On the super edge-magic deficiency of a star forest*

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Abstract

Let $G = (V, E)$ be a finite, simple and undirected graph of order p and size q. A super edge-magic total labeling of a graph G is a bijection $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$, where the vertices are labeled with the numbers $1, 2, \ldots, p$ and there exists a constant t such that $f(x) + f(xy) + f(y) = t$, for every edge $xy \in E(G)$. The super edge-magic deficiency of a graph G, denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge-magic total labeling, or it is ∞ if there exists no such n.

In this paper, we study the super edge-magic deficiency of a forest consisting of stars.

Keywords: *super edge-magic total deficiency, disjoint union of stars.*

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1 Introduction

In this paper, we consider finite, simple and undirected graphs. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$ respectively, where $|V(G)| = p$ and $|E(G)| = q$. An *edge-magic total labeling* of a graph G is a bijection $\lambda : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$, where there exists a constant t such that $f(x) + f(xy) + f(y) = t$, for every edge $xy \in E(G)$. The constant t is called the *magic constant* and a graph that admits an edge magic total labeling is called an *edge-magic total graph.* An edge-magic total labeling λ is called *super edge-magic total* if the vertices are labeled with the smallest possible numbers, i.e. $1, 2, \ldots, p$.

The concept of edge-magic total labeling was given by Kotzig and Rosa [7] in 1970. They proved that for any graph G there exists an edge-magic total graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n. This fact leads to the concept of *edge-magic total deficiency* of a graph G [7], which is the minimum nonnegative integer n such that $G \cup nK_1$ is edge-magic total. The edge-magic deficiency of G is denoted by $\mu(G)$. In particular,

$$\mu(G) = min\{n \geq 0 : G \cup nK_1 \text{ is edge-magic}\}.$$

In the same paper, Kotzig and Rosa gave the upper bound of the edge-magic deficiency of a graph G with n vertices,

$$\mu(G) \leq F_{n+2} - 2 - n - \frac{1}{2}n(n-1)$$

where F_n is the nth Fibonacci number.

Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno *et al.* [3] defined a similar concept for the super edge-magic total labelings. The *super edge-magic deficiency* of a graph G, denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge-magic total labeling, or ∞ if there exists no such n. More precisely, if

$$M(G) = \{n \geq 0 : G \cup nK_1 \text{ is a super edge-magic graph}\},$$

then

$$\mu_s(G) = \begin{cases} minM(G), & \text{if } M(G) \neq \emptyset, \\ \infty, & \text{if } M(G) = \emptyset. \end{cases}$$

It is easy to see that for every graph G, $\mu(G) \leq \mu_s(G)$.

In [5, 3] Figueroa-Centeno *et al.* showed the exact values of the super edge-magic deficiencies of several classes of graphs, such as cycles, complete graphs, 2-regular graphs and complete bipartite graphs $K_{2,m}$. They also proved that all forests have finite deficiency. In particular, they proved that

$$\mu_s(nK_2) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

In [9] Ngurah, Simanjuntak and Baskoro proved some upper bound for the super edge-magic deficiency of fans, double fans and wheels. In [4] Figueroa-Centeno *et al.* proved

$$\mu_s(P_m \cup K_{1,n}) = \begin{cases} 1, & \text{if } m = 2 \text{ and } n \text{ is odd or} \\ & m = 3 \text{ and } n \not\equiv 0 \pmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

In the same paper, they proved that

$$\mu_s(K_{1,m} \cup K_{1,n}) = \begin{cases} 0, & \text{if } m \text{ is a multiple of } n+1 \text{ or} \\ & n \text{ is a multiple of } m+1, \\ 1, & \text{otherwise.} \end{cases}$$

They also conjectured that every forest with two components has the super edge-magic deficiency less or equal to 1.

For a positive integer n, let $St(n)$ be a star with n leaves. Lee and Kong [8] use $St(n_1, n_2, \ldots, n_k)$ to denote the disjoint union of the k stars $St(n_1)$, $St(n_2)$,..., $St(n_k)$. They proved that the following graphs are super edge-magic: $St(m, n)$ where $n \equiv 0 \pmod{(m+1)}$, $St(1,1,n)$, $St(1,2,n)$, $St(1,n,n)$, $St(2,2,n)$, $St(2,3,n)$, $St(1,1,2,n)$ for $n \geq 2$, $St(1,1,3,n)$, $St(1,2,2,n)$ and $St(2,2,2,n)$. They conjectured that $St(n_1, n_2, \ldots, n_k)$ is super edge-magic when k is odd.

It is known that if a graph G with p vertices and q edges is super edge-magic, then $q \leq 2p - 3$, see [1].

In this paper, we will deal with the super edge-magic deficiencies of the forests formed by stars. In proving the results in this paper, we frequently use this below proposition.

Proposition 1. [2] *A graph G with p vertices and q edges is super edge-magic total if and only if there exists a bijective function $\lambda : V(G) \to \{1, 2, \ldots, p\}$ such that the set $S = \{\lambda(x) + \lambda(y) | xy \in E(G)\}$ consists of q consecutive integers. In such a case, λ extends to a super edge-magic total labeling of G.*

2 Super edge-magic deficiency of disjoint union of stars

Let $n_1, n_2, n_3, \ldots, n_k$ be positive integers. We denote the vertex set and edge set of the graph $St(n_1, n_2, n_3, \ldots, n_k)$ such that

$$V\left(St(n_1, n_2, n_3, \ldots, n_k)\right) = \{v_{j,i} : i = 0, 1, \ldots, n_j;\ j = 1, 2, ..., k\}$$

and

$$E(St(n_1, n_2, n_3, \ldots, n_k)) = \{v_{j,0}v_{j,i} : i = 1, 2, ..., n_j;\ j = 1, 2, ..., k\}.$$

For the better clarity see Figure 1.

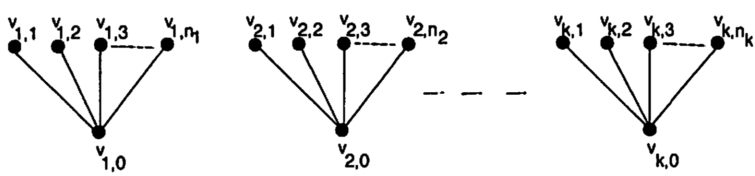

Figure 1: The graph $St(n_1, n_2, n_3, \ldots, n_k)$.

For the forest consisting of three stars we get

Theorem 1. *For the positive integers n_1, n_2, n_3, the super edge-magic deficiency of $St(n_1, n_2, n_3)$ is*

$$\mu_s(St(n_1, n_2, n_3)) \leq 1.$$

Proof. Let n_1, n_2, n_3 be the positive integers, $n_1 \leq n_2 \leq n_3$. Let $G \cong St(n_1, n_2, n_3) \cup K_1$. To show that $\mu_s(G) = 0$, we define the labeling λ_1, $\lambda_1(V(G)) \to \{1, 2, \ldots, n_1 + n_2 + n_3 + 4\}$ in the following way

$$\begin{aligned}
&\lambda_1(v_{1,0}) = 2,\ \lambda_1(v_{2,0}) = 3,\ \lambda_1(v_{3,0}) = 1,\\
&\lambda_1(v_{1,i}) = 1 + 3i, \quad 1 \leq i \leq n_1,\\
&\lambda_1(v_{2,i}) = 2 + 3i, \quad 1 \leq i \leq n_1,\\
&\lambda_1(v_{3,i}) = 3 + 3i, \quad 1 \leq i \leq n_1,\\
&\lambda_1(v_{1,i}) = n_1 + 2i + 3, \quad n_1 + 1 \leq i \leq n_2,\\
&\lambda_1(v_{2,i}) = n_1 + 2i + 2, \quad n_1 + 1 \leq i \leq n_2,\\
&\lambda_1(v_{1,i}) = n_1 + n_2 + i + 4, \quad n_2 + 1 \leq i \leq n_3,\\
&\lambda_1(z_1) = 3 + n_1 + 2n_2,
\end{aligned}$$

where z_1 is the isolated vertex. It is easy to see that λ_1 extends to a super edge-magic total labeling of G since

$$\{\lambda_1(x) + \lambda_1(y) | xy \in E(G)\} = \{6, 7, 8, \dots, 5 + n_1 + n_2 + n_3\}.$$

□

Now we are able to prove the following Theorem.

Theorem 2. *Let $n_1, n_2, \dots, n_k$, $k \geq 3$ be the positive integers. The super edge-magic deficiency of $St(n_1, n_2, \dots, n_{k-1}, n_k)$ is*

$$\mu_s(St(n_1, n_2, \dots, n_{k-1}, n_k)) \leq k - 2.$$

Proof. Let k be a positive integer, $k \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_k$. We will prove the theorem by induction. By H_k we denote the union of $St(n_1, n_2, \dots, n_k)$ and $k - 2$ isolated vertices, i.e.

$$H_k \cong St(n_1, n_2, \dots, n_k) \cup (k-2)K_1 \cong H_{k-1} \cup St(n_k) \cup K_1.$$

According to Theorem 1, we have that H_3 is super edge-magic total. Thus $\mu_s(H_3) = 0$.

By induction, from the existence of a super edge-magic total labeling of a graph H_k, we show that there exist a super edge-magic total labeling of a graph H_{k+1}. As H_k is super edge-magic total, then according to Proposition 1, there exists a labeling λ_k, $\lambda_k : V(H_k) \to \{1, 2, \dots, (n_1 + n_2 + \cdots + n_k) + 2k - 2\}$ with the property

$$\begin{aligned} &\{\lambda_k(x) + \lambda_k(y) | xy \in E(H_k)\} \\ &= \{2k, 2k+1, 2k+2, \dots, 2k - 1 + n_1 + n_2 + \cdots + n_k\}. \end{aligned}$$

We define the labeling λ_{k+1}, $\lambda_{k+1} : V(H_{k+1}) \to \{1, 2, \dots, 2k + n_1 + n_2 + \cdots + n_{k+1}\}$ of the graph H_{k+1} in the following way

$$\lambda_{k+1}(v) = \begin{cases} \lambda_k(v) + 1, & \text{if } v \in V(H_k), \\ 1, & \text{if } v = v_{k+1,0}, \\ n_1 + n_2 + \cdots + n_k + 2k + i, & \text{if } v = v_{k+1,i},\ i = 1, 2, \dots, n_{k+1}. \end{cases}$$

The new isolated vertex, denoted by z_{k-1}, is labeled such that

$$\lambda_{k+1}(z_{k-1}) = n_1 + n_2 + \cdots + n_k + 2k.$$

If $xy \in E(H_k)$, then

$$\lambda_{k+1}(x) + \lambda_{k+1}(y) = (\lambda_k(x) + 1) + (\lambda_k(y) + 1) = \lambda_k(x) + \lambda_k(y) + 2.$$

Thus the set of edge sums is

$$\{2k+2, 2k+3, \dots, 2k+1+n_1+n_2+\cdots+n_k\}.$$

If $xy \in E(St(n_{k+1}))$ then,

$$\lambda_{k+1}(x)+\lambda_{k+1}(y) = 1+n_1+n_2+\cdots+n_k+2k+i,$$

for $i = 1, 2, \dots, n_{k+1}$. The edge sums is the set of $\{n_1+n_2+\cdots+n_k+2k+2, n_1+n_2+\cdots+n_k+2k+3, \dots, n_1+n_2+\cdots+n_{k+1}+2k+1\}$.

It means that the edge sums of H_{k+1} under the labeling λ_{k+1} are consecutive integers. By Proposition 1, we can extend the labeling λ_{k+1} to a super edge-magic total labeling of H_{k+1}. As the graph H_{k+1} contains $k-1$ isolated vertices, we obtain the following result

$$\mu_s(St(n_1, n_2, \dots, n_k, n_{k+1})) \leq k-1.$$

□

The Theorem 2 gives an upper bound for the super edge-magic total deficiency of forests formed by the stars. However, in some special cases can prove that the super edge-magic total deficiency is equal to 0.

Now we will deal with the forests consisting of four stars.

Theorem 3. *For every positive integer n the graph $St(1, n, n, n+2)$ is super edge-magic total.*

Proof. Let n be a positive integer. We define the vertex labeling λ, $\lambda : V(St(1, n, n, n+2)) \to \{1, 2, \dots, 3n+7\}$ such that

$$\begin{aligned}
&\lambda(v_{1,0}) = 1,\ \lambda(v_{2,0}) = 2,\ \lambda(v_{3,0}) = 3,\\
&\lambda(v_{4,0}) = 4,\ \lambda(v_{1,1}) = 10,\ \lambda(v_{2,1}) = 7,\\
&\lambda(v_{3,1}) = 5,\ \lambda(v_{4,1}) = 6,\ \lambda(v_{4,2}) = 8,\\
&\lambda(v_{2,i}) = 3i+7, \quad 2 \leq i \leq n,\\
&\lambda(v_{3,i}) = 3i+5, \quad 2 \leq i \leq n,\\
&\lambda(v_{4,i}) = 3i, \quad 3 \leq i \leq n+2.
\end{aligned}$$

For the edge sums we get

$$\lambda(v_{1,0}) + \lambda(v_{1,1}) = 11$$

$$\lambda(v_{2,0}) + \lambda(v_{2,i}) = \begin{cases} 9, & \text{if } i = 1, \\ 9 + 3i, & \text{if } 2 \leq i \leq n, \end{cases}$$

$$\lambda(v_{3,0}) + \lambda(v_{3,i}) = \begin{cases} 8, & \text{if } i = 1, \\ 8 + 3i, & \text{if } 2 \leq i \leq n, \end{cases}$$

$$\lambda(v_{3,0}) + \lambda(v_{3,i}) = \begin{cases} 10, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 4 + 3i, & \text{if } 3 \leq i \leq n + 2. \end{cases}$$

It is easy to see that the edge sums of $St(1, n, n, n + 2)$ under the labeling λ are consecutive integers $8, 9, \ldots, 3n + 10$. Thus, according to Proposition 1, the graph $St(1, n, n, n + 2)$ is super edge-magic total. □

Theorem 4. *For every nonnegative integer n the graph $St(1, n + 5, 2n + 6, n + 1)$ is super edge-magic total.*

Proof. Let n be a nonnegative integer. According to Proposition 1 it is sufficient to prove that there exists a vertex labeling with the property that the edge sums under this labeling are consecutive integers. It is easy to see that the following labeling λ, $\lambda : V(St(1, n + 5, 2n + 6, n + 1)) \to \{1, 2, \ldots, 4n + 17\}$ has the desired property.

$$\lambda(v_{1,0}) = 5,\ \lambda(v_{2,0}) = 6,\ \lambda(v_{3,0}) = 1,\ \lambda(v_{4,0}) = 4,$$
$$\lambda(v_{1,1}) = 8,\ \lambda(v_{2,1}) = 2,\ \lambda(v_{3,1}) = 9,\ \lambda(v_{4,1}) = 7,$$
$$\lambda(v_{2,2}) = 3,\ \lambda(v_{3,2}) = 11,\ \lambda(v_{4,2}) = 20,$$
$$\lambda(v_{2,3}) = 10,\ \lambda(v_{2,4}) = 12,\ \lambda(v_{3,3}) = 13,\ \lambda(v_{3,4}) = 14,$$
$$\lambda(v_{3,5}) = 16,\ \lambda(v_{3,6}) = 18,\ \lambda(v_{3,7}) = 19,\ \lambda(v_{3,8}) = 21,$$
$$\lambda(v_{3,3}) = 14,\ \lambda(v_{4,3}) = 12,\ \lambda(v_{4,4}) = 13,$$
$$\lambda(v_{2,i}) = 22 + 4(i - 7), \quad 7 \leq i \leq n + 5,$$
$$\lambda(v_{3,i}) = \begin{cases} 6 + 2i, & \text{if } i \equiv 1 \pmod 2, 9 \leq i \leq 2n + 2, \\ 5 + 2i, & \text{if } i \equiv 0 \pmod 2, 9 \leq i \leq 2n + 2, \end{cases}$$
$$\lambda(v_{4,i}) = 23 + 4(i - 3), \quad 3 \leq i \leq n + 1.$$

The set of all edge sums forms a consecutive integer sequence 8,9,..., 4n+20.

□

By the symbol $St^k(n)$ we denote the union of k isomorphic copies of a star $St(n)$, i.e. $St^k(n) = kSt(n)$.

Theorem 5. *For every positive integers n, k, where k is even, the super edge-magic deficiency of $St^k(n)$ is*

$$\mu_s(St^k(n)) \leq 1.$$

Proof. Let n, k be the positive integers and let $G \cong St^k(n) \cup K_1$ be the graph with the vertex set and edge set as follows

$$V(G) = \{v_{j,i} : 0 \leq i \leq n, 1 \leq j \leq k\} \cup \{z\},$$

and

$$E(G) = \{v_{j,0}v_{j,i} : 1 \leq i \leq n, 1 \leq j \leq k\}.$$

For k even, to show that G is super edge-magic total, we define a labeling λ, $\lambda : V(G) \to \{1, 2, \dots, kn + k + 1\}$ in the following way

$$\lambda(v_{j,0}) = \begin{cases} \frac{k}{2} + 1 - j, & \text{if } j = 1, 2, \dots, \frac{k}{2}, \\ \frac{3}{2}k + 2 - j, & \text{if } j = \frac{k}{2} + 1, \frac{k}{2} + 2, \dots, k, \end{cases}$$

$$\lambda(v_{j,i}) = \begin{cases} 1 + ki + 2j, & \text{if } j = 1, 2, \dots, \frac{k}{2} \text{ and } i = 1, 2, \dots, n, \\ (i-1)k + 2j, & \text{if } j = \frac{k}{2} + 1, \frac{k}{2} + 2, \dots, k \text{ and } i = 1, 2, \dots, n, \end{cases}$$

$$\lambda(z) = \frac{k}{2} + 1.$$

For the edge sum of the edge $v_{j,0}v_{j,i}$, $j = 1, 2, \dots, k$; $i = 1, 2, \dots, n$, we get

$$\lambda(v_{j,0}v_{j,i}) = \frac{k}{2} + ki + j + 2.$$

It is easy to see that all the edge sums are consecutive integers. It means, according to Proposition 1, for k even the graph G is super edge-magic total. □

Enomoto *et al.* [1] proved that the complete bipartite graph is super edge-magic total if and only if it is isomorphic to the star. In [6] Figueroa-Centeno, Ichishima and Muntaner-Batle proved

Proposition 2. [6] *If G is a (super) edge-magic total bipartite or tripartite graph and k is odd, then kG is (super) edge-magic total.*

Immediately from this results we obtain that the odd number of copies of the star $St(n)$ is a super edge-magic total graph. Summarizing the previous results we obtain.

Theorem 6. *Let k, n be positive integers. Then the super edge-magic deficiency of $St^k(n)$ is either 0 if k is odd or it is at most 1 if k is even.*

Moreover, from the proof of Theorem 5, it is easy to get that also the following graphs are super edge-magic total:

- the graph $St(n, n, \ldots, n, n+1)$ for every positive integer n,
- the graph $St(n, n, \ldots, n, n+1, n+m, n+m)$ for every positive integers n and m,
- the graph $St(n, n, \ldots, n, n+1, n+m, n+m+1)$ for every positive integers n and m,
- the graph $St^k(n) \cup St(n+m)$ for $n \geq 1$, $m \geq 0$ and $k \equiv 0 \pmod 2$.

Note, that in [6] it is proved that $St(m) \cup St^{2k}(2)$ is super edge-magic total for all positive integers n and m.

3 Open problem

According to Theorem 6 we have that the super edge-magic total deficiency of the graph $St^k(n)$ is 0 for k odd and it is at most 1 for k even. However, we do not know what is the exact value in this case. Thus we would like to introduce the following open problem

Open Problem 1. *For k even, determine the exact value of the super edge-magic deficiency of the disjoint union of k isomorphic copies of a star.*

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