Total and adjacent vertex-distinguishing total chromatic numbers of augmented cubes *

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Abstract

A total coloring of a graph G is a coloring of both the edges and the vertices. A total coloring is proper if no two adjacent or incident elements receive the same color. An adjacent vertex-distinguishing total coloring h of a simple graph $G = (V, E)$ is a proper total coloring of G such that $H(u) \neq H(v)$ for any two adjacent vertices u and v, where $H(u) = \{h(wu)|wu \in E(G)\} \cup \{h(u)\}$ and $H(v) = \{h(xv)|xv \in E(G)\} \cup \{h(v)\}$. The minimum number of colors required for a proper total coloring (resp. an adjacent vertex-distinguishing total coloring) of G is called the total chromatic number (resp. adjacent vertex-distinguishing total chromatic number) of G and denoted by $\chi_t(G)$ (resp. $\chi_{at}(G)$). The Total Coloring Conjecture (TCC) states that for every simple graph G, $\Delta(G) + 1 \leq \chi_t(G) \leq \Delta(G) + 2$. G is called Type 1 (resp. Type 2) if $\chi_t(G) = \Delta(G) + 1$ (resp. $\chi_t(G) = \Delta(G) + 2$). In this paper, we prove that the augmented cubes AQ_n is of Type 1 for $n \geq 4$. We also consider the adjacent vertex-distinguishing total chromatic number of AQ_n, prove that

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$\chi_{at}(AQ_n) = \Delta(AQ_n) + 2$ for $n \geq 3$.

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1 Introduction

The augmented cube AQ_n is an important topology structure in network. The properties of AQ_n, such as the fault-tolerance, the pancyclicity and so on have been intensively considered. Since some network problems can be converted to coloring problem, we consider some coloring indices of AQ_n in this paper. A *k-total coloring* of a graph $G = (V, E)$ is an assignment of k colors to both the edges and the vertices of G. The total coloring is called *a proper k-total coloring* if no incident or adjacent elements (vertices or edges) receive the same color. The *total chromatic number* of G, $\chi_t(G)$, is the least integer k for which G admits a proper k-total coloring. Let $\Delta(G)$ be the maximum degree of G, Behzad [1] and Vizing [6] proposed independently the following famous conjecture, which is known as the *Total Coloring Conjecture* (TCC).

Conjecture 1. For any graph G, $\Delta(G) + 1 \leq \chi_t(G) \leq \Delta(G) + 2$. □

The lower bound of this conjecture is obvious, the upper bound remains to be proved. Using probabilistic methods, Molloy and Reed (1998) showed that the total chromatic number of a simple graph G is at most $\Delta(G)+10^{26}$, provided that $\Delta(G)$ is sufficiently large. Apart from this result, not much progress has been made on the conjecture. If G satisfies TCC and $\chi_t(G) = \Delta(G) + 1$ (resp. $\chi_t(G) = \Delta(G) + 2$), then G is of Type 1 (resp. Type 2).

In [7], Zhang *et al.* proposed a new concept, namely adjacent vertex-distinguishing total coloring. For a *k-total coloring* $h : V \cup E \rightarrow \{1, 2, \cdots, k\}$ of a graph G, let $h(uv)$ and $h(v)$ be the color of the edge $uv \in E(G)$ and the vertex v, respectively. Denote the color set of a vertex v in G by $H(v) = \{h(uv) | uv \in E(G)\} \cup \{h(v)\}$. If h is a proper k-total coloring, and $H(u) \neq H(v)$ for any edge $uv \in E(G)$, then h is called a k-adjacent vertex-distinguishing total coloring of graph G (abbreviated *k-AVDTC* of G). The minimum number of colors required for an adjacent vertex-distinguishing total coloring of G is called *the adjacent vertex-distinguishing total chromatic number* of G and denoted by $\chi_{at}(G)$. The following theorem was obtained by Zhang *et al.* [7].

Theorem 2 [7]. If a graph G has two vertices of maximum degree which are adjacent, then $\chi_{at}(G) \geq \Delta(G) + 2$.

In this paper, we prove that for $n \geq 4$, the augmented cube AQ_n is of Type 1. We get the result by the following method: first, decompose the augmented cubes into 2^{n-3} 3-dimensional cubes, color the edges and the vertices of each of these 3-dimensional cubes properly by four colors such that any two adjacent vertices in augmented cubes are colored differently; second, the uncolored edges form $2n-4$ perfect matchings of augmented cubes, they can be colored properly by $2n-4$ colors. In this paper, we also consider the adjacent vertex-distinguishing total chromatic number of AQ_n. Since AQ_n is a regular graph, by Theorem 2, we have $\chi_{at}(AQ_n) \geq \Delta(AQ_n) + 2$. We prove that $\chi_{at}(AQ_n) = \Delta(AQ_n) + 2$ for $n \geq 2$, which attains the lower bound of Theorem 2.

2 The Total Chromatic Number of AQ_n

In this section, we consider the total chromatic number of AQ_n. We would like to begin with some definitions and known results first.

The n-dimensional augmented cube, denoted by AQ_n, has 2^n vertices, each of which corresponds to an n-bit binary string. It can be defined recursively as follows: AQ_1 is a complete graph K_2 with the vertex set $\{0,1\}$. For $n \geq 2$, let AQ_{n-1}^0 and AQ_{n-1}^1 be two copies of AQ_{n-1} with $V(AQ_{n-1}^k) = \{ku_{n-1}u_{n-2}\cdots u_1 | u_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq n-1\}$ for $k \in \{0,1\}$. Then AQ_n is constructed by connecting AQ_{n-1}^0 and AQ_{n-1}^1 with 2^n edges so that a vertex $u = 0u_{n-1}u_{n-2}\cdots u_1$ in AQ_{n-1}^0 is adjacent to a vertex $v = 1v_{n-1}v_{n-2}\cdots v_1$ in AQ_{n-1}^1 if and only if one of the following two conditions holds: (i) $v_i = u_i$ for all $1 \leq i \leq n-1$; (ii) $v_i = \overline{u}_i = 1 - u_i$ for all $1 \leq i \leq n-1$. Denote the adjacent vertex in condition (ii) by $\overline{u}$. The augmented cubes AQ_1, AQ_2, AQ_3 are illustrated in Fig. 1.

Many topological properties related to cycle and path embedding in augmented cubes, such as pancyclicity [5], panconnectedness [5], Hamil-

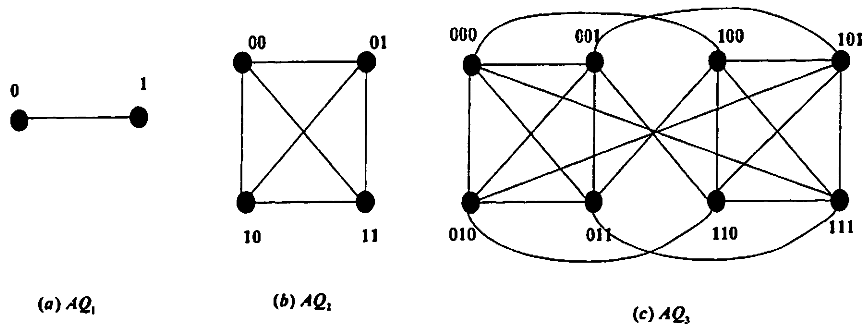

Figure 1: The augmented cubes AQ_1, AQ_2 and AQ_3.

tonian connectedness [4], panpositionable panconnectedness [3] have been investigated. The following lemma gives a property of AQ_n that will be used repeatedly in the proofs to come.

Lemma 3 [2]. For $n \geq 1$, the augmented cubes AQ_n are Cayley graphs, $AQ_n \cong G(Z_2^n, S)$ where $S = \{10\cdots000, 01\cdots000, \ldots, 00\cdots001, 00\cdots011, 00\cdots111, \ldots, 11\cdots111\}$ with $2n-1$ elements.

We find that the edges form $2n-1$ perfect matchings of AQ_n, each perfect matching is generated by one of the elements in S. Denote the vertex with i-th (from right to left) position 1 and other positions 0 by e_n^i. If $n \geq 4$, let $S_1 = \{e_n^1, e_n^2, e_n^4\} \subseteq S$. In fact, the subgraph generated by S_1 form 2^{n-3} 3-dimensional cubes. Clearly, if $n = 4$, then subgraph generated by $S_1 = \{e_4^1, e_4^2, e_4^4\}$ is two 3-dimensional cubes, denote them by D_4^1 and D_4^2, which are illustrated in Fig. 2. For $n \geq 5$, let $i_{n-4} \cdots i_1 D_4^l$ be 2^{n-4} copies of D_4^l with $V(i_{n-4} \cdots i_1 D_4^l) = \{i_{n-4} \cdots i_1 u | u \in V(D_4^l)$ and $i_j = 0$ or 1 for $1 \leq j \leq n-4\}$ where $l \in \{0, 1\}$. Since $l \in \{0, 1\}$, so there are 2^{n-3} 3-dimensional cubes in $i_{n-4} \cdots i_1 D_4^l$, each is isomorphic to D_4^1 or D_4^2. Obviously, the edges of $i_{n-4} \cdots i_1 D_4^l$ are edges generated by S_1. Furthermore, there are $2^{n-3} \times 12 = 2^{n-1} \times 3$ edges in $i_{n-4} \cdots i_1 D_4^l$, which is equal to the number of edges generated by S_1. So the edges generated by S_1 form 2^{n-3} 3-dimensional cubes. Denote them by D_n^1, D_n^2, $\ldots$, $D_n^{2^{n-3}}$.

In the following, we will prove that the edges and the vertices of the 2^{n-3} 3-dimensional cubes $\bigcup_{k=1}^{2^{n-3}} D_n^l$ can be properly colored with four colors such that any two vertices adjacent in AQ_n are colored differently.

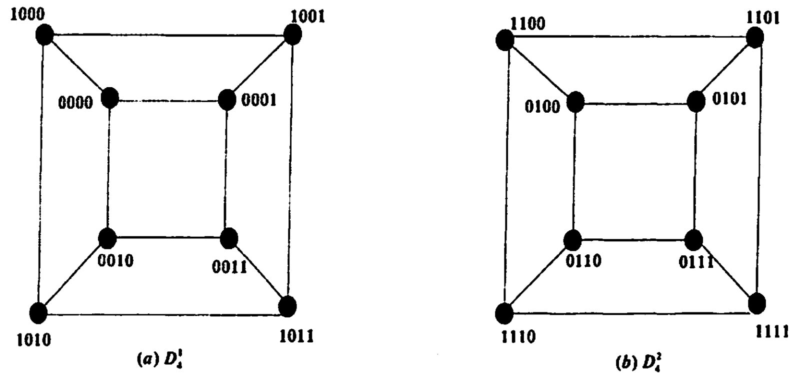

Figure 2: The two 3-dimensional cubes D_4^1 and D_4^2 in AQ_4.

Lemma 4. There exists a proper 4-total coloring f_n for $\bigcup_{l=1}^{2^{n-3}} D_n^l$ such that any two vertices adjacent in AQ_n are colored differently, where $n \geq 4$.

Proof. For any $v \in V(AQ_n)$, let $(F_n(v))$ denote the ordered color set $(f_n(v), f_n\{v, v+e_n^1\}, f_n\{v, v+e_n^2\}, f_n\{v, v+e_n^4\})$ of v. A 4-total coloring f_n of $\bigcup_{l=1}^{2^{n-3}} D_n^l$ can be coded in $\{(v, (F_n(v)))\}$. We will prove Lemma 4 by induction on n.

If $n = 4$, let $\{(v, (F_4(v)))\}$ be as follows: $\{(0000, (1,3,4,2)), (0001, (4,3,2,1)), (0011, (3,1,2,4)), (0010, (2,1,4,3)), (1000, (3,4,1,2)), (1001, (2,4,3,1)), (1011, (1,2,3,4)), (1010, (4,2,1,3)), (0100, (4,1,2,3)), (0101, (3,1,4,2)), (0111, (2,3,4,1)), (0110, (1,3,2,4)), (1100, (2,4,1,3)), (1101, (1,4,3,2)), (1111, (4,2,3,1)), (1110, (3,2,1,4))\}$. Clearly, f_4 is a proper 4-total coloring of D_4^1 and D_4^2. Furthermore, we can verify that two vertices adjacent in AQ_4 are colored differently by f_4.

For $n-1 \geq 4$, suppose there exists a proper 4-total coloring f_{n-1} for $D_{n-1}^1, D_{n-1}^2, \ldots, D_{n-1}^{2^{n-4}}$ such that two vertices adjacent in AQ_{n-1} are colored differently by f_{n-1}.

If $n \geq 5$, then D_n^j has the form $0D_{n-1}^l$ or $1D_{n-1}^l$, where $j \in \{1, 2, \cdots, 2^{n-3}\}$ and $l \in \{1, 2, \cdots, 2^{n-4}\}$. So the vertices of D_n^j have the form $0x$ or $1x$, where $x \in V(D_{n-1}^l)$; and the edges of D_n^j have the form $(0x, 0y)$ or $(1x, 1y)$, where $(x, y) \in E(D_{n-1}^l)$. Define f_n as follows: let $f_n(0x) = f_{n-1}(x)$, $f_n(1x) = f_{n-1}(x+e_{n-1}^{n-1})$; let $f_n\{0x, 0y\} = f_{n-1}\{x, y\}$, $f_n\{1x, 1y\} =$

$f_{n-1}\{x+e_{n-1}^{n-1}, y+e_{n-1}^{n-1}\}$.

Claim 1. Any two vertices adjacent in AQ_n are colored different by f_n.

Let $(u,v) \in E(AQ_n)$, in order to show that $f_n(u) \neq f_n(v)$. We classify into three cases.

Case 1. Let $u, v \in V(AQ_{n-1}^0)$. Assume $u = 0x$ and $v = 0y$, where $x, y \in V(AQ_{n-1})$. Clearly, x and y are adjacent in AQ_{n-1}. By definition, $f_n(u) = f_{n-1}(x)$ and $f_n(v) = f_{n-1}(y)$. By assumption, we have $f_{n-1}(x) \neq f_{n-1}(y)$ since f_{n-1} distinguishes any two adjacent vertices in AQ_{n-1}. Therefore, $f_n(u) \neq f_n(v)$.

Case 2. Let $u, v \in V(AQ_{n-1}^1)$. Assume $u = 1x$ and $v = 1y$, where $x, y \in V(AQ_{n-1})$. Then x and y are adjacent in AQ_{n-1}. Also, $x + e_{n-1}^{n-1}$ and $y + e_{n-1}^{n-1}$ are adjacent in AQ_{n-1} since (x, y) and $(x + e_{n-1}^{n-1}, y + e_{n-1}^{n-1})$ are generated by the same generator of S. By definition, $f_n(u) = f_{n-1}(x + e_{n-1}^{n-1})$ and $f_n(v) = f_{n-1}(y + e_{n-1}^{n-1})$. By induction assumption, we have $f_{n-1}(x + e_{n-1}^{n-1}) \neq f_{n-1}(y + e_{n-1}^{n-1})$. Therefore, $f_n(u) \neq f_n(v)$.

Case 3. Let $u \in V(AQ_{n-1}^0)$, $v \in V(AQ_{n-1}^1)$. Let $u = 0x$, where $x \in V(AQ_{n-1})$. By definition $v = 1x$ or $v = 1\overline{x}$. Then $f_n(u) = f_n(0x) = f_{n-1}(x)$. If $v = 1x$, then $f_n(v) = f_n(1x) = f_{n-1}(x + e_{n-1}^{n-1})$. Otherwise, $v = 1\overline{x}$ and $f_n(v) = f_n(1\overline{x}) = f_{n-1}(\overline{x} + e_{n-1}^{n-1})$. Since x is adjacent to both $x + e_{n-1}^{n-1}$ and $\overline{x} + e_{n-1}^{n-1}$ in AQ_{n-1}, so $f_{n-1}(x) \neq f_{n-1}(x + e_{n-1}^{n-1})$ and $f_{n-1}(x) \neq f_{n-1}(\overline{x} + e_{n-1}^{n-1})$. In both case, we have $f_n(u) \neq f_n(v)$.

Claim 2. The coloring f_n is a proper 4-total coloring for $\bigcup_{l=1}^{2^{n-3}} D_n^l$.

In order to prove claim 2, by claim 1, we only need to prove that any edge in $\bigcup_{l=1}^{2^{n-3}} D_n^l$ is colored different from its end vertices and its adjacent edges in $\bigcup_{l=1}^{2^{n-3}} D_n^l$.

Case 1. Let $(u, v) \in E(0D_{n-1}^l)$ for some $l \in \{1, 2, \cdots, 2^{n-4}\}$. Assume $(u, w) \in E(0D_{n-1}^l)$. Let $u = 0x$, $v = 0y$ and $w = 0z$. Then x and y are adjacent in D_{n-1}^l; also, x and z are adjacent in D_{n-1}^l. By definition, $f_n\{u, v\} = f_{n-1}\{x, y\}$, $f_n\{u, w\} = f_{n-1}\{x, z\}$, $f_n(v) = f_{n-1}(y)$. By induction assumption, $f_{n-1}\{x, y\} \neq f_{n-1}(y)$, $f_{n-1}\{x, y\} \neq f_{n-1}\{x, z\}$. Therefore, $f_n\{u, v\} \neq f_n(v)$, $f_n\{u, v\} \neq f_n\{u, w\}$.

Case 2. Let $(u, v) \in E(1D_{n-1}^l)$ for some $l \in \{1, 2, \cdots, 2^{n-4}\}$. Assume $(u, w) \in E(1D_{n-1}^l)$. Let $u = 1x$, $v = 1y$ and $w = 1z$. Then we can get that x and y are adjacent in D^l_{n-1}; also, x and z are adjacent in D^l_{n-1}. It is easy to see that (x, y) and $(x + e^{n-1}_{n-1}, y + e^{n-1}_{n-1})$ are generated by the same generator of S_1. So $x + e^{n-1}_{n-1}$ and $y + e^{n-1}_{n-1}$ are adjacent in some D^j_{n-1}. Moreover, $x + e^{n-1}_{n-1}$ and $z + e^{n-1}_{n-1}$ are adjacent in the same D^j_{n-1}. By definition, $f_n\{u, v\} = f_{n-1}\{x + e^{n-1}_{n-1}, y + e^{n-1}_{n-1}\}$, $f_n\{u, w\} = f_{n-1}\{x+e^{n-1}_{n-1}, z+e^{n-1}_{n-1}\}$, $f_n(v) = f_{n-1}(y+e^{n-1}_{n-1})$. By induction assumption, $f_{n-1}\{x + e^{n-1}_{n-1}, y + e^{n-1}_{n-1}\} \neq f_{n-1}(y + e^{n-1}_{n-1})$, $f_{n-1}\{x + e^{n-1}_{n-1}, y + e^{n-1}_{n-1}\} \neq f_{n-1}\{x+e^{n-1}_{n-1}, z+e^{n-1}_{n-1}\}$. Therefore, $f_n\{u, v\} \neq f_n(v)$, $f_n\{u, v\} \neq f_n\{u, w\}$. The proof of Claim 2 is now complete.

Combining claim 1 with claim 2, we can conclude that f_n is a proper 4-total coloring for $\bigcup_{l=1}^{2^{n-3}} D^l_n$ such that any two vertices adjacent in AQ_n are colored differently. □

Next is the main result of this section.

Theorem 5. For each $n \geq 4$, $\chi_t(AQ_n) = 2n$.

Proof. Since AQ_n is $(2n - 1)$-regular, then $\chi_t(AQ_n) \geq 2n$, and in order to prove $\chi_t(AQ_n) = 2n$, we only need to prove that AQ_n has a proper $2n$-total coloring for $n \geq 4$.

First, by Lemma 4, color the edges generated by S_1 and the vertices of AQ_n properly by four colors such that any two adjacent vertices in augmented cube are colored differently. Second, the uncolored edges form $2n - 4$ perfect matchings of AQ_n, they can be colored properly by another $2n - 4$ colors. This yields a proper $2n$-total coloring of AQ_n. Hence, $\chi_t(AQ_n) = 2n$. □

3 The Adjacent Vertex-distinguishing Total Chromatic Number of AQ_n

In this section, we consider the adjacent vertex-distinguishing total chromatic number of AQ_n. We begin with the following definition.

If $n \geq 3$, let $S_2 = \{e^1_n, e^2_n\} \subseteq S$. In fact, the subgraph generated by S_2 form 2^{n-2} four cycles. Clearly, if $n = 3$ then subgraph generated by $S_2 = \{e^1_3, e^2_3\}$ is two four cycles, denote them by C^1_3 and C^2_3. For $n \geq 4$, let $i_{n-3} \cdots i_1 C^l_3$ be 2^{n-3} copies of C^l_3 with $V(i_{n-3} \cdots i_1 C^l_3) = \{i_{n-3} \cdots i_1 u | u \in$

$V(C_3^l)$ and $i_j = 0$ or 1 for $1 \le j \le n-3\}$ where $l \in \{0,1\}$. Since $l \in \{0,1\}$, so there are 2^{n-2} four cycles in $i_{n-3}\cdots i_1 C_3^l$, each is isomorphic to C_3^1 or C_3^2. And the edges of $i_{n-3}\cdots i_1 C_3^l$ are generated by S_2. Denote the 2^{n-2} four cycles by $C_n^1, C_n^2, \dots, C_n^{2^{n-2}}$.

Let h_n be a k-total coloring of $\bigcup_{l=1}^{2^{n-2}} C_n^l$. For any $v \in V(AQ_n)$, let $H_n(v)$ (resp. $(H_n(v))$) denote the color multiset $\{h_n(v), h_n\{v, v+e_n^1\}, h_n\{v, v+e_n^2\}\}$ (resp. the ordered color multiset $(h_n(v), h_n\{v, v+e_n^1\}, h_n\{v, v+e_n^2\})$) of v. Next, we will prove that there exists a proper 4-total coloring h_n for $\bigcup_{l=1}^{2^{n-2}} C_n^l$ such that $h_n(u) \neq h_n(v)$ and $H_n(u) \neq H_n(v)$ for any $(u,v) \in E(AQ_n)$.

Lemma 6. For $n \ge 3$, there exists a proper 4-total coloring h_n for $\bigcup_{l=1}^{2^{n-2}} C_n^l$ such that $h_n(u) \neq h_n(v)$ and $H_n(u) \neq H_n(v)$ for any $(u,v) \in E(AQ_n)$.

Proof. A 4-total coloring h_n of $\bigcup_{l=1}^{2^{n-2}} C_n^l$ can be coded in $\{(v, (H_n(v)))\}$. We will prove Lemma 6 by induction on n.

If $n = 3$, let $\{(v, (H_3(v)))\}$ be as follows: $\{(v, (H_3(v)))\} = \{(000, (2,1,4)), (001, (3,1,2)), (010, (1,3,4)), (011, (4,3,2)); (100, (1,4,3)), (101, (2,4,1)), (110, (4,2,3)), (111, (3,2,1))\}$. Clearly, h_3 is a proper 4-total of $C_3^1 \cup C_3^2$. Furthermore, $h_3(u) \neq h_3(v)$ and $H_3(u) \neq H_3(v)$ for any $(u,v) \in E(AQ_3)$.

For $n-1 \ge 3$, assume h_{n-1} is a proper 4-total coloring of $\bigcup_{l=1}^{2^{n-3}} C_{n-1}^l$ such that $h_{n-1}(u) \neq h_{n-1}(v)$ and $H_{n-1}(u) \neq H_{n-1}(v)$ for any $(u,v) \in E(AQ_{n-1})$.

We need only to prove that $\bigcup_{l=1}^{2^{n-2}} C_n^l$ has a proper 4-total coloring such that $h_n(u) \neq h_n(v)$ and $H_n(u) \neq H_n(v)$ for any $(u,v) \in E(AQ_n)$.

If $n \ge 4$, then C_n^j has the form $0C_{n-1}^l$ or $1C_{n-1}^l$, where $j \in \{1, 2, \cdots, 2^{n-2}\}$ and $l \in \{1, 2, \cdots, 2^{n-3}\}$. So the vertices of C_n^j have the form $0x$ or $1x$, where $x \in V(C_{n-1}^l)$; and the edges of C_n^j have the form $(0x, 0y)$ or $(1x, 1y)$, where $(x,y) \in E(C_{n-1}^l)$. Define h_n as follows: let $h_n(0x) = h_{n-1}(x)$, $h_n(1x) = h_{n-1}(x + e_{n-1}^{n-1})$; let $h_n\{0x, 0y\} = h_{n-1}\{x, y\}$, $h_n\{1x, 1y\} = h_{n-1}\{x + e_{n-1}^{n-1}, y + e_{n-1}^{n-1}\}$.

By the definition of h_n and the induction assumption of h_{n-1}, we can prove that h_n is a proper 4-total coloring for $\bigcup_{l=1}^{2^{n-2}} C_n^l$ and any two adjacent vertices in AQ_n are colored differently by h_n similar to Lemma 4. We omit the proof here.

In order to prove Lemma 6, we only need to show that $H_n(u) \neq H_n(v)$ for any $(u,v) \in E(AQ_n)$. We classify into three cases to prove that $H_n(u) \neq H_n(v)$.

Case 1. Let $u, v \in V(AQ^0_{n-1})$. Assume $u = 0x$ and $v = 0y$, where $x, y \in V(AQ_{n-1})$. Clearly, x and y are adjacent in AQ_{n-1}. By definition, $h_n(u) = h_{n-1}(x)$, $h_n\{u, u+e^1_n\} = h_{n-1}\{x, x+e^1_{n-1}\}$, $h_n\{u, u+e^2_n\} = h_{n-1}\{x, x+e^2_{n-1}\}$. So $H_n(u) = H_{n-1}(x)$. Similarly, we can get $H_n(v) = H_{n-1}(y)$. By assumption, we have $H_{n-1}(x) \neq H_{n-1}(y)$ since any two vertices adjacent in AQ_{n-1} are coded with different color set. Therefore, $H_n(u) \neq H_n(v)$.

Case 2. Let $u, v \in V(AQ^1_{n-1})$. Assume $u = 1x$ and $v = 1y$, where $x, y \in V(AQ_{n-1})$. Then x and y are adjacent in AQ_{n-1}. Also, $x + e^{n-1}_{n-1}$ and $y + e^{n-1}_{n-1}$ are adjacent in AQ_{n-1} since (x,y) and $(x+e^{n-1}_{n-1}, y+e^{n-1}_{n-1})$ are generated by the same generator of S. By definition, $h_n(u) = h_{n-1}(x+e^{n-1}_{n-1})$, $h_n\{u, u+e^1_n\} = h_{n-1}\{x+e^{n-1}_{n-1}, x+e^{n-1}_{n-1}+e^1_{n-1}\}$, $h_n\{u, u+e^2_n\} = h_{n-1}\{x+e^{n-1}_{n-1}, x+e^{n-1}_{n-1}+e^2_{n-1}\}$. So $H_n(u) = H_{n-1}(x+e^{n-1}_{n-1})$. Similarly, we can get $H_n(v) = H_{n-1}(y+e^{n-1}_{n-1})$. By assumption, we have $H_{n-1}(x) \neq H_{n-1}(y)$. Therefore, $H_n(u) \neq H_n(v)$.

Case 3. Let $u \in V(AQ^0_{n-1})$, $v \in V(AQ^1_{n-1})$. Assume $u = 0x$, where $x \in V(AQ_{n-1})$. By definition $v = 1x$ or $v = 1\overline{x}$. Then by case 1, $H_n(u) = H_{n-1}(x)$. If $v = 1x$, then $H_n(v) = H_{n-1}(x+e^{n-1}_{n-1})$ by case 2. Otherwise, $v = 1\overline{x}$ and $H_n(v) = H_{n-1}(\overline{x}+e^{n-1}_{n-1})$. Since x is adjacent to both $x+e^{n-1}_{n-1}$ and $\overline{x}+e^{n-1}_{n-1}$ in AQ_{n-1}, so $H_{n-1}(x) \neq H_{n-1}(x+e^{n-1}_{n-1})$ and $H_{n-1}(x) \neq H_{n-1}(\overline{x}+e^{n-1}_{n-1})$. In both case, we have $H_n(u) \neq H_n(v)$. □

Since AQ_n is a $(2n-1)$-regular graph, by Theorem 2, we have $\chi_{at}(AQ_n) \geq \Delta(AQ_n)+2 = 2n+1$. In the following, we will prove that $\chi_{at}(AQ_n) = 2n+1$ for $n \geq 3$.

Theorem 7. For each $n \geq 3$, $\chi_{at}(AQ_n) = 2n+1$.

Proof. Since $\chi_{at}(AQ_n) \geq 2n+1$, in order to prove $\chi_{at}(AQ_n) = 2n+1$, we only need to give a $(2n+1)$-adjacent vertex-distinguishing total coloring of AQ_n for $n \geq 3$.

Let h_n be the proper 4-total coloring for $\bigcup_{l=1}^{2^{n-2}} C^l_n$ provided by Lemma 6. The uncolored edges of AQ_n form $2n-3$ perfect matchings. We can extend h_n to be a proper $(2n+1)$-total coloring of AQ_n by coloring each perfect matching a new color with colors $5, 6, \ldots, 2n+1$. Let $\overline{H}_n(v)$ be the color set of v and its incident edges in AQ_n. Then $\overline{H}_n(v) = H_n(v) \cup \{5, 6, \ldots, 2n+1\}$. For any $(u, v) \in E(AQ_n)$, by Lemma 6, $H_n(u) \neq H_n(v)$. So $\overline{H}_n(u) \neq \overline{H}_n(v)$. This yields a $(2n+1)$-adjacent vertex-distinguishing total coloring of AQ_n. Hence, $\chi_{at}(AQ_n) = 2n+1$. $\square$

References

[1] M. Behzad, Graphs and their chromatic numbers, Ph.D. Thesis, Michigan State University, 1965.

[2] S.A. Choudum, V. Sunitha, Augmented cubes, Networks 40 (2) (2002) 71-84.

[3] T.-L. Kung, Y.-H. Teng, L.-H. Hsu, The panpositionable panconnectedness of augmented cubes, Information Sciences 180 (2010) 3781-3793.

[4] C.-M. Lee, Y.-H. Teng, J.J.M. Tan, L.-H. Hsu, Embedding Hamiltonian paths in augmented cubes with a required vertex in a fixed position, Comput. Math. Appl. 589 (2009) 1762-1768.

[5] M. Ma, G. Liu, J.-M. Xu, Panconnectivity and edge-fault-tolerant pancyclicity of augmented cubes, Parallel Comput. 33 (2007) 36-42.

[6] V.G. Vizing, Some unsolved problems in graph theory, Uspekhi Mat. Nauk 23 (1968) 117-134. (in Russian).

[7] Z. Zhang, X. Chen and J. Li *et al.*, On the adjacent vertex-distinguishing total coloring of graphs, Science in China Ser. A Mathematics 48 (2005) 289-299.