Lattices associated with partial linear maps of finite vector spaces

Junli Liu* Wei Jiang Baohuan Zhang Qiuli Xu
Math. and Inf. College, Langfang Teachers' College, Langfang, 065000, China

Abstract Let $\mathbb{F}_q^{(n)}$ denote the n-dimensional row vector space over the finite field $\mathbb{F}_q$ with $n \geq 2$. An l-partial linear map of $\mathbb{F}_q^{(n)}$ is a pair (V, f), where V is an l-dimensional subspace of $\mathbb{F}_q^{(n)}$ and $f : V \to \mathbb{F}_q^{(n)}$ is a linear map. Let $\mathscr{L}$ be the set of all partial linear maps of $\mathbb{F}_q^{(n)}$ containing $\hat{1}$. Ordered $\mathscr{L}$ by ordinary and reverse inclusion, two families of finite posets are obtained. This paper proves that these posets are lattices, discusses their geometricity and computes their characteristic polynomials.

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1 Introduction

It is well known that lattice is an important part of poset's theory, its theory plays an important role in many branches of mathematics, such as computer logical design and procedure theory. In recent times there has been great interest in constructing more kinds of practical lattices. For example, the results on the lattices generated by transitive sets of subspaces under finite classical groups may be found in Huo, Liu and Wan [5, 6, 7]. In [2], Guo discussed the lattices associated with finite vector spaces and finite affine spaces. In [15], Wang and Li discussed the lattice $\mathscr{L}(n,d) = \mathcal{P} \cup \{\mathbb{F}_q^{(n)}\}$, where $\mathcal{P}$ is the set of all the subspaces of $\mathbb{F}_q^{(n)}$ intersecting trivially with a given $(n-d)$-dimensional subspace of $\mathbb{F}_q^{(n)}$. The lattices generated by the orbits of subspaces under finite classical groups have been obtained in a series of papers by Huo and Wan [8], Guo, Li and Wang [3], Wang and Feng [12], Wang and Guo [13, 14], Guo and Nan [4, 9].

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a prime power. For a non-negative integer $n \geq 2$, let $\mathbb{F}_q^{(n)}$ denote the n-dimensional row vector space over $\mathbb{F}_q$. An l-partial linear map of $\mathbb{F}_q^{(n)}$ is a pair (V, f), where V is an l-dimensional subspace of $\mathbb{F}_q^{(n)}$ and $f : V \to \mathbb{F}_q^{(n)}$ is a linear map. Let $\mathscr{L}$ be the set of all partial linear maps of $\mathbb{F}_q^{(n)}$ containing $\hat{1}$. For any $x, y \in \mathscr{L} \setminus \{\hat{1}\}$, we define $\hat{1}$ *includes* x, and y *includes* x if $A \subseteq B$ and $g|_A = f$ where $x = (A, f)$, $y = (B, g)$. By ordering $\mathscr{L}$ by ordinary and reverse inclusion, two families of finite posets are obtained, denoted $\mathscr{L}_O$ and $\mathscr{L}_R$ respectively. We prove that $\mathscr{L}_O$ and $\mathscr{L}_R$ are lattices, discusse their geometricity and compute their characteristic polynomials.

This paper is organized as follows. In Section 2, we introduce some definitions and terminologies about finite posets, and prove that $\mathscr{L}_O$ and $\mathscr{L}_R$ are

*junli810@emails.bjut.edu.cn

finite lattices. In Section 3, we discuss the geometricity of $\mathscr{L}_O$ and compute its characteristic polynomial. The same problem of $\mathscr{L}_R$ is discussed in Section 4.

2 Preliminaries

In this section, we first recall some definitions and terminologies about finite posets and lattices. The reader is referred to [1, 10] for details. And then introduce two families of finite lattices generated by partial linear maps of $\mathbb{F}_q^{(n)}$.

Let P be a poset with partial order $\leq$. As usual, we write $a < b$ whenever $a \leq b$ and $a \neq b$. For any two elements $a, b \in P$, we say b *covers* a, denoted by $a \lessdot b$, if $a < b$ and there exists no element $c \in P$ such that $a < c < b$. If P has the minimum (resp. maximum) element, then we denote it by 0 (resp. 1). In this case we say that P is a poset with 0 (resp. 1).

Let P be a finite poset with 0. By a *rank function* on P, we mean a function r from P to the set of all the nonnegative integers such that

(i) $r(0) = 0$;

(ii) $r(b) = r(a) + 1$ whenever $a \lessdot b$.

Note that the rank function on P is unique if it exists.

Let P be a finite poset with 0 and 1. The polynomial

$$\chi(P, t) = \sum_{a \in P} \mu(0, a) t^{r(1) - r(a)}$$

is called the *characteristic polynomial* of P, where r is the rank function on P.

A poset L is said to be a *lattice* if both $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$ exist for any two elements $a, b \in L$. Let L be a finite lattice with 0. By an *atom* in L, we mean an element in L covering 0. We say L is *atomic* if any element in $L \setminus \{0\}$ is a union of atoms. A finite atomic lattice L is said to be *geometric* if L admits a rank function r satisfying $r(a \wedge b) + r(a \vee b) \leq r(a) + r(b)$ for any two distinct elements $a, b \in L$.

Lemma 2.1 ([11]) *The number of m-dimensional subspace of* $\mathbb{F}_q^{(n)}$ *containing a given k-dimensional subspace of* $\mathbb{F}_q^{(n)}$ *is* $\begin{bmatrix} n-k \\ m-k \end{bmatrix} = \frac{\prod_{i=n-m+1}^{n-k}(q^i-1)}{\prod_{i=1}^{m-k}(q^i-1)}$.

Definition 2.1 *Let V be a subspace of* $\mathbb{F}_q^{(n)}$, *and $f : V \longrightarrow \mathbb{F}_q^{(n)}$ be a linear map. Then the pair (V, f) is said to be a partial linear map of* $\mathbb{F}_q^{(n)}$. *In particular, we write $(V, f) = \hat{0}$ if $V = \{0\}$.*

Let $\mathscr{L}$ denote the set of all partial linear maps of $\mathbb{F}_q^{(n)}$ containing $\hat{1}$, i.e.,

$$\mathscr{L} = \{(V, f) \mid (V, f) \text{ is a partial linear map of } \mathbb{F}_q^{(n)}\} \cup \{\hat{1}\}.$$

For any elements $x, y \in \mathscr{L}$, if we define $x \leq y \iff y$ includes x, then $\mathscr{L}$ is a finite poset, denoted by $\mathscr{L}_O$. If we define $x \leq y \iff x$ includes y, then $\mathscr{L}$ is also a finite poset, denoted by $\mathscr{L}_R$.

Proposition 2.2 *The poset $\mathscr{L}_O$ (resp. $\mathscr{L}_R$) is a finite lattice.*

Proof. For any two elements $x, y \in \mathscr{L}_O$, if $x = \hat{1}$ or $y = \hat{1}$, then both the least upper bound and the greatest lower bound of x, y exist. Let (A, f), $(B, g) \in \mathscr{L}_O \setminus \{\hat{1}\}$. Then we assert that

$$\begin{aligned}(A,f)\vee(B,g) &= \begin{cases}(A+B,h), h|_A=f, h|_B=g, & \text{if } f|_{A\cap B}=g|_{A\cap B};\\ \hat{1}, & \text{otherwise},\end{cases}\\ (A,f)\wedge(B,g) &= (D,h), h=f|_D=g|_D,\end{aligned}$$

where D is the maximum subspace in the set $\{C\subseteq A\cap B \mid f|_C=g|_C\}$. In fact, there are three cases to be considered:

Case 1. $f|_{A\cap B}=g|_{A\cap B}$. Assume that $(C,\varphi)\in\mathscr{L}_O$ is an upper bound of (A,f) and (B,g). Then $A\subseteq C$, $B\subseteq C$ and $\varphi|_A=f$, $\varphi|_B=g$, which implies that $A+B\subseteq C$ and $\varphi|_{A+B}=h$. Thus the desired result follows.

Case 2. $f|_{A\cap B}\neq g|_{A\cap B}$. Assume that $(C,\varphi)<\hat{1}$ is an upper bound of (A,f) and (B,g). Then $A\cap B\subseteq C$, $f|_{A\cap B}=\varphi|_{A\cap B}=g|_{A\cap B}$, a contradiction.

Case 3. Assume that $(C,\varphi)\in\mathscr{L}_O$ is a lower bound of (A,f) and (B,g), then $C\subseteq A\cap B$ and $f|_C=g|_C=\varphi$, which implies that C belongs to $\{C\subseteq A\cap B|f|_C=g|_C\}$. Thus the desired result follows.

Similarly, $\mathscr{L}_R$ is also a lattice. □

3 The lattice $\mathscr{L}_O$

The lattice $\mathscr{L}_O$ has the maximum element $\hat{1}$ and minimum element $\hat{0}$. Since the set of all the atoms of $\mathscr{L}_O$ consists of all the partial linear map (A,f) with $\dim A=1$, $\mathscr{L}_O$ is a finite atomic lattice. Now we will discuss the geometricity of $\mathscr{L}_O$ and compute its characteristic polynomial.

Lemma 3.1 *The the rank function r_O on $\mathscr{L}_O$ is*

$$r_O(x)=\begin{cases}\dim A, & \text{if } x=(A,f)\in\mathscr{L}_O\setminus\{\hat{1}\};\\ n+1, & \text{if } x=\hat{1}.\end{cases}$$

Proof. By the definition of r_O, it is easy to see that $r_O(\hat{0})=0$. For any (A,f),$(B,g)\in\mathscr{L}_O$, if $(A,f)\lessdot(B,g)\neq\hat{1}$, then $(B,g)\leq(A,f)$ and $(B,g)\neq(A,f)$. Thus $0<\dim B-\dim A\leq 1$. Since, if $\dim B-\dim A\geq 2$, then there exists an non-zero vector e in $B\backslash A$ such that $(A,f)<(A+\langle e\rangle,h)<(B,g)$, where $\langle e\rangle$ is the subspace spanned by e and $h=g|_{A+\langle e\rangle}$, a contradiction. If $(A,f)\lessdot\hat{1}$, then $\dim A=n$. Therefore, the desired result follows. □

Theorem 3.2 *$\mathscr{L}_O$ is not a geometric lattice with $n\geq 2$.*

Proof. Let $x=(\langle e_1\rangle,f_1)$, $y=(\langle e_1\rangle,f_2)\in\mathscr{L}_O$ with $f_1|_{\langle e_1\rangle}\neq f_2|_{\langle e_1\rangle}$. Then $x\vee y=\hat{1}$ and $x\wedge y=\hat{0}$. It follows that

$$r_O(x\vee y)+r_O(x\wedge y)=n+1>r_O(x)+r_O(y).$$

Hence, $\mathscr{L}_O$ is not a geometric lattice with $n\geq 2$. □

Proposition 3.3 ([10]) *Let n be a nonnegative integer, and $q\neq 1$. Then*

$$\prod_{i=0}^{n-1}(1+q^i x)=\sum_{m=0}^{n} q^{\binom{m}{2}}\begin{bmatrix}n\\m\end{bmatrix}x^m. \tag{1}$$

Corollary 3.4 *Let n be a nonnegative integer, and $q \neq 1$. Then*

$$\prod_{i=0}^{n-1}(t-q^iq^n)=\sum_{m=0}^{n} q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix} (-q^n)^m t^{n-m}.$$

Proof. Let $x=-\frac{q^n}{t}$ as in (1). Then the desired result follows. □

Lemma 3.5 *For any $x, y \in \mathscr{L}_O$, the Möbius function μ_O of $\mathscr{L}_O$ is*

$$\mu_O(x,y)=\begin{cases} 0, & \text{if } x \not\leq y; \\ (-1)^\tau q^{\binom{\tau}{2}}, & \text{if } x \leq y \neq \hat{1} \text{ or } x=y=\hat{1}; \\ -\prod_{i=0}^{n-r_O(x)-1}(1-q^iq^n), & \text{if } x<y=\hat{1}, \end{cases}$$

where $\tau = r_O(y)-r_O(x)$.

Proof. In order to prove that μ_O is the Möbius function of $\mathscr{L}_O$, we only need to show that $\sum_{x\leq z\leq y}\mu_O(x,z)=0$ for any $x,y\in\mathscr{L}_O$ with $x<y$. There are the following two cases to be considered:

Case 1. $y \neq \hat{1}$. Let $r_O(y)-r_O(x)=m$. By Lemma 2.1 and Proposition 3.3, we have

$$\begin{aligned}\sum_{x\leq z\leq y}\mu_O(x,z) &= (-1)^0q^{\binom{0}{2}}\begin{bmatrix} m \\ 0 \end{bmatrix}+(-1)^1q^{\binom{1}{2}}\begin{bmatrix} m \\ 1 \end{bmatrix}+(-1)^2q^{\binom{2}{2}}\begin{bmatrix} m \\ 2 \end{bmatrix} \\ &\quad +\cdots+(-1)^mq^{\binom{m}{2}}\begin{bmatrix} m \\ m \end{bmatrix} \\ &= \sum_{k=0}^{m}(-1)^kq^{\binom{k}{2}}\begin{bmatrix} m \\ k \end{bmatrix} \\ &= \prod_{i=0}^{m-1}(1-q^i) \\ &= 0.\end{aligned}$$

Case 2. $y=\hat{1}$. Let $r_O(x)=m$. Then

$$\begin{aligned}\sum_{x\leq z\leq \hat{1}}\mu_O(x,z) &= 1+(-1)^1q^{\binom{1}{2}}\begin{bmatrix} n-m \\ 1 \end{bmatrix}q^n+\cdots \\ &\quad +(-1)^{n-m}q^{\binom{n-m}{2}}\begin{bmatrix} n-m \\ n-m \end{bmatrix}q^{n(n-m)}+\mu_O(x,\hat{1}) \\ &= \prod_{i=0}^{n-m-1}(1-q^nq^i)-\prod_{i=0}^{n-m-1}(1-q^iq^n) \qquad \text{(by Proposition 3.3)} \\ &= 0.\end{aligned}$$

Therefore, the desired result follows. □

Theorem 3.6 *The characteristic polynomial of $\mathscr{L}_O$ is*

$$\chi(\mathscr{L}_O,t)=t\prod_{i=0}^{n-1}(t-q^nq^i)-\prod_{i=0}^{n-1}(1-q^nq^i).$$

Proof. By Lemma 3.5 we obtain

$$\begin{aligned}\chi(\mathscr{L}_O, t) &= \sum_{a\in\mathscr{L}_O} \mu_O(\hat{0}, a) t^{r_O(\hat{1})-r_O(a)} \\ &= t^{n+1} + q^n \begin{bmatrix} n \\ 1 \end{bmatrix} (-1)^1 q^{\binom{1}{2}} t^n + q^{2n} \begin{bmatrix} n \\ 2 \end{bmatrix} (-1)^2 q^{\binom{2}{2}} t^{n-1} \\ &\quad + \cdots + q^{n^2} \begin{bmatrix} n \\ n \end{bmatrix} (-1)^n q^{\binom{n}{2}} t + \mu_O(\hat{0}, \hat{1}) \\ &= t \prod_{i=0}^{n-1} (t - q^i q^n) - \prod_{i=0}^{n-1} (1 - q^n q^i) \qquad \text{(by Corollary 3.4)}\end{aligned}$$

as desired. □

4 The lattice $\mathscr{L}_R$

The lattice $\mathscr{L}_R$ has the maximum element $\hat{0}$ and minimum element $\hat{1}$. In this section, we will show that $\mathscr{L}_R$ is an atomic lattice, but not a geometric lattice with $n \geq 2$, and compute the characteristic polynomial of $\mathscr{L}_R$.

Theorem 4.1 *$\mathscr{L}_R$ is an atomic lattice, but not a geometric lattice with $n \geq 2$.*

Proof. For any $x \in \mathscr{L}_R$, define

$$r_R(x) = \begin{cases} n + 1 - \dim A, & \text{if } x = (A, f) \in \mathscr{L}_R \setminus \{\hat{1}\}; \\ 0, & \text{if } x = \hat{1}, \end{cases}$$

then r_R is the rank function on $\mathscr{L}_R$.

Note that all the partial linear map $(\mathbb{F}_q^{(n)}, f)$ are atoms of $\mathscr{L}_R$. For any $x = (A, h) \in \mathscr{L}_R \setminus \{\hat{1}\}$, there are the following two cases to be considered:

Case 1. $x \neq \hat{0}$. Let $\{a_1, a_2, \cdots, a_m\}$ be a basis for A and $\{a_1, a_2, \cdots, a_m, a_{m+1}, \cdots, a_n\}$ be a basis for $\mathbb{F}_q^{(n)}$. Let linear maps f and g as follows:

$$f: \mathbb{F}_q^{(n)} \longrightarrow \mathbb{F}_q^{(n)},\ a_i \longmapsto h(a_i)(1 \leq i \leq m),\ a_{m+j} \longmapsto a_{m+j}(1 \leq j \leq n-m);$$

$$g: \mathbb{F}_q^{(n)} \longrightarrow \mathbb{F}_q^{(n)},\ a_i \longmapsto h(a_i)(1 \leq i \leq m),\ a_{m+j} \longmapsto 0(1 \leq j \leq n-m).$$

Thus, $(\mathbb{F}_q^{(n)}, f)$ and $(\mathbb{F}_q^{(n)}, g)$ are the atoms of $\mathscr{L}_R$, and $(A, h) = (\mathbb{F}_q^{(n)}, f) \vee (\mathbb{F}_q^{(n)}, g)$.

Case 2. $x = \hat{0}$. Let linear maps f and g as follows:

$$f: \mathbb{F}_q^{(n)} \longrightarrow \mathbb{F}_q^{(n)},\ a \longmapsto 0; \quad g: \mathbb{F}_q^{(n)} \longrightarrow \mathbb{F}_q^{(n)},\ a \longmapsto a. \tag{2}$$

Then $(\mathbb{F}_q^{(n)}, f)$ and $(\mathbb{F}_q^{(n)}, g)$ are the atoms of $\mathscr{L}_R$, and $\hat{0} = (\mathbb{F}_q^{(n)}, f) \vee (\mathbb{F}_q^{(n)}, g)$. Therefore, $\mathscr{L}_R$ is an atomic lattice.

Let linear maps f and g as in (2). Write $x = (\mathbb{F}_q^{(n)}, f)$, $y = (\mathbb{F}_q^{(n)}, g)$. Then $x \vee y = \hat{0}$ and $x \wedge y = \hat{1}$. It follows that

$$r_R(x \vee y) + r_R(x \wedge y) = n + 1 > r_R(x) + r_R(y).$$

Hence, $\mathscr{L}_R$ is not a geometric lattice with $n \geq 2$. □

Lemma 4.2 *For any $x, y \in \mathscr{L}_R$, the Möbius function μ_R of $\mathscr{L}_R$ is*

$$\mu_R(x,y) = \begin{cases} 0, & \text{if } x \not\leq y; \\ (-1)^\tau q^{\binom{\tau}{2}}, & \text{if } \hat{1} \neq x \leq y \text{ or } x = y = \hat{1}; \\ -\prod_{i=0}^{r_R(y)-2}(1-q^i q^n), & \text{if } \hat{1} = x \leq y, r_R(y) \geq 2; \\ -1, & \text{if } x = \hat{1}, r_R(y) = 1, \end{cases}$$

where $\tau = r_R(y) - r_R(x)$.

Proof. Suppose $x \neq \hat{1}$. Let $r_R(y) - r_R(x) = m$. By Lemma 2.1 and Proposition 3.3, we have

$$\begin{aligned} \sum_{x \leq z \leq y} \mu_R(z,y) &= (-1)^0 q^{\binom{0}{2}} \begin{bmatrix} m \\ 0 \end{bmatrix} + (-1)^1 q^{\binom{1}{2}} \begin{bmatrix} m \\ 1 \end{bmatrix} + \cdots + (-1)^m q^{\binom{m}{2}} \begin{bmatrix} m \\ m \end{bmatrix} \\ &= \sum_{k=0}^{m} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix} \\ &= \prod_{i=0}^{m-1} (1-q^i) \qquad \text{(by Proposition 3.3)} \\ &= 0. \end{aligned}$$

Suppose $x = \hat{1}$. Let $r_R(y) = m \geq 2$. Then

$$\begin{aligned} \sum_{\hat{1} \leq z \leq y} \mu_R(z,y) &= \mu_R(\hat{1},y) + (-1)^{m-1} q^{\binom{m-1}{2}} \begin{bmatrix} m-1 \\ m-1 \end{bmatrix} q^{n(m-1)} + \cdots \\ &\quad + (-1)^1 q^{\binom{1}{2}} \begin{bmatrix} m-1 \\ 1 \end{bmatrix} q^n + 1 \\ &= (-\prod_{i=0}^{r_R(y)-2} (1-q^i q^n)) + \sum_{k=0}^{m-1} (-q^n)^k q^{\binom{k}{2}} \begin{bmatrix} m-1 \\ k \end{bmatrix} \\ &= \prod_{i=0}^{m-2} (1-q^i q^n) - \prod_{i=0}^{m-2} (1-q^n q^i) \qquad \text{(by Proposition 3.3)} \\ &= 0. \end{aligned}$$

Therefore, the desired result follows. □

Theorem 4.3 *The characteristic polynomial of $\mathscr{L}_R$ is*

$$\chi(\mathscr{L}_R, t) = t^{n+1} - \left(\begin{bmatrix} n \\ n \end{bmatrix} q^{n^2} t^n + \sum_{k=0}^{n-1} \prod_{i=0}^{n-k-1} (1-q^i q^n) \begin{bmatrix} n \\ k \end{bmatrix} t^k \right).$$

Proof. For a given subspace A of $\mathbb{F}_q^{(n)}$, there are $q^{n \dim A}$ partial linear maps in $\mathscr{L}_R$. By Lemma 2.1 and 4.2 we obtain

$$\chi(\mathscr{L}_R, t) = \sum_{a \in \mathscr{L}_R} \mu_R(\hat{1}, a) t^{r_R(\hat{0}) - r_R(a)}$$

$$= t^{n+1} + \sum_{(A,f)\in \mathcal{L}_R \setminus \{\hat{1}\}} \mu_R(\hat{1},(A,f))t^{\dim A}$$

$$= t^{n+1} - \left(\begin{bmatrix} n \\ n \end{bmatrix} q^{n^2} t^n + \sum_{k=0}^{n-1} \prod_{i=0}^{n-k-1} (1 - q^i q^n) \begin{bmatrix} n \\ k \end{bmatrix} t^k \right)$$

as desired. □

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