UTILITY AND EXPANDABILITY OF CHANNEL ASSIGNMENTS*

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ABSTRACT. The Channel Assignment Problem is often modeled by integer vertex-labelings of graphs. We will look at $L(2,1)$-labelings that realize the span λ of a simple, connected graph $G = (V, E)$. We define the *utility* of G to be the number of possible *expansions* that can occur on G, where an expansion refers to an opportunity to add a new vertex u to G, with label $\ell(u)$, such that

(1) edges are added between u and v and
(2) edges are added between u and the neighbors of v, and
(3) the resulting labeling of the graph is a valid $L(2,1)$-labeling.

Building upon results of Griggs, Jin, and Yeh, we use known values of λ to compute utility for several infinite families and analyze the utility of specific graphs that are of interest elsewhere.

1. INTRODUCTION

A considerable amount of research has been done on $L(2,1)$-labelings and the more general $L(h,k)$-labelings. The origin of this category of problems is *channel assignment*, in which broadcast channels for various nodes are assigned such that there is no interference with each other, while minimizing the frequency spectrum used. For a survey of the problem, see [2] as well as [3] and [4]. The goal of this paper is to examine how "expandable" a particular graph labeling is without changing the span of all labels. We call our measure of expandability the *utility* of a labeling.

When we consider the original problem of assigning channels to nodes while trying to minimize both interference and span, we often find many possible ways to label nodes while realizing the same span. If the labeled graph represents an actual network of nodes and channels, we seek to determine if another node may be added and assigned a channel within the current span that does not require existing nodes to change their channels. While many labelings may realize the same span, only a few of those labelings permit the addition of more nodes in the above manner. In some

*This research was supported by the NSF REU, "Groups, Graphs, and Geometry," held at UNC Asheville, Summer 2008.

sense, then, our measure distinguishes some channel assignments as "more expandable" than others.

Given a simple connected graph $G = (V, E)$, we define a *channel assignment*, or an $L(2,1)$-*labeling*, to be a function $\ell : V(G) \to \{0, 1, 2, \ldots\}$ such that whenever $u, v \in V$ are adjacent, $|\ell(u) - \ell(v)| \geq 2$, and whenever u and v are distance 2 from each other then $|\ell(u) - \ell(v)| \geq 1$. The *span* of G is the smallest λ such that there is a valid $L(2,1)$-*labeling* $\ell : V(G) \to \{0, 1, 2, \ldots, \lambda\}$, and is denoted by $\lambda(G)$. Throughout this paper, we will consider only finite graphs unless we explicitly say otherwise. Note that the following definitions apply in both cases.

For $v \in V$ we let $N(v)$ denote the set of *neighbors* of v (those vertices adjacent to v). Let $\ell : V(G) \to \{0, 1, \ldots, \lambda\}$ be a labeling. Let $\#$ denote the operation in which a new vertex u is attached to $v \in V(G)$ such that edges are drawn from u both to v and to all $v' \in N(v)$, and the resulting graph, using some value $\ell(u) \leq \lambda$, realizes an $L(2,1)$-labeling. We define the *utility* of a labeled graph $U(G, \ell)$ as the number of $v \in V(G)$ such that $u\#v$. Thus $0 \leq U(G, \ell) \leq |V(G)|$.

In measuring utility, we may permit attachments independently to v_1 and v_2, yet it may be the case that these attachments cannot occur at once. In order to measure the maximal number of attachments that can occur simultaneously, we introduce the notion of *simultaneous utility*, $U_s(G, \ell)$. We define $U_s(G, \ell)$ to be the maximum number of $u\#v \in V(G)$ such that all may occur simultaneously.

Notes. In such attachments, if vertices u_1 and u_2 are simultaneously attached to the same vertex v_0, then an edge $[u_1, u_2]$ must be included. Because of this, we cannot have more than $\frac{\lambda}{2}$ attachments at any vertex. Thus $0 \leq U_s(G) \leq \frac{\lambda}{2}n$.

We also note that while utility counts vertices of the original graph, simultaneous utility counts vertices that are attached to the original graph. This is the fundamental distance between the two measures.

We now define utility and simultaneous utility for the graph G alone:

$$U(G) = \max_{\ell} U(G, \ell) \qquad \text{and} \qquad U_s(G) = \max_{\ell} U_s(G, \ell),$$

where the maxima are taken over all $L(2,1)$-labelings of G.

Note that if $U(G, \ell) = 0$ then $U_s(G, \ell) = 0$; however, it is possible that $U_s(G, \ell) > U(G, \ell)$ in the case where multiple expansions occur on a single vertex. We see that this is realized by the graph in Figure 1.

This article will proceed as follows. In Section 2, we will consider the utility of paths, prisms, and other basic families of graphs. In Section 3, we will expand our search to include several families of infinite, regular graphs. In Section 4, we will examine bounds on utility and simultaneous utility as well as graphs that realize exceptionally high utility and simultaneous

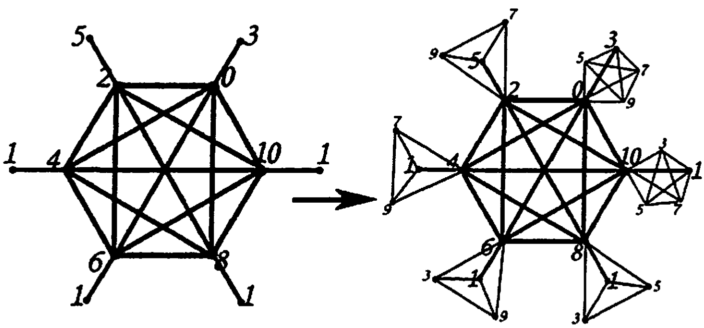

FIGURE 1. $|V(G)| = 12$ and $U_s(G, \ell) = 14$

utility. Finally, in Section 5 we list several open problems and natural extensions of this research.

The authors extend special thanks to Professor Jerrold Griggs at University of South Carolina for his insights, suggestions, and editorial remarks. Thanks are also due to UNC Asheville Professors Samuel Kaplan and David Peifer for helpful conversations and suggestions, and to the referee for help in indicating some errors in the original manuscript.

2. UTILITY FOR SEVERAL INFINITE FAMILIES

In this section, we examine the utility $U(G)$ and simultaneous utility $U_s(G)$ of some basic graphs of general interest, including complete graphs K_n, complete k-partite graphs $K_{n_1,n_2,\ldots,n_k}$, wheels W_n, cycles C_n, paths P_n, prisms Pr_n, and ladders L_n. Throughout this section, let $G = (V, E)$ be a simple, connected graph, let $\ell(v)$ denote the label of vertex v and let u be the vertex to be added to the graph in expansion. As usual, let the *order* of G be $|V(G)|$.

Proposition 2.1. *The utility $U(G) = 0$ if $G = K_n$, $K_{n_1,n_2,\ldots,n_k}$, W_n or C_n.*

Proof.

Complete Graphs: Let $v_i, v_j \in V(K_n) = \{v_1, v_2, \ldots, v_n\}$. We know that there is an edge connecting v_i to v_j. In constructing a labeling λ starting with $\ell(v_1) = 0$, each successive label must be 2 higher than the previous, thus $\lambda(K_n) = 2(n-1)$. Let $v \in V(K_n)$. If $u \# v$, the resulting graph is K_{n+1}. As we just saw, $\lambda(K_{n+1}) = 2((n+1)-1) = 2n$, but $2n > 2(n-1)$. Thus $U(K_n) = 0$.

Complete k-Partite Graphs: Let $G = K_{n_1,n_2,\ldots,n_k}$ and let N_i be the classes of the partition of $V(G)$. By [9], $\lambda(G) = |V(G)| + k - 2$.

Lemma 2.2. *The labeling ℓ realizes $\lambda(G)$ if and only if $\ell(N_i)$ are disjoint intervals in $[0, 1, \dots, \lambda]$ for every $i = 1, \dots, k$, each of size $|N_i|$ and the difference between the maximum label in $\ell(N_i)$ and the minimum label in its successor is exactly 2, as is the difference between the minimum label in $\ell(N_i)$ and the maximum label in its predecessor.*

Proof. ($\Leftarrow$) A straightforward calculation shows that if ℓ has the given form, then ℓ realizes $\lambda(G)$.
($\Rightarrow$) Note that $\text{diam}(G) = 2$, so all labels $\ell(v)$ must be distinct. Since $|\{0, 1, \dots, \lambda\}| = |V(G)| + k - 1$, $k - 1$ of the values in $[0, \lambda]$ are omitted by ℓ. Note that if $u_i \in N_i, u_j \in N_j, i \neq j$, such that $\ell(u_i) < \ell(u_j)$ and there is no $u \in V(G)$ such that $\ell(u_i) < \ell(u) < \ell(u_j)$, then $\ell(u_j) - \ell(u_i) \geq 2$. That is, any time we transition from one class of the partition to another, we must skip at least one value. Since there are only $k-1$ values omitted by ℓ, at most $k-1$ transitions are allowed. but this forces all $\ell(N_i)$s to be intervals separated by 2 because there are k classes. □

Let the labeling ℓ realize $\lambda(G)$ and let $v \in V(G)$. Suppose $u\#v$. By Lemma 2.2, $\ell(u) \notin \ell(V(G))$, so it must be one of the skipped labels. However, u must be adjacent to vertices in the class of either the successor or predecessor, and thus $\ell(u)$ must be separated by at least 2 from the successor's maximum label or the predecessor's minimum label. Since $\ell(u)$ is a skipped label, it is separated from both the successor and predecessor by only 1, which is a contradiction. Therefore, $U(K_{n_1,n_2,\dots,n_k}) = 0$.

Wheels: From [8], $\lambda(W_n) = n + 1$. The complete k-partite case shows that $U(K_{1,n}) = 0$. It follows at once from $K_{1,n} \leq W_n$ and $\lambda(K_{1,n}) = \lambda(W_n)$ that since $U(K_{1,n}) = 0$, $U(W_n) = 0$ as well, as if no vertex can be attached to $K_{1,n}$, the same is clearly true for W_n.

Cycles: From [9], $\lambda(C_n) = 4$. No matter where u is attached, it will be connected to 3 vertices: the initial vertex v and its two neighbors. Suppose $u\#v_i$ with neighbors v_{i-1} and v_{i+1}. This forces

$$\ell(u) \leq \min\{\ell(v_{i-1}), \ell(v_i), \ell(v_{i+1})\} - 2 \text{ or } \ell(u) \geq \max\{\ell(v_{i-1}), \ell(v_i), \ell(v_{i+1})\} + 2.$$

However, from examining all possible optimal labelings for 3 consecutive vertices in a cycle, we see that each must contain a 0 or 1 as well as a 3 or 4. Since $\ell(u)$ cannot be less than 0 or greater than λ, $U(C_n) = 0$.

□

Thus far, we have seen several graphs for which $U(G) = 0$ regardless of $|V(G)|$. We will now examine two families, prisms and paths, for which

$U(G) > 0$ if $|V(G)|$ has certain values. Our arguments rest on finding "blocks" of labels, from which we may construct labels yielding optimal utility.

Theorem 2.3. *For any finite path P_n,*

$$U(P_n) = \begin{cases} 0 & \text{if } n < 5, \\ 1 & \text{if } 5 \le n < 10,\ n \ne 6, \text{ and} \\ 2 & \text{if } n \ge 10 \text{ or } n = 6. \end{cases}$$

Proof. First, we check the cases $n < 5$. Note that $P_1 = K_1$, $P_2 = K_2$, $P_3 = K_{1,2}$, and an optimal labeling of P_4 uses the labels $2, 0, 3, 1$. In these four cases, we the utility is 0, by Proposition 2.1. For $n \ge 5, \lambda(P_n) = 4$ with a *canonical* cyclical labeling $\{0, 2, 4, 0, 2, 4, \dots\}$. Henceforth for simplicity we will indicate a labeling sequence by a concatenated string of integers, so the canonical labeling is 024024.... For all interior vertices along the path, the cases for expansion are the same as those for cycles (Proposition 2.1.3) and thus, no expansions can be made. Terminal vertices do not allow for expansions, and thus the canonical labeling has 0 utility.

Now let us consider non-canonical labelings of P_n. Let $v_1, v_2, \dots, v_n$ correspond to consecutive vertices on P_n where v_1 and v_n represent the two terminal points. Let $\ell(v_i) = \ell_i$. We will now construct a labeling of P_5 such that $U(P_5) > 0$. To avoid the canonical labeling, we force $\ell_3 \in \{1, 3\}$, which also forces $\ell_2 \in \{0, 4\}$. There are 4 possibilities of $\ell_1\ell_2\ell_3\ell_4\ell_5$ that allow for $u\#v_1$:

$$\alpha_1 = 20314, \quad \alpha_2 = 24130, \quad \alpha_3 = 40314, \quad \text{and } \alpha_4 = 04130.$$

Suppose that ℓ is an optimal labeling and let $\ell'(v) = \lambda - \ell(v)$ be the *inverse labeling.* We claim that ℓ' is optimal. Indeed ℓ realizes the minimal span λ, thus $\ell'(v) \in [0, \lambda]$ for all $v \in V$ as well. Hence, ℓ' is also an optimal labeling.

Thus we may consider the inverse pairs $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$. Without loss of generality, we look at only one in each pair, say α_1 and α_3. Note that for $n > 5$, ℓ_6 only depends on $\ell_4\ell_5$. The label pair $\ell_4\ell_5$ is the same in both α_1 and α_3 (41). To obtain possible *ending sequences* $\ell_{n-4}\ell_{n-3}\ell_{n-2}\ell_{n-1}\ell_n$ so that $u\#v_n$, we take the α_is and reverse the order of their labels:

$$\beta_1 = 41302, \quad \beta_2 = 03142, \quad \beta_3 = 41304, \quad \text{and } \beta_4 = 03140.$$

Given labels $\ell_1\ell_2 \dots \ell_{n-5}$, to determine if β_i can be used to label $\ell_{n-4}...\ell_n$, we need only examine $\ell_{n-4}\ell_{n-3}$, which have possible values of 03 and 41.

For the case $n = 5$, using α_1 we see that 20314 allows for $u\#v_1$ with $\ell(u) = 4$ and that if $u\#v_5, \ell(u) > \lambda$. Thus $U(P_5) = 1$.

We begin with α_1.

For $n = 6$ we see that if $\ell_6 = 0$, we have 203140, and thus $U(P_6) = 2$. Also note that β_4 is used for the last 5 vertex labels.

Since the starting sequence α_1 is fixed and only one of the β_is can be used for the last 5 labels, we need only check if a sequence of length n can be built that begins with α_1 and ends with a β_i. Since the last 3 digits of α_1, 314, do not begin any β_i, if $n = 7$ we can expand on the one terminal vertex but not the other, thus $U(P_7) = 1$. By the same argument, since 14 does not begin any β_i, $U(P_8) = 1$. For the case $n = 9$, note that 4 does begin β_1 and β_3, however, this sequence puts a 1 on either side of the 4, thus $U(P_9) = 1$. If $n = 10$, the length of the path is long enough that starting and ending sequences do not overlap, and we check combinations that may lie adjacent to each other. Indeed, the sequence $\alpha_1\beta_2$ witnesses $U(P_{10}) = 2$.

For $n > 10$, we seek to find labelings satisfying either $\alpha_1 P \beta_1$ or $\alpha_1 P \beta_2$ where $P = p_1 \dots p_{n-10}$ is a string of labelings such that $p_1 \in \{2, 0\}$ and $p_{n-10} \notin \{1, 3\}$. For the next 6 cases, we need only find a witness P for each case.

For $n = 11$, note that $P = 2$ gives us $\alpha_1 2 \beta_2$, so $U(P_{11}) = 2$.

For $n = 12$, $P = 20$ gives us $\alpha_1 20 \beta_1$, so $U(P_{12}) = 2$.

For $n = 13$, $P = 024$ gives us $\alpha_1 024 \beta_2$, so $U(P_{13}) = 2$.

For $n = 14$, $P = 0314$ gives us $\alpha_1 0314 \beta_2$, so $U(P_{14}) = 2$.

For $n = 15$, $P = 20314$ gives us $\alpha_1 20314 \beta_2$, so $U(P_{15}) = 2$.

For $n = 16$, $P = 024024$ gives us $\alpha_1 024024 \beta_2$, so $U(P_{16}) = 2$.

We now note that every integer ≥ 6 can be written as a linear combination of 3 and 4 with positive coefficients. We have "building blocks" $A = 024$ and $B = 0314$. Thus for for $n \geq 16$ we may construct a sequence of the form $\alpha_1 C_1 C_2 \dots C_r \beta_2$ where $C_i \in \{A, B\}$, for all i that witnesses $U(P_n) = 2$. The value we obtain is an upper bound as well as a lower bound because any other blocking would result in less efficient use of the path's length.

Thus for $n \geq 16, U(P_n) = 2$. Therefore if $n \geq 10$, $U(P_n) = 2$. □

We remind the reader that a *prism* Pr_n is graph consisting of two equal-length cycles C_n and C'_n in which each vertex on C_n is connected to the corresponding vertex on C'_n. We will refer to labels on prisms by pairs of vertices $\begin{smallmatrix} a \\ b \end{smallmatrix}$ where a is the label of a vertex on C_n and b is the label of the vertex adjacent to a on C'_n.

Theorem 2.4. *Let $n \geq 3$. Then for the prism Pr_n,*

$$U(Pr_n) \geq \begin{cases} 0, & \text{if } n < 10 \text{ or } n \equiv 0 \ (mod\ 3) \\ \lfloor \frac{n}{3} - 2 \rfloor, & \text{if } n \geq 10 \text{ and } n \not\equiv 0 \ (mod\ 3) \end{cases}$$

Proof. For this proof, let all congruences be mod 3. As with paths, we wish to find blocks of labels that can be strung together to form a labeling admitting $U(Pr_n) > 0$. For an arbitrary attachment $u\#v$ in the prism we

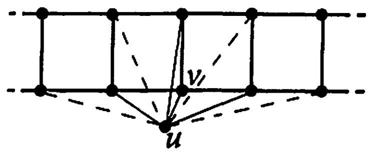

FIGURE 2. Prism $u\#v$

find that the smallest block of labels admitting an attachment must contain at least 3 vertex pairs.

By exhaustive computer search, for $n < 10, U(Pr_n) = 0$. Thus we assume $n \geq 10$. By Georges and Mauro [5], we have $\lambda(Pr_n) = \begin{cases} 5, & \text{if } n \equiv 0 \\ 6, & \text{if } n \not\equiv 0 \end{cases}$.
We have 3 cases corresponding to congruence classes mod 3.

Case 1 ($n \equiv 0$). By [5], if $n \equiv 0, \lambda(Pr_n) = 5$. This case uses the labeling $\alpha = \begin{matrix} 1\,3\,5 \\ 4\,0\,2 \end{matrix}$ and repeats cyclically around the whole prism. For this case, $U(Pr_n, \alpha) = 0$, and since the only other optimal labelings of Pr_n are permutations of α, we have that $U(Pr_n) = 0$, when $n \equiv 0$.

Case 2 ($n \equiv 1$). We let A and B be labeling blocks $A = \begin{matrix} 1\,5\,3 \\ 4\,0\,6 \end{matrix}$ and $B = \begin{matrix} 1\,5\,0\,3\,6\,0\,3 \\ 4\,2\,6\,1\,4\,2\,6 \end{matrix}$ of lengths with 3 vertex pairs and 7 vertex pairs respectively. (A computer search yielded no label blocks of shorter length that could be appended to each other.) Note that for A, if $\ell(v) = 0, u\#v$ where $\ell(u) = 2$. We also see that if A is placed adjacent to itself, the sequence AA may extend to a valid labeling.

From this we can build a labeling $BAA\ldots A$ for the prism Pr_n where A is repeated $\frac{n-7}{3}$ times. For each A there is a vertex where an expansion can occur, $U(Pr_n) = \frac{n-7}{3}$. Since $n \equiv 1, \exists k$ such that $3k+1 = n$. This gives us $\frac{(3k+1)-7}{3} = k - 2 = \lfloor k - 2 + \frac{1}{3} \rfloor = \lfloor \frac{3k+1}{3} - 2 \rfloor = \lfloor \frac{n}{3} - 2 \rfloor$. Thus for $n \equiv 1, U(Pr_n) \geq \lfloor \frac{n}{3} - 2 \rfloor$.

Case 3 ($n \equiv 2$). We let $C = \begin{matrix} 0\,5\,2\,4\,6\,1\,5\,3 \\ 4\,1\,6\,0\,2\,4\,0\,6 \end{matrix}$ be a labeling block of 8 vertex pairs, noting that no shorter labels for $n \equiv 2$ were found by a computer search. From this we can build a labeling $CAA\ldots A$ for the prism Pr_n where A is repeated $\frac{n-8}{3}$ times. For each A there is a vertex where an expansion can occur, $U(Pr_n) = \frac{n-8}{3}$. Since $n \equiv 2, \exists k$ such that $3k+2 = n$. This gives us $\frac{(3k+2)-8}{3} = k - 2 = \lfloor k - 2 + \frac{2}{3} \rfloor = \lfloor \frac{3k+2}{3} - 2 \rfloor = \lfloor \frac{n}{3} - 2 \rfloor$. Thus for $n \equiv 2, U(Pr_n) \geq \lfloor \frac{n}{3} - 2 \rfloor$. □

Note. Proving the equality $U(Pr_n) = \lfloor \frac{n}{3} - 2 \rfloor$ for $n \not\equiv 0$ would require showing that our blocking scheme is optimal, which we have not here done.

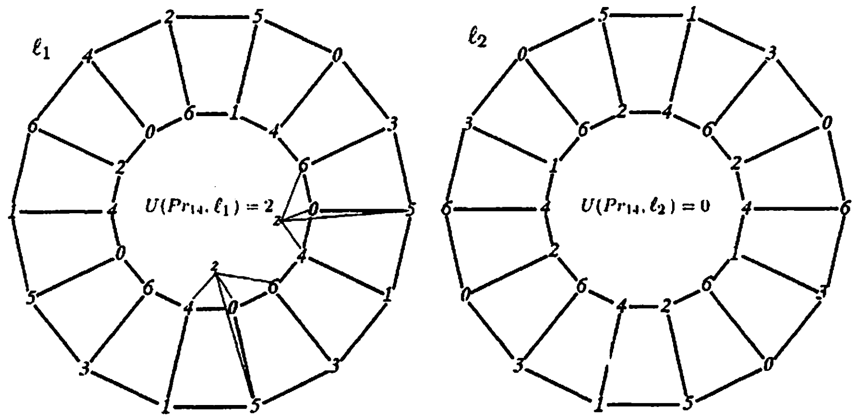

FIGURE 3. $U(Pr_{14}, \ell_1)$ and $U(Pr_{14}, \ell_2)$

To see how different labelings realizing the span of a graph can yield different utility, we look at the prism Pr_{14} in Figure 3. We use the labeling blocks A, B, and C from Proposition 2.4. In Figure 3, ℓ_1 is given by AAC and ℓ_2 is given by BB. However, in the first case $U(Pr_{14}, \ell_1) = 2$ while in the latter $U(Pr_{14}, \ell_2) = 0$.

Corollary 2.5. *For a prism $Pr_n, n \geq 3$, $U_s(Pr_n) = U(Pr_n)$.*

Proof. Suppose we have the label pattern A used consecutively. By this we know that exactly one expansion occurs inside the label block A, and since the expansion points are both separated by distance 3, they can occur simultaneously. Thus $U_s(Pr_n) \geq U(Pr_n)$. Moreover, since all expansions on Pr_n occur inside blocks of vertices labeled using A and since inside any one block we can append no more than one new vertex, in fact we have $U_s(Pr_n) = U(Pr_n)$. □

We remind the reader that a *ladder* L_n is graph consisting of two equal-length paths P_n and P'_n where each vertex on P_n is connected to a corresponding vertex on P'_n. We will refer to labels on ladders by pairs of vertices, as we did for prisms.

Corollary 2.6. *For a ladder $L_n, U(L_n) = 0$.*

Proof. For a subgraph $H \leq G$, it is easily shown that $\lambda(H) \leq \lambda(G)$. So for a ladder $L_n, \lambda(L_n) \leq \lambda(Pr_n)$. By using the same labeling block $\alpha = \begin{smallmatrix} 1 & 3 & 5 \\ 4 & 0 & 2 \end{smallmatrix}$ that was used for a prism Pr_n in case $n \equiv 0 \pmod 3$, we obtain $\lambda(L_n) \leq 5$.

Starting by labeling any vertex $\ell(v) = 0$ it is easy to see that $\lambda(L_n)$ cannot equal 4. Thus for $n \in \mathbb{N}$, $\lambda(L_n) = 5$. We also see that, as before, the only valid labelings are all permutations of α. Since it was previously shown that no vertices in this labeling can be expanded, we need only check what occurs at the terminal points of the ladder. For any valid permutation of the labels in α, utility is still 0. Since no other labelings can be used and no permutation α permits expansions, we have $U(L_n) = 0$. □

3. Infinite Regular Graphs

Up to this point, we have considered only finite graphs; we now consider a few infinite graphs. For the following translation-invariant graphs, utility and simultaneous utility are easily computed because attachments look the same at all vertices. Let $\Gamma_\triangle, \Gamma_\square$, and Γ_H, be the infinite Euclidean lattices whose regions are triangles, squares, and hexagons, respectively.

Proposition 3.1. *The utility $U(G) = 0$ if $G = \Gamma_\triangle, \Gamma_\square$, or Γ_H.*

Proof. From Griggs and Jin [7], $\lambda(\Gamma_H) = 5, \lambda(\Gamma_\square) = 6$, and $\lambda(\Gamma_\triangle) = 8$.

- **Hexagonal Lattice:** For the infinite hexagonal lattice Γ_H we examine at a potential expansion on an arbitrary vertex $v \in V(\Gamma_H)$. The subgraph induced by v and $N(v)$ is isomorphic to $K_{1,3}$, thus $K_{1,3} \leq \Gamma_H$. We know that $\lambda(K_{1,3}) = 4$, and we will show that even if we allow labels in $\{0, 1, \dots, 5\}$, no expansion is possible. We see that for 4 vertices and 6 possible labels, exactly 2 labels must be omitted. If $\ell(v) \in [1, 4]$, then the 2 omitted labels must be $\ell(v) + 1$ and $\ell(v) - 1$, which does not permit $u\#v$. If $\ell(v) = 0, 5$, then 1 or 4 is omitted, respectively, and then there are 4 choices for the other omitted label. Regardless of the combination of omitted labels used, $u\#v$ cannot occur. Since an expansion cannot be made for any arbitrary vertex, $U(\Gamma_H) = 0$.
- **Square Lattice:** Similarly, $K_{1,4} \leq \Gamma_\square$. While $\lambda(K_{1,4}) = 5$, we allow labels in $[0, 6]$. As before, we attempt to label $K_{1,4}$ using this longer interval. By applying the same argument, we again see that $u\#v$ is impossible and thus $U(\Gamma_\square) = 0$.
- **Triangular Lattice:** In this case, $K_{1,6} \leq \Gamma_\triangle$. We know that $\lambda(K_{1,6}) = 7$, but we allow labels in $[0, 8]$. Arguing as before, we obtain $U(\Gamma_\triangle) = 0$.

□

Indeed, the argument used in the above proof can be generalized to prove the following

Corollary 3.2. *If a graph G is k-regular and $\lambda(G) = k+2$, then $U(G) = 0$.*

There are large general classes of graphs with this property. For example, we may consider the Cayley graphs of Artin groups. For elements s, t in the generating set of a group, we let $w_m(s,t)$ denote the alternating product $stst\cdots$ consisting of exactly m terms. An *Artin group* is a group A given by presentation $\langle S \mid R\rangle$ in which R consists of pairs $w_{m(s,t)}(s,t) = w_{m(s,t)}(t,s)$, for $s,t \in S$ and $m(s,t)$ satisfying $m(s,t) = m(t,s) \in \mathbb{N}\cup\{\infty\}$ and $m(s,t) = 1$ if and only if $s = t$. If $m(s,t) = \infty$ we mean that there is no relation involving s and t. The set S is called the *fundamental* generating set. If all $m(s,t)$ are even or infinite, we call A an *even* Artin group.

For example, the integer lattice is the Cayley graph of the even Artin group $\langle s,t \mid st = ts\rangle$. More generally, any $2k$-regular tiling of the hyperbolic plane by p-gons for which $p \equiv 0 \bmod 4$ gives rise to a graph that can be realized as a subgraph of the Cayley graph of an even Artin group.

The following is proven in [1].

Proposition 3.3. *If Γ is a subgraph of the Cayley graph of an even Artin group, relative to the fundamental generating set, then $\lambda(\Gamma) = \Delta + 2$.*

From this we have

Corollary 3.4. *If Γ is a subgraph of the Cayley graph of an even Artin group, relative to the fundamental generating set, then $U(\Gamma) = 0$.*

Proof. Apply Corollary 3.2 and Proposition 3.3. □

4. Bounds on Utility

In this section we will examine the bounds on utility and simultaneous utility. To do this, we need the following lemma. Roughly it says that the utility of a graph $U(G)$ can also be used as a measurement of a label's "flexibility," *i.e.* the extent to which the label of a specific vertex can be altered so that the resulting labeling is also a valid $L(2,1)$-labeling.

Lemma 4.1. *If $U(G) > 0$ then $u\#v_0 \Leftrightarrow \exists k \in \mathbb{Z}, |k| \geq 2$, such that $\ell'(v) = \begin{cases} \ell(v)+k, & \text{if } v = v_0 \\ \ell(v), & \text{if } v \neq v_0 \end{cases}$ also realizes a $L(2,1)$-labeling.*

Proof. ($\Rightarrow$) Suppose $u\#v$ and let $\ell'(v) = \ell(u)$. Let $k = \ell'(v) - \ell(v)$. Then $|k| = |\ell'(v) - \ell(u)| \geq 2$ because $u\#v$. Similarly, $|\ell(w) - \ell(v)| \geq 2$ for all $w \in N(v)$. Thus ℓ' realizes a proper labeling.

($\Leftarrow$) Suppose for some $|k| \geq 2$ and $\ell'(v) = \ell(v) + k$ that ℓ' realizes a proper labeling. Let $\ell(u) = \ell'(v)$ and let $w \in N(v)$. To show $u\#v$, we need $|\ell(w) - \ell(u)| \geq 2$ and $|\ell(u) - \ell(v)| \geq 2$. Since ℓ' is a proper labeling, $|\ell(w) - \ell(u)| = |\ell(w) - \ell'(v)| \geq 2$. Likewise $|\ell(v) - \ell(u)| = |k| \geq 2$. Hence $u\#v$ is valid. □

Theorem 4.2. *If G has order n, then $U(G) \leq n - 1$.*

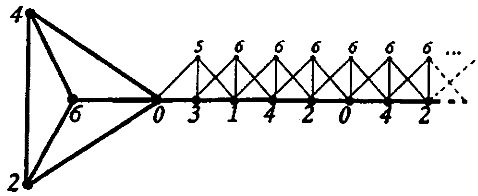

FIGURE 4. G_n with expansions

Proof. To show $U(G) < n$, assume to the contrary that $U(G) = n$. (By definition $U(G) \leq n$.) In this case we may attach to any $v \in V$. We know that for some $v_i \in V(G)$, $\ell(v_i) = \lambda$. Because $u \# v_i$, by Lemma 4.1 we may relabel v_i with ℓ' such that for some $|k| \geq 2$, $\ell'(v_i) = \ell(v_i) + k$. Since we cannot allow $\ell'(v_i) > \lambda$, we have that $k \leq -2$ and so $\ell'(v_i) < \ell(v_i)$. This is true for all v such that $\ell(v) = \lambda$, and thus we could create a new labeling of G with a span less than λ, which contradicts $\lambda(G)$ realizing the span of G. Thus $U(G) < n$ and so $U(G) \leq n - 1$. □

As indicated in Figure 4, there exists a graph G_n with order n such that $U(G_n) = n - 4$. This shows that the sharp upper bound for $U(G)$ must lie in $[n-4, n-1]$. This G_n has the form of K_4 with a pendant path of length $n - 4$. We see that $\lambda(G_n) = 6$. Note that expansions can occur, shown here with the smaller labels and thinner edges, at all vertices but the 4 contained in K_4. We emphasize that the expansions shown in Figure 4 are not to be performed simultaneously; they are all shown together merely to indicate the vertices at which expansion can occur.

Theorem 4.3. *Let G be a graph with maximum degree Δ and order n. Then*

$$U_s(G) \leq \frac{\lambda - 2}{2}(n-1).$$

Proof. We saw in Figure 1 that $U_s(G) > n$ is possible when multiple attachments are made to a given vertex. When p attachments are made to a single vertex v_0, then v_0, a single neighbor of v_0 and p attached vertices induce K_{p+2}. From Proposition 2.1, we know that $\lambda(K_{p+2}) = 2(p+1)$. By Theorem 4.2 we know that there are at most $n - 1$ vertices where attachments can be made. By the pigeonhole principle we can ensure that p vertices can be attached to a single vertex only if $U_s(G) \geq p(n-1)$.

Thus to find the maximum number of attachments that can be made to a vertex, we look at how many attachments, p, can be made before adding another induces a complete graph that requires a higher λ. By taking a vertex and a single neighbor, we seek a number of attachments p to a vertex that induces K_{p+2} such that $\lambda(K_{p+2}) > \lambda(G)$, forcing a contradiction. In terms of $\lambda(G)$, we get $p > \frac{\lambda-2}{2}$, which gives us $U_s(G) \leq \frac{\lambda-2}{2}(n-1)$. □

We now show that there exists a graph G such that $U_s(G) = \frac{1}{2}n(\frac{\lambda-6}{2})$, so that our bound above is of the right order in terms of n and λ.

Let the graph G_k with order n have the structure of K_k with a pendant edge at each vertex (Figure 1), thus $n = 2k$. Let V' be the set of vertices $v_i' \in V(G_k), i = 1, \ldots, k$ such that $\deg(v_i) = 1$. As in Proposition 2.1.1, label the vertices $v_i \in V(G_k), i = 1, \ldots, k$ that form K_k such that $\ell(v_1) = 0, \ell(v_2) = 2, \ldots, \ell(v_k) = 2(k-1)$. Let $\ell(v_i') = \begin{cases} 3, & \text{if } i = 1 \\ 5, & \text{if } i = 2 \\ 1, & \text{if } 3 \leq i \leq k \end{cases}$.

Note that this is a valid labeling of G, $\lambda(G) = 2k-2$, $U(G_k) = k$, and attachments can be made on each $v_i' \in V(G_k)$ for $i = 1, \ldots, k$. The number of expansions that can occur simultaneously at each v_i' is at least $k-4$. For v_1', v_k', the number of simultaneous expansions is $k-3$. This gives $U_s(G_k) = k(k-4)+2$. Putting this in terms of n and λ we have $U_s(G_k) = \frac{1}{2}n(\frac{\lambda-6}{2})$.

Corollary 4.4. *Let G be a graph with maximum degree Δ and order n. Then*

$$U_s(G) \leq \frac{1}{2}(\Delta^2 + \Delta - 4)(n-1) \leq \frac{1}{2}(n^3 - 2n^2 - 3n + 4).$$

Proof. By Theorem 4.3 we have $U_s(G) \leq \frac{1}{2}(\lambda(G) - 2)(n-1)$. From Gonçalves [6] we know that $\lambda(G) \leq \Delta^2 + \Delta - 2$. By a substitution, we get a bound $U_s(G) \leq \frac{1}{2}(\Delta^2+\Delta-4)(n-1)$ in terms of maximum degree Δ and order n. From here, we can get a bound depending only on n by noting that $\Delta \leq n-1$. This substitution gives $U_s(G) \leq \frac{1}{2}((n-1)^2+(n-1)-4)(n-1)$, which simplifies to $U_s(G) \leq \frac{1}{2}(n^3 - 2n^2 - 3n + 4)$. □

Note that Griggs and Yeh have conjectured [9] that $\lambda(G) \leq \Delta^2$, which, if proven true, improves the bound to $U_s(G) \leq \frac{1}{2}(\lambda - 2)(n-1) \leq \frac{1}{2}(\Delta^2 - 2)(n-1) \leq \frac{1}{2}(n^3 - 3n^2 + n + 1)$.

5. Directions for Further Research

Several areas remain open for exploration. The following section mentions several that are closely related to results found in this paper.

Question 1. What is $U(Q_n)$, for the cube Q_n with 2^n vertices? Clearly, for $n \leq 3, U(Q_n) = 0$. For larger n, however, it is unknown.

Question 2. What is $U(M_n)$, for the Möbius ladder M_n? These graphs are similar to prisms, but it appears likely that $\lambda(M_n) \geq \lambda(Pr_n)$. Because of its similarities to prisms, if $\lambda(M_n) = \lambda(Pr_n)$, then it is likely that $U(M_n) \approx$

$U(Pr_n)$ for sufficiently large n. If $\lambda(M_n) > \lambda(Pr_n)$, then it is very possible that $U(M_n) > U(Pr_n)$.

Question 3. What is the utility of finite subgraphs of the lattices $G_{\square}, G_{\triangle}, G_H$? Here, expansions would have to occur on the perimeter. It would be useful to show if a particular grid structure has $U(G) > 0$ because such knowledge could lead to more efficient methods of setting up grids that allow for expandability of real-world networks.

Question 4. What is the utility of a given tree? Trees seem to have a relatively high utility compared to their size. Finding a bound on a tree's utility compared to its size, maximal degree, and span would be a significant result.

Question 5. Can we obtain a sharp bound for utility or simultaneous utility? The bounds presented in Theorems 4.2, 4.3, and 4.4 are clearly not optimal, as our examples have shown. Finding better bounds may give rise to other methods of determining the utility of a graph. Also, showing the existence of graphs that have higher utility or simultaneous utility than the ones mentioned in Figure 4 and Figure 1 may be useful in discovering the most flexible labelings for a particular graph or class of graphs.

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