Construction and enumeration of a special class of transitive relations*

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Abstract

In this paper, we deal with the transitive relations on a finite n-element set. The transitive relations are interpreted as Boolean matrices. A special class of transitive relations are constructed and enumerated, which can generate all transitive relations on a finite n-element set by intersection operation. Besides, several necessary and sufficient conditions that a relation R is transitive are given.

Keywords: Binary relations; Transitive relations; Boolean matrices; Enumeration

1 Introduction

Binary relations, especially equivalence and various ordering relations, play an important role in modeling different kinds of fundamental concepts

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related to social sciences, decision-making, domain theory and measurement theory. In all cases, the transitivity of a binary is a crucial property [6].

Let X be a finite n-element set. A binary relation (or relation) R on X is a subset of $X \times X$. A relation R on X is called transitive if for all $x, y, z \in X$, the conditions $(x, y) \in R$ and $(y, z) \in R$ imply that $(x, z) \in R$. Counting the transitive relations on a finite n-element set is a long-standing open problem [10]. There is no known general formula for the number of such a simply-stated relations until now. In 1975, Kleitman and Rothschild [9] showed that the number of transitive relations on a finite n-element set is asymptotically 2^n times the number of partial orders. In 1997, Klaska [11] proved that there is a one-to-one correspondence between transitive relations and partial orders. Further, he deduced a recurrence formula for their enumeration. In fact, the enumeration of all finite posets is also a long-standing open problem. We defer to [12, 14] for a historical survey. No reasonable explicit or recursive formula for the numbers of posets on n elements is known. Though various algorithms [1–5, 14, 15] have been proposed and applied for the constructive enumeration of partial posets, the best result now is up to $n = 16$ [1] due to huge numbers of them and the quick exponential growth of their numbers. Based on these algorithms, we can also determine the number of all transitive relations up to $n = 16$.

The purpose of this paper is to construct and enumerate a special class of transitive relations, which has the similar performance of a basis of linear space, to generate all transitive relations on a finite n-element set.

In this paper, we interpret transitive relations as Boolean matrices to research their structure and properties. The remainder of this paper is organized as follows. In Section 2, we discuss the properties of transitive relations, and give two necessary and sufficient conditions that a relation is transitive. In Section 3, we construct a special class of transitive relations on a finite n-element set, and obtain a formula for the number of them. In Section 4, we show that any transitive relation can be generated by these special class of transitive relations. Finally, the paper concludes in Section 5.

2 Some properties of transitive relations

In this section, some properties of transitive relations will be recalled and obtained. Let $I_n = \{1, 2, \cdots, n\}$, and $\mathscr{B}_n = \{R = (R_{ij})_{n\times n} : R_{ij} \in \{0, 1\}, (i, j) \in I_n \times I_n\}$ be the set of all $n \times n$ Boolean matrices with the usual matrix multiplication except that we assume $1+1=1$ [7,8]. It is isomorphic (in a natural way) to the set of all binary relations on finite n-element set, where the operation is the composition of relations(see [13], P.4).

Definition 2.1 [10] If $R, Q \in \mathscr{B}_n$, then $R \leq Q$ if and only if $R_{ij} \leq Q_{ij}$ for all $(i,j) \in I_n \times I_n$. In particular, $R = Q$ if and only if $R_{ij} = Q_{ij}$ for all $(i,j) \in I_n \times I_n$.

Definition 2.2 [10] Let $R \in \mathscr{B}_n$. We define $R^2 \in \mathscr{B}_n$, given by

$$R^2_{ij} = \bigvee_{p=1}^{n} (R_{ip} \wedge R_{pj})$$

for all $(i,j) \in I_n \times I_n$, with $\vee$ and $\wedge$ denoting the usual maximum and minimum, respectively.

Definition 2.3 [10] Let $R, Q \in \mathscr{B}_n$. We define $R \cap Q \in \mathscr{B}_n$, given by $(R \cap Q)_{ij} = R_{ij} \wedge Q_{ij}$ for all $(i,j) \in I_n \times I_n$.

Theorem 2.1 [10] R is a transitive relation if and only if $R^2 \leq R$.

Theorem 2.2 [10] If R and Q are transitive relations, so is $R \cap Q$.

For any $(i,j) \in I_n \times I_n$, let $L_R(i) = \{h \in I_n : R_{ih} = 1\}$ and $C_R(j) = \{k \in I_n : R_{kj} = 1\}$.

Theorem 2.3 $R^2 \leq R$ if and only if $R_{ij} = 0$ implies that $L_R(i) \cap C_R(j) = \emptyset$ for all $(i,j) \in I_n \times I_n$.

Proof The first part. For any $(i,j) \in I_n \times I_n$, if $R_{ij} = 0$, then $R^2_{ij} = \bigvee_{p=1}^{n}(R_{ip} \wedge R_{pj}) = 0$ by $R^2 \leq R$. Therefore, $R_{ip} \wedge R_{pj} = 0$ for all $p \in I_n$. Suppose that $L_R(i) \cap C_R(j) \neq \emptyset$, then there must exist a $k \in L_R(i) \cap C_R(j)$ such that $R_{ik} \wedge R_{kj} = 1$, a contradiction. Hence, $L_R(i) \cap C_R(j) = \emptyset$.

For the converse implication. To verify $R^2 \leq R$, it is only to verify that for any $(i,j) \in I_n \times I_n$, $R_{ij} = 0$ implies $R^2_{ij} = 0$. If $R_{ij} = 0$, then $L_R(i) \cap C_R(j) = \emptyset$. That is, $p \notin L_R(i) \cap C_R(j)$ and $R_{ip} \wedge R_{pj} = 0$ for all $p \in I_n$. Therefore, $R^2_{ij} = \bigvee_{p=1}^{n}(R_{ip} \wedge R_{pj}) = 0$.

Corollary 2.1 $R^2 \leq R$ if and only if for any $(i,j) \in I_n \times I_n$, if $L_R(i) \neq I_n$, then $L_R(i) \cap (\cup_{j \notin L_R(i)} C_R(j)) = \emptyset$.

Proof Immediately from Theorem 2.3.

Example 2.1 Let $R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. Obviously, R is transitive by Theorem 2.1. It is easy to see that $L_R(2) = L_R(3) = \{2,3\}$, $L_R(1) = L_R(4) = C_R(2) = C_R(3) = I_n$, $C_R(1) = C_R(4) = \{1,4\}$, $R_{21} = R_{24} = R_{31} = R_{34} = 0$. We have $L_R(2) \cap C_R(1) = \emptyset$, $L_R(2) \cap C_R(4) = \emptyset$, $L_R(3) \cap$

$C_R(1) = \emptyset$, $L_R(3) \cap C_R(4) = \emptyset$, $L_R(2) \cap (\cup_{j \notin L_R(2)} C_R(j)) = \emptyset$ and $L_R(3) \cap (\cup_{j \notin L_R(3)} C_R(j)) = \emptyset$.

3 A special class of transitive relations

This section will investigate a special class of transitive relations and show a method to construct them.

Definition 3.1 Let $i \in I_n$ and $X \subseteq I_n$.

(1) If $R^2 \le R$ and $L_R(i) = X$, then R is called an $(i, X)-$transitive relation.

(2) If the following conditions hold:

(i) R is an $(i, X)-$transitive relation;

(ii) For any $Q \in \mathscr{B}_n$, if Q is an $(i, X)-$transitive relation, then $R \ge Q$;

then R is called the maximum $(i, X)-$transitive relation.

For any $i \in I_n$, $X \subseteq I_n$, if there exists the maximum $(i, X)-$transitive relation, then it is denoted by $T^{(i,X)}$.

Remark 3.1 It is easy to see that a transitive relation on a finite n-element set has n different denotations according to Definition 3.1. For example, let $R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ such that $R^2 \le R$, then R is an $(1, \{1\})-$transitive relation, and it is also a $(2, \emptyset)-$transitive relation. However, the maximum $(1, \{1\})$-transitive relation is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and the maximum $(2, \emptyset)$-transitive relation is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Remark 3.2 Let $R \in \mathscr{B}_n$, $R^2 \le R$ and $h \in I_n$. If $T^{(h, L_R(h))}$ exists, then $L_{T^{(h,L_R(h))}}(h) = L_R(h)$.

Theorem 3.1 For any $i \in I_n$, $X \subseteq I_n$,

$$T_{hk}^{(i,X)} = \begin{cases} 0, & h \in (X \cup \{i\}) \text{ and } k \in I_n \backslash X, \\ 1, & \text{otherwise} \end{cases}$$

for all $(h, k) \in I_n \times I_n$.

Proof For any $i \in I_n$, $X \subseteq I_n$, we can construct a $Q \in B^{n \times n}$ as follows:

$$Q_{hk} = \begin{cases} 0, & h \in (X \cup \{i\}) \text{ and } k \in I_n \backslash X, \\ 1, & \text{otherwise} \end{cases}$$

for all $(h, k) \in I_n \times I_n$.

(I) At first, we prove that Q is an $(i, X)-$transitive relation.

(i) Since $i \in (X \cup \{i\})$, it is easy to see that $L_Q(i) = X$.

(ii) For any $h \in I_n$, if $L_Q(h) \neq I_n$, then $h \in (X \cup \{i\})$ and $L_Q(h) = X$. Again, $C_Q(k) = I_n \backslash (X \cup \{i\})$ for all $k \notin L_Q(h) = X$. It follows that $L_Q(h) \cap (\cup_{k \notin L_Q(h)} C_Q(k)) = X \cap [I_n \backslash (X \cup \{i\})] = \emptyset$, thus $Q^2 \leq Q$ by Corollary 2.1.

(i) and (ii) imply that Q is an (i, X)−transitive relation by Theorem 2.1 and Definition 3.1.

(II) Then, we prove that $Q = T^{(i,X)}$. For any (i, X)−transitive relation $R \in I_n \times I_n$, we want to prove that $R \leq Q$ by Definition 3.1. Obviously, we only want to verify that $Q_{h_0 k_0} = 0$ implies that $R_{h_0 k_0} = 0$ for all $(h_0, k_0) \in I_n \times I_n$.

For any $(h_0, k_0) \in I_n \times I_n$, if $Q_{h_0 k_0} = 0$, then $h_0 \in (X \cup \{i\})$ and $k_0 \in I_n \backslash X$. Again, $L_R(i) = X$ since R is an (i, X)−transitive relation, thus $h_0 \in (L_R(i) \cup \{i\})$ and $k_0 \notin L_R(i)$. There are two cases:

Case 1: $h_0 = i$. Then $R_{h_0 k_0} = 0$ since $k_0 \notin L_R(i)$.

Case 2: $h_0 \in L_R(i)$. In this situation, we have $h_0 \in L_R(i)$ but $k_0 \notin L_R(i)$, i.e., $R_{ih_0} = 1$ but $R_{ik_0} = 0$. If $R_{h_0 k_0} = 1$, then $R_{ih_0} \wedge R_{h_0 k_0} = 1$. This implies that $R^2_{ik_0} = 1 > R_{ik_0}$, a contradiction to Theorem 2.1. Therefore, $R_{h_0 k_0} = 0$.

From Theorem 3.1, we have:

Corollary 3.1 For any $X \subseteq I_n$, $i_1, i_2 \in I_n$, the following statements hold:

(i) If $i_1, i_2 \in X$, then $T^{(i_1,X)} = T^{(i_2,X)}$.

(ii) If $i_1 \in X$ and $i_2 \notin X$, then $T^{(i_1,X)} \geq T^{(i_2,X)}$ but $T^{(i_1,X)} \neq T^{(i_2,X)}$.

(iii) If $i_1, i_2 \notin X$, then $T^{(i_1,X)} \neq T^{(i_2,X)}$.

Corollary 3.2 For any $i \in I_n$, $X_1, X_2 \subseteq I_n$, the following statements hold:

(i) If $X_1 \neq X_2$, then $T^{(i,X_1)} \neq T^{(i,X_2)}$.

(ii) $X_1 \subset X_2$ is not necessary for $T^{(i_1,X)} \leq T^{(i_2,X)}$.

Example 3.1 Let $n = 4$ and $X_1 = \{2, 3\}, X_2 = \{1, 3\}, X_3 = \{1, 2, 3\}$. By Theorem 3.2, we have:

$$T^{(1,X_1)} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, T^{(2,X_1)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$T^{(3,X_1)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, T^{(4,X_1)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$T^{(1,X_2)} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, T^{(1,X_3)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Obviously, $T^{(2,X_1)} = T^{(3,X_1)} \geq T^{(1,X_1)}$, $T^{(2,X_1)} \neq T^{(1,X_1)}$, $T^{(1,X_1)} \neq T^{(4,X_1)}$, $T^{(1,X_1)} \neq T^{(1,X_2)}$, $T^{(1,X_1)} \leq T^{(1,X_3)}$, but $T^{(1,X_2)}$ and $T^{(1,X_3)}$ are incomparable.

Remark 3.3 Theorem 3.1 implies that for any $i \in I_n$, $X \subseteq I_n$, $T^{(i,X)}$ always exists.

Let $\mathscr{T}_n^X = \{T^{(i,X)} : i \in I_n\}$, $\mathscr{T}_n^* = \bigcup_{X \subseteq I_n} \mathscr{T}_n^X$ and $|X|$ be the cardinal number of X. By Theorem 3.1 and Corollary 3.1, we have:

Corollary 3.3 For any $X \subseteq I_n$, if $|X| \neq 0$, then $|\mathscr{T}_n^X| = n - |X| + 1$; otherwise, $|\mathscr{T}_n^X| = n$.

Corollary 3.4 $|\mathscr{T}_n^*| = n + \sum_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (n - k + 1)$, where $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$ is the combination formula.

Example 3.2 Let $n = 5$ and $X = \{1, 3\}$. Consider $\mathscr{T}_5^{\{1,3\}}$.

Solution By Theorem 3.1 and Corollary 3.3, we have:

$$|\mathscr{T}_5^{\{1,3\}}| = 4, \mathscr{T}_5^{\{1,3\}} = \{R^{(1,\{1,3\})}, R^{(2,\{1,3\})}, R^{(4,\{1,3\})}, R^{(5,\{1,3\})}\},$$

where

$$R^{(2,\{1,3\})} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, R^{(4,\{1,3\})} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$R^{(5,\{1,3\})} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}, R^{(1,\{1,3\})} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

$$R^{(3,\{1,3\})} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = R^{(1,\{1,3\})}.$$

4 An method to obtain all the transitive relations on $I_n \times I_n$

In this section, we will investigate the relationship between the maximum transitive relations and the general ones. That is, we can construct all the transitive relations from those maximum ones by intersection operation.

Theorem 4.1 If $R^2 \leq R$, then there exits a $Q \in \mathscr{T}_n^*$ such that $Q \geq R$.

Proof By Definition 3.1, it is easy to see that R is an $(i, L_R(i))$-transitive relation, and $R \leq T^{(i,L_R(i))} \in \mathscr{T}_n^*$ for all $i \in I_n$.

Theorem 4.2 If $R^2 \leq R$, then there exists an $M \subseteq \mathscr{T}_n^*$ such that $R = \cap_{Q \in M} Q$.

Proof (i) Obviously, $R \leq \cap_{i \in I_n} T^{(i,L_R(i))}$ by the proof of Theorem 4.1.

(ii) For any $(h,k) \in I_n \times I_n$, if $R_{hk} = 0$, then $T_{hk}^{(h,L_R(h))} = 0$ since $L_R(h) = L_{T^{(h,L_R(h))}}(h)$. It follows that $(\cap_{i \in I_n} T^{(i,L_R(i))})_{hk} = 0$, i.e., $R \geq \cap_{i \in I_n} T^{(i,L_R(i))}$. (i) and (ii) imply that $R = \cap_{i \in I_n} T^{(i,L_R(i))}$. Let $M = \{T^{(i,L_R(i))} : i \in I_n\}$, then $R = \cap_{Q \in M} Q$.

Let $\mathscr{T}_n$ be the set of all the transitive relations on $I_n \times I_n$ and $T_n = |\mathscr{T}_n|$. By the Theorems 2.2 and 4.2, the following two theorems hold:

Theorem 4.3 $R^2 \leq R$ if and only if $R = \cap_{i \in I_n} T^{(i,L_R(i))}$.

Theorem 4.4 $\mathscr{T}_n = \{\cap_{R \in M} R : M \subseteq \mathscr{T}_n^*\}$.

Theorem 4.5 Let $R \in \mathscr{B}_n$,

$$\cap_{h \in I_n} L_{T^{(h,L_R(h))}}(i) = L_R(i) \cap (\cap_{i \in L_R(h), i \neq h} L_R(h))$$

for all $i \in I_n$.

Proof For any $i, h \in I_n$, there are three cases:

Case 1: $h = i$. Clearly, $L_{T^{(h,L_R(h))}}(i) = L_R(h) = L_R(i)$ by Remark 3.2.

Case 2: $h \neq i$ and $i \in L_R(h)$. By Theorem 3.1, we have $L_{T^{(h,L_R(h))}}(i) = L_R(h)$.

Case3: $h \neq i$ and $i \notin L_R(h)$. $L_{T^{(h,L_R(h))}}(i) = I_n$ immediately from Theorem 3.1.

Therefore, $\cap_{h \in I_n} L_{T^{(h,L_R(h))}}(i) = L_R(i) \cap (\cap_{i \in L_R(h), i \neq h} L_R(h))$.

Theorem 4.6 $R^2 \leq R$ if and only if $L_R(i) \subseteq \cap_{i \in L_R(h), i \neq h} L_R(h)$ for all $i \in I_n$.

Proof Necessity. By Theorem 4.3, $R = \cap_{i \in I_n} T^{(i,L_R(i))}$ since $R^2 \leq R$. This implies that $L_R(i) = \cap_{h \in I_n} L_{T^{(h,L_R(h))}}(i)$ for all $i \in I_n$. Again, $L_R(i) = L_R(i) \cap (\cap_{i \in L_R(h), i \neq h} L_R(h))$ by Theorem 4.5. Hence, $L_R(i) \subseteq \cap_{i \in L_R(h), i \neq h} L_R(h)$.

Sufficiency. For any $i \in I_n$, $\cap_{h\in I_n} L_{R^{(h,L_R(h))}}(i) = L_R(i) \cap (\cap_{i\in L_R(h), i\neq h} L_R(h))$ by Theorem 4.4, thus $\cap_{h\in I_n} L_{T^{(h,L_R(h))}}(i) = L_R(i)$ since $L_R(i) \subseteq \cap_{i\in L_R(h), i\neq h} L_R(h)$.

It follows that $R = \cap_{i\in I_n} T^{(i,L_R(i))}$. Therefore, $R^2 \leq R$ by Theorem 4.3.

Remark 4.1 By Corollary 3.4 and Theorem 4.3, we have

$$T_n \leq \sum_{k=1}^{|\mathscr{T}_n^*|} \binom{|\mathscr{T}_n^*|}{k},$$

where $|\mathscr{T}_n^*| = n + \sum_{k=1}^{n} \binom{n}{k} (n-k+1)$.

From Theorem 4.4, we can give a method to construct all the transitive relations on $I_n \times I_n$.

Algorithm 4.1

Step1 Obtain $\mathscr{T}_n^X$ for all $X \in I_n$ according to Theorem 3.1 and Corollary 3.1;

Step2 $\mathscr{T}_n^* = \bigcup_{X\subseteq I_n} \mathscr{T}_n^X$;

Step3 $\mathscr{T}_n = \{\cap_{R\in M} R : M \subseteq \mathscr{T}_n^*\}$;

Step4 $T_n = |\mathscr{T}_n|$;

Step5 Stop.

Remark 4.2 In fact, there always exists a general algorithm to obtain all the transitive relations on $I_n \times I_n$. Since $\mathscr{T}_n \subseteq \mathscr{B}_n$, we can test the Boolean matrices in $\mathscr{B}_n$ one by one to determine whether it is a transitive relation or not within 2^{n^2} steps. On the other hand, one can easily see that the key part of Algorithm 4.1 is to construct the maximum transitive relations. From Corollaries 3.3 and 3.4, we can obtain them in $|\mathscr{T}_n^*| \leq n\sum_{k=0}^{n} \binom{n}{k} = n2^n$ steps. Though it is not a polynomial algorithm we expected, it has much less computational complexity.

By means of the Algorithm 4.1 we have shown how to construct all the transitive relations on $I_n \times I_n$. Now we demonstrate this algorithm by a suitable example.

Example 4.1 Let $n = 2$. Construct all the transitive relations on $I_2 \times I_2$ and count them.

$$\text{Step1 } \mathscr{T}_2^{\emptyset} = \{\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\}, \mathscr{T}_2^{\{1\}} = \{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\},$$

$$\mathscr{T}_2^{\{2\}} = \{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\}, \mathscr{T}_2^{\{1,2\}} = \{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\}.$$

Step2 $\mathscr{T}_2^* = \{\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\}.$

Step3 $\mathscr{T}_2 = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\}.$

Step4 $T_2 = 13$.

Step5 Stop.

5 Conclusion

This paper studied the transitive relations on a finite n-element set. Several necessary and sufficient conditions that a relation R is transitive are given. A special type of transitive relations are constructed and counted, which can be used to obtain all the transitive relations. We hope that these results can provide some help for future research on related problems, especially the enumeration of all finite transitive relations.

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