 180-degree rotation on M. But this means that M is a ro-symmetrical magic square.

(ii) Suppose n is even.

Consider $M(2 \times 2)$, the central 2×2 subsquare of M.

Suppose a is the entry at the $(1,1)$-cell of $M(2 \times 2)$. Since $\sigma(M) = \varphi(M)$, $\overline{a}$ must be at the corner cell of $M(2 \times 2)$. There are only two cases to be considered.

Case (1) $\bar{a}$ is at the $(1,2)$-cell or the $(2,1)$-cell of $M(2 \times 2)$.

Without loss of generality, assume that $\bar{a}$ is at the $(1,2)$-cell of $M(2\times 2)$. In this case, if b is the entry at the $(2,1)$-cell of $M(2 \times 2)$, then $\bar{b}$ can only be at the $(2,2)$-cell of $M(2 \times 2)$. But this implies that σ has to be the reflection along the central vertical line of M.

Now, if w is the entry at the (i,j)-cell of M, then $\overline{w}$ must be at the $(i, n-j+1)$-cell of M because σ is the reflection along the central vertical line of M. But this means that $\overline{M} = \varphi(M)$ is obtained from M by a central vertical reflection. That is, M is a ref-symmetrical magic square.

Case (2) $\bar{a}$ is at the $(2,2)$-cell of $M(2 \times 2)$.

In this case, if b is the entry at the $(1,2)$-cell of $M(2 \times 2)$, then $\bar{b}$ can only be at the $(2,1)$-cell of $M(2 \times 2)$. But this implies that σ has to be a 180-degree rotation on M.

Moreover, if w is the entry at the (i,j)-cell of M, then $\overline{w}$ must be at the $(n-i+1, n-j+1)$-cell of M because σ is a 180-degree rotation on M. But this means that M is a ro-symmetrical magic square. It is well-known that a ro-symmetrical magic square of order n exists if and only if $n \not\equiv 2 \pmod 4$. (See page 203 of [1]). Hence $n \equiv 0 \pmod 4$. □

Remark 1: We now know that for a self-complementary magic square of order n, any two integers a and b from $\{1, 2, \ldots, n^2\}$ which add up to be $n^2 + 1$ must always be either at their π-rotation positions or else at their ρ-reflection positions depending on whether it is ro-symmetrical or ref-symmetrical. Therefore, in describing a self-complementary magic square of order n, we need only to describe the locations of those positive integers $a \leq \lceil \frac{n^2}{2} \rceil$ since the rest will be uniquely determined by their corresponding positions. This is illustrated by the example shown in Figure 4 which is a ref-symmetrical magic square of order 6.

	3		15		16
	1		9		10
			11	2	12
8		6		17	
7		5		18	
			13	4	14

Figure 4: A ref-symmetrical magic square of order 6

3 Construction

Note that, in the literature, ro-symmetrical magic squares are also called *symmetrical* or *associative* (or *regular*) magic squares. It is well-known that De la Loubére's method produces a magic square of order n which is ro-symmetrical if n is odd. If $n \equiv 0 \pmod 4$, then the method described on pages 199-200 of [1] produces a ro-symmetrical magic square of order n. As far as we know, the construction of ref-symmetrical magic squares of order n seems to be unknown. The purpose of this section is to present a method of construction for ref-symmetrical magic squares of order n when $n \geq 4$ is even.

Theorem 3 *If there is a ref-symmetrical magic square of order n, then there is a ref-symmetrical magic square of order $n + 4$.*

PROOF: By construction. Let M denote a ref-symmetrical magic square of order n whose vertical reflection is $\overline{M}$. Then n is even by Theorem 2. We shall construct a ref-symmetrical magic square of order $n + 4$ using M.

For this purpose, let M^* denote an $(n+4) \times (n+4)$ square that has been partitioned as shown in Figure 5. Place the integers $3n+1, 3n+2, \ldots, 3n+8$ in M^* as shown.

Now, add $4n + 8$ to each entry of M and place the resulting square (denoted by $M + (4n + 8)$) at the central $n \times n$ subsquare of M^*.

Arrange the $2n$ integers $1, 2, \ldots, 2n$ into four columns c_1, c_2, c_3 and c_4 having equal number of integers such that the sums of numbers in c_1 and c_2 (respectively c_3 and c_4) are equal.

Then put those integers in c_1 (respectively c_2) in the column L_1 (respectively R_1) such that no integers from c_1 and c_2 are put in those positions which are ρ-reflectional to one another. Do the same to the integers in c_3 and c_4 by putting them in the columns L_2 and R_2 respectively. One easy way of doing this is to put all of the integers from c_1 (respectively c_3) in the upper half of L_1 (respectively L_2) and all of the integers from c_2 (respectively c_4) in the lower half of R_1 (respectively R_2).

For each $i = 1, 2, \ldots, n$, let $A_i = \{2n + i, 4n + 9 - i\}$ and put the two integers from A_i in the same column that belongs to T_1, T_2, B_1, or B_2 such that no two integers from two different $A_i's$ are put in those positions which are ρ-reflectional to one another. One easy way to do this is to put $A_1, A_2, \ldots, A_{\frac{n}{2}}$ in the $\frac{n}{2}$ left-hand most columns of T_1 and T_2 and put $A_{\frac{n}{2}+1}, A_{\frac{n}{2}+2}, \ldots, A_n$ in the $\frac{n}{2}$ right-hand most columns of B_1 and B_2.

$M^* =$

$3n+1$	$3n+4$	T_1		
		T_2	$3n+2$	$3n+3$
L_1	L_2	$M+(4n+8)$	R_2	R_1
$3n+8$	$3n+5$	B_2		
		B_1	$3n+7$	$3n+6$

Figure 5: The square M^* and its partition

It is then routine to check that the resulting square is a ref-symmetrical magic square of order $n+4$ whose vertical reflection is its complement. □

Remark 2: In the above construction, we may adopt the following arrangements for c_1, c_2, c_3, c_4 (although other arrangements are also possible).

When $n \equiv 0 \pmod 4$, let

c_1	c_2	c_3	c_4
1	2	3	4
8	7	6	5
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2n-7$	$2n-6$	$2n-5$	$2n-4$
$2n$	$2n-1$	$2n-2$	$2n-3$

and we see that all the $c_i's$ have equal sum $\frac{n(2n+1)}{4}$. Alternatively, we may adopt the following arrangement

c_1	c_2	c_3	c_4
1	2	5	6
4	3	8	7
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2n-7$	$2n-6$	$2n-3$	$2n-2$
$2n-4$	$2n-5$	$2n$	$2n-1$

in which case, we see that c_1 and c_2 (respectively c_3 and c_4) both have sum equal to $\frac{2n^2-3n}{4}$ (respectively $\frac{2n^2+5n}{4}$).

When $n \equiv 2 \pmod 4$, let

c_1'	c_2'	c_3'	c_4'
1	2	3	4
8	7	6	5
9	10	11	12
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2n-11$	$2n-10$	$2n-9$	$2n-8$
$2n-4$	$2n-5$	$2n-6$	$2n-7$
$2n-3$	$2n-2$	$2n-1$	$2n$

and let $c_1 = c_1'$, $c_4 = c_4'$. Further, let c_2 and c_3 be obtained from c_2' and c_3' respectively by interchanging exactly one pair of integers of the form $8i-1$ and $8i-2$ (from c_2' and c_3' respectively) (for example 7 and 6 or 15 and 14). Then it is easy to see that c_1 and c_2 (respectively c_3 and c_4) both have sum equal to $\frac{2n^2+n-6}{4}$ (respectively $\frac{2n^2+n+6}{4}$).

The following examples serve to illustrate the proof of Theorem 3.

13	16	9	10				
		24	23			14	15
1	3	25	26				
8	6			27	29		
			31		28	4	2
		32		30		5	7
20	17			11	12		
				22	21	19	18

13	16	9	23				
				12	11	14	15
	1	25	26				4
3				27	29	2	
	8		31		28		5
6		32		30		7	
20	17	24		21			
			10		22	19	18

Figure 6: Two self-complementary magic squares of order 8.

Example 1: Let M denote the ref-symmetrical magic square of Figure 2. Then $n = 4$ and we may arrange the integers $1, 2, \ldots, 8$ into four columns

c_1, c_2, c_3, c_4 in the following way.

c_1	c_2	c_3	c_4
1	2	3	4
8	7	6	5

Also, $A_1 = \{9, 24\}, A_2 = \{10, 23\}, A_3 = \{11, 22\}$ and $A_4 = \{12, 21\}$. With these we can obtain several ref-symmetrical magic squares of order 8. Two such squares are shown in Figure 6. Here, the square bounded by the thick lines is the square $M + 24$.

Example 2: Let M denote the ref-symmetrical magic square of Figure 4. Then $n = 6$ and we may arrange the integers $1, 2, \ldots, 12$ into four columns c_1, c_2, c_3, c_4 in the following way.

c_1	c_2	c_3	c_4
1	2	3	4
8	6	7	5
9	10	11	12

Also, $A_1 = \{13, 32\}, A_2 = \{14, 31\}, A_3 = \{15, 30\}, A_4 = \{16, 29\}, A_5 = \{17, 28\}$ and $A_6 = \{18, 27\}$. With these we can obtain several ref-symmetrical magic squares of order 10. One such square is shown in Figure 7.

Remark 3: (i) Note that, when $n = 0$, the square M^* of Figure 5 is a ref-symmetrical magic square of order 4. Here M is an empty square.

(ii) The method of construction in the proof of Theorem 3 is independent of the magic square M of order n. Moreover, when a central vertical reflection is made to the central square M, we obtain another ref-symmetrical magic square of order $n + 4$.

(iii) Since there is a ref-symmetrical magic square of order 4 (for example the square mentioned in Remark 3(i)) and one of order 6 (for example the square of Figure 4), by Theorem 3, we can use these squares to construct ref-symmetrical magic squares of any even order at least 4.

(iv) Since ro-symmetrical magic squares of any odd order and any order a multiple of 4 can be constructed, Theorems 2 and 3 lead to the following result.

19	22	13	14	15					
		32	31	30				20	21
1	3		35		47		48		
8	7		33		41		42		
9	11				43	34	44		
		40		38		49		4	2
		39		37		50		5	6
					45	36	46	12	10
26	23				16	17	18		
					29	28	27	25	24

Figure 7: A ref-symmetrical magic square of order 10

Corollary 1 *For every integer $n \geq 3$, there is a self-complementary magic square of order n.*

References

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[4] Magic squares – from Wolfram Mathworld, *http : //mathworld. wolfram.com/MagicSquare.html*