RECOGNIZING TENACIOUS GRAPHS IS NP-HARD

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Abstract

In this paper we settle a long-standing open problem. We prove that it is NP-hard to recognize T-tenacious graphs for any fixed positive rational number T.

Keywords: Tenacity, Tenacious, NP-Completeness.

AMS subject Classification: 68R10, 05C38.

1. Introduction

We consider only finite undirected graphs without loops and multiple edges. Let G be a graph. We denote by $V(G)$, $E(G)$ and $|V(G)|$ the set of vertices, the set of edges and the order of G, respectively. The concept of tenacity of a graph G was introduced in [4,5], as a useful measure of the "vulnerability" of G. In [5] Cozzens et al. calculated tenacity of the first and second case of the Harary Graphs but they didn't show the complete proof of the third case. In [18] we showed a new and complete proof for case three of the Harary Graphs. In [12], we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [3 - 26], the authors studied more about this new invariant. The tenacity of a

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graph G, $T(G)$, is defined by $T(G) = min\{\frac{|S|+\tau(G-S)}{k(G-S)}\}$, where the minimum is taken over all vertex cutsets S of G. We define $\tau(G-S)$ to be the number of the vertices in the largest component of the graph $G-S$, and $k(G-S)$ be the number of components of $G-S$. A connected graph G is called T-tenacious if $\mid S \mid + \tau(G-S) \geq Tk(G-S)$ holds for any subset S of vertices of G with $k(G-S) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p ($p \geq 1$). A set $S \subseteq V(G)$ is said to be a T-set of G if $T(G) = \frac{|S|+\tau(G-S)}{k(G-S)}$.

The Mix-tenacity $T_m(G)$ of a graph G is defined as

$$T_m(G) = \min_{A \subset E(G)} \{\frac{\mid A \mid + \tau(G-A)}{k(G-A)}\}$$

where $\tau(G-A)$ denotes the order (the number of vertices) of a largest component of $G-A$ and $k(G-A)$ is the number of components of $G-A$. Cozzens et al. in [4], called this parameter Edge-tenacity, but Moazzami changed the name of this parameter to Mix-tenacity. It seems Mix-tenacity is a better name for this parameter. $T(G)$ and $T_m(G)$ turn out to have interesting properties.

After the pioneering work of Cozzens, Moazzami, and Stueckle in [4,5], several groups of researchers have investigated tenacity, and its related problems. In [19] and [20] Piazza et al. used the $T_m(G)$ as Edge-tenacity. But this parameter is a combination of cutset $A \subset E(G)$ and the number of vertices of a largest component, $\tau(G-A)$. It may be observed that in the definition of $T_m(G)$, the number of edges removed is added to the number of vertices in a largest component of the remaining graph. Also this parameter didn't seem very satisfactory for Edge-tenacity. Thus Moazzami and Salehian introduced a new measure of vulnerability, the Edge-tenacity, $T_e(G)$, in [16]. The Edge-tenacity $T_e(G)$ of a graph G is defined as

$$T_e = \min_{A \subset E(G)} \{\frac{\mid A \mid + \tau(G-A)}{k(G-A)}\}$$

where $\tau(G-A)$ denotes the order (the number of edges) of a largest component of $G-A$ and $k(G-A)$ is the number of components of $G-A$. This new measure of vulnerability involves edges only and thus is called the Edge-tenacity. Since 1992 there were several interesting questions. But the question " How difficult is it to recognize T-tenacious graphs? " has remained an interesting open problem for some time. The question was first raised by Moazzami in [11]. Our purpose here is to show that for any fixed positive rational number T, it is NP-hard to recognize T-tenacious graphs. To prove this we will show that it is NP-hard to recognize T-tenacious graphs by reducing a well-known NP-complete variant of INDEPENDENT SET. Any undefined terms can be found in the standard references on graph theory, including Bondy and Murty [1].

2. Main results

We begin by considering the following problem.

Not T-tenacious

Instance: An undirected graph G and a fixed positive rational number T.

Question: Does there exists $X \subseteq V(G)$ with $k(G-X) > 1$ such that $Tk(G-X) > |X| + \tau(G-X)$?

Claim: Not T-tenacious is NP-complete.
To prove this, we will reduce the following problem, which is known to be NP-complete [2, p.194].

INDEPENDENT MAJORITY.

Instance: An undirected graph G.

Question: Does G contain an independent set $I \subseteq V(G)$ with $|I| \geq \left\lceil \frac{1}{2}|V(G)| \right\rceil$?

Clearly Not T-tenacious $\in NP$, and we prove only that Not T-tenacious is NP-hard. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. Suppose $T = c$, any fixed positive rational number. We consider the following two cases.

Theorem 1. *Not T-tenacious is NP-complete, where $T = c < 1$.*
Proof: Construct G' from G as follows. Add to G, n disjoint copies of K_{n-1} by $G_1, \ldots, G_n$, and join vertex $v_i \in V(G)$ to any vertex in G_i, $1 \leq i \leq n$.
Then add a star graph, $K_{1,m}$, where $m = \left\lfloor \frac{n}{c} + \frac{\lfloor \frac{n}{2} \rfloor}{c} - n + \frac{2}{c} \right\rfloor$, and join $s \in K_{1,m}$ to every vertex of $V(G) \cup G_i$, $1 \leq i \leq n$, (Fig.1), where s is the center of $K_{1,m}$.

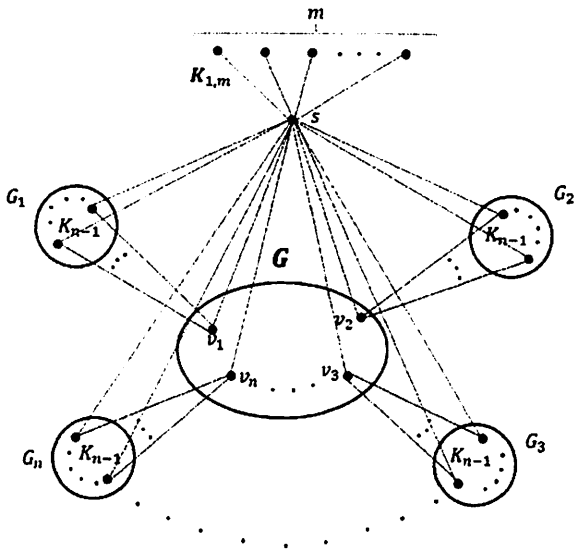

Fig.1. Graph G', combination of $K_{1,m}$ and $V(G) \cup G_i$.

To complete the proof, it suffices to show that G contains an independent set I with $|I| \geq \lceil \frac{n}{2} \rceil$ if and only if G' is not T-tenacious.
Suppose first that G contains an independent set $I \subseteq V(G)$ with $|I| \geq \lceil \frac{n}{2} \rceil$. Define $X' \subseteq V(G')$ by $X' = (V(G) - I) \cup \{s\}$. Note that $|X'| \leq \lfloor \frac{n}{2} \rfloor + 1$. But it is easy to verify that $k(G' - X') = n + m$ and $\tau(G' - X') = n$, thus

$$Tk(G' - X') = Tn + Tm > Tn + T\left(\frac{n}{c} + \frac{\lfloor \frac{n}{2} \rfloor}{c} - n + \frac{2}{c} - 1\right)$$

$$= T(\frac{n}{c}) + \frac{T\lfloor \frac{n}{2} \rfloor}{c} + \frac{2T}{c} - T$$

$$= n + \lfloor \frac{n}{2} \rfloor + 2 - c$$

Since $c < 1$, we have

$$n + \lfloor \frac{n}{2} \rfloor + 2 - c > \lfloor \frac{n}{2} \rfloor + 1 + n$$

Thus

$$Tk(G' - X') > |X'| + \tau(G' - X')$$

and therefore G' is not T-tenacious.

Conversely, suppose G' is not T-tenacious. Then there exists $X' \subseteq V(G')$ with $k(G'-X') > 1$ such that $Tk(G'-X') > |X'| + \tau(G'-X')$. Since $k(G'-X') > 1$, we have $s \in X'$.
We may assume $|X'| \geq 2$, otherwise $X' = \{s\}$ and $k(G'-X') = 1+m$, $\tau(G'-X') = n^2$ (see Fig.1). In this event, we have $T\Big(1+(\frac{n}{c}+\frac{\lfloor\frac{n}{2}\rfloor}{c}-n+\frac{2}{c})\Big) \geq T(1+m) = Tk(G'-X') > |X'| + \tau(G'-X') = 1+n^2$, and then $c(1-n)+n+\lfloor\frac{n}{2}\rfloor+1 > n^2$, and this is a contradiction for $n \geq 2$. Therefore $|X'| \geq 2$.
We may also assume $X' \cap G_i = \phi$, $1 \leq i \leq n$, since otherwise suppose $B = X' \cap G_i \neq \phi$. When we remove $s \in K_{1,m}$, the complete component G_i is only connected to $v_i \in V(G)$, $1 \leq i \leq n$. Thus removing B from G' does not make any component. Therefore we have $k(G'-(X'-B)) \geq k(G'-X')$ and $\tau(G'-(X'-B)) \leq |B| + \tau(G'-X')$.
Then

$$\begin{aligned} Tk(G'-(X'-B)) &\geq Tk(G'-X') \\ &> |X'| + \tau(G'-X') \\ &= |X'-B| + |B| + \tau(G'-X') \\ &\geq |X'-B| + \tau(G'-(X'-B)) \end{aligned}$$

and we could use $X'-B$ instead of X'. Therefore $B = X' \cap G_i = \phi$, $\tau(G'-X') \geq n-1$, and $k(G'-X') \leq n+m$. On the other hand $\tau(G'-X') \leq n$, since otherwise there exist at least two components G_i and G_j that are connected by a path from v_i to v_j. Then $k(G'-X') \leq n-1+m$ and $\tau(G'-X') \geq 2n$.
Thus

$$\begin{aligned} T\Big(n-1+(\frac{n}{c}+\frac{\lfloor\frac{n}{2}\rfloor}{c}-n+\frac{2}{c})\Big) \geq T(n-1+m) &\geq Tk(G'-X') \\ &> |X'| + \tau(G'-X') \\ &\geq 2+2n \end{aligned}$$

$$\begin{aligned} -c+n+\lfloor\frac{n}{2}\rfloor+2 &> 2+2n \\ n+\lfloor\frac{n}{2}\rfloor &> 2n+c \\ \lfloor\frac{n}{2}\rfloor &> n+c \end{aligned}$$

And this is a contradiction.
We claim that $\tau(G' - X') \neq n-1$, since otherwise $X' \cap V(G) = V(G)$ and $|X'| \geq n+1$.
Thus

$$T\left(n + (\frac{n}{c} + \frac{\lfloor\frac{n}{2}\rfloor}{c} - n + \frac{2}{c})\right) \geq T(n+m) \geq Tk(G' - X')$$
$$> |X'| + \tau(G' - X')$$
$$\geq (n+1) + (n-1)$$

$$n + \lfloor\frac{n}{2}\rfloor + 2 > 2n$$
$$\lfloor\frac{n}{2}\rfloor + 2 > n$$

and for $n > 2$, this is a contradiction. Thus $\tau(G' - X') = n$. Now assume that $X' \cap K_{1,m} = s$, since otherwise, suppose that $D = K_{1,m} - s$. (We note that $\tau(G' - X') = \tau(G' - (X' - D)) = n$).

$$Tk(G' - (X' - D)) \geq Tk(G' - X')$$
$$> |X'| + \tau(G' - X')$$
$$> |X' - D| + \tau(G' - (X' - D))$$

and we could use $X' - D$ instead of X'. Thus $X' \cap K_{1,m} = s$.
Therefore $k(G' - X') = n + m$, $X' \subset V(G) \cup \{s\}$.
Then

$$T\left(n + (\frac{n}{c} + \frac{\lfloor\frac{n}{2}\rfloor}{c} - n + \frac{2}{c})\right) \geq T(n+m)$$
$$= Tk(G' - X')$$
$$> |X'| + n$$

Thus

$$n + \lfloor\frac{n}{2}\rfloor + 2 > |X'| + n$$
$$\lfloor\frac{n}{2}\rfloor + 2 > |X'|$$

and since $|X'|$ is an integer number, we have $|X'| \leq \lfloor\frac{n}{2}\rfloor + 1$.
Let $X = X' - s$, then $|X| \leq \lfloor\frac{n}{2}\rfloor$, and $X \subset V(G)$. We have shown that no two vertices $v_i, v_j \in V(G)$ are connected by a path in $G' - X'$. Thus $G - X$

contains at least $\lceil \frac{n}{2} \rceil$ components. Choosing one vertex in each component of $G - X$ yields a set of at least $\lceil \frac{n}{2} \rceil$ independent vertices in G.

Theorem 2. *Not T-tenacious is NP-complete, where $T = c \geq 1$.*
Proof: Construct G' from G as follows. Add to G, n disjoint copies of K_p by $G_1, \ldots, G_n$, where $p = \lceil 2nc - \lfloor \frac{n}{2} \rfloor - 3 \rceil$, and join any vertex of G_i to v_i in G, $1 \leq i \leq n$.
Then add a star graph, $K_{1,n}$, and join $s \in K_{1,n}$ to every vertex of $V(G) \cup G_i$, $1 \leq i \leq n$, (Fig.2), where s is the center of $K_{1,n}$.

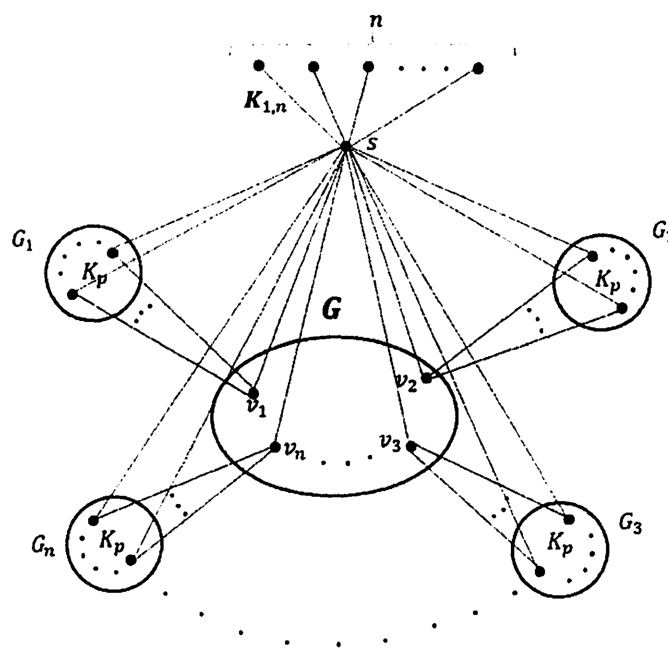

Fig.2. Graph G', combination of $K_{1,n}$ and $V(G) \cup G_i$.

To complete the proof, it suffices to show that G contains an independent set I with $|I| \geq \lceil \frac{n}{2} \rceil$ if and only if G' is not T-tenacious. Suppose first that G contains an independent set $I \subseteq V(G)$ with $|I| \geq \lceil \frac{n}{2} \rceil$. Define $X' \subseteq V(G')$ by $X' = (V(G) - I) \cup \{s\}$. Note that $|X'| \leq \lfloor \frac{n}{2} \rfloor + 1$. But it is easy to verify that $k(G' - X') = 2n$ and $\tau(G' - X') = \lceil 2nc - \lfloor \frac{n}{2} \rfloor - 2 \rceil$
Thus

$$\begin{aligned} Tk(G' - X') = 2nT &= (\lfloor \frac{n}{2} \rfloor + 1) + (2nc - \lfloor \frac{n}{2} \rfloor - 2) + 1 \\ &> (\lfloor \frac{n}{2} \rfloor + 1) + \lceil 2nc - \lfloor \frac{n}{2} \rfloor - 2 \rceil \end{aligned}$$

$$\geq |X'| + \tau(G' - X')$$

and therefore G' is not T-tenacious.

Conversely, suppose G' is not T-tenacious. Then there exists $X' \subseteq V(G')$ with $k(G' - X') > 1$ such that $Tk(G' - X') > |X'| + \tau(G' - X')$. Since $k(G' - X') > 1$, we have $s \in X'$. We may assume that $|X'| \geq 2$ since otherwise we have $k(G' - X') = 1 + n$ and $\tau(G' - X') = n(p + 1)$.
Thus

$$\begin{aligned} T(n+1) &= Tk(G' - X') \\ &> |X'| + \tau(G' - X') \\ &= 1 + n(p+1) \\ &\geq 1 + n(2nc - \lfloor \frac{n}{2} \rfloor - 2) \end{aligned}$$

We have

$$T + nT > 1 + 2n^2c - n\lfloor \frac{n}{2} \rfloor - 2n$$
$$(c-1) + n(\lfloor \frac{n}{2} \rfloor + 2) > nc(2n-1)$$

and this is a contradiction for $c \geq 1$ and $n > 2$. Therefore $|X'| \geq 2$.
We may assume that $G_i \cap X' = \phi$, $1 \leq i \leq n$, otherwise suppose $B = (G_i \cap X') \neq \phi$. Now G_i is only connected to $v_i \in V(G)$, and G_i is complete component. Therefore it is clear that removing B from G' does not make any component. Then we have $k(G' - (X' - B)) \geq k(G' - X')$ and $\tau(G' - (X' - B)) \leq |B| + \tau(G' - X')$ then

$$\begin{aligned} Tk(G' - (X' - B)) &\geq Tk(G' - X') \\ &> |X'| + \tau(G' - X') \\ &= |X' - B| + |B| + \tau(G' - X') \\ &\geq |X' - B| + \tau(G' - (X' - B)) \end{aligned}$$

and we could use $X' - B$ instead of X'. Therefore $B = G_i \cap X' = \phi$, $\tau(G' - X') \geq \lceil 2nc - \lfloor \frac{n}{2} \rfloor - 3 \rceil$ and $k(G' - X') \leq 2n$. We may assume that $\tau(G' - X') \leq \lceil 2nc - \lfloor \frac{n}{2} \rfloor - 2 \rceil$, since otherwise there exist at least two components G_i and G_j which are connected by a path from v_i to v_j. Then

$k(G'-X') \leq (n-1)+n$ and $\tau(G'-X') \geq 2\lceil 2nc - \lfloor \frac{n}{2} \rfloor - 2 \rceil$.
Thus

$$\begin{aligned} T(2n-1) &\geq Tk(G'-X') \\ &> |X'| + \tau(G'-X') \\ &\geq 2 + 2\lceil 2nc - \lfloor \frac{n}{2} \rfloor - 2 \rceil \\ &\geq 2 + 2(2nc - \lfloor \frac{n}{2} \rfloor - 2) \end{aligned}$$

Therefore, we have

$$2 + 2\lfloor \frac{n}{2} \rfloor > 2nc + c$$

and this is a contradiction for $c \geq 1$.
We claim that $\tau(G'-X') \neq \lceil 2nc - \lfloor \frac{n}{2} \rfloor - 3 \rceil$, since otherwise $X' \cap V(G) = V(G)$ and $|X'| \geq n+1$ and

$$\begin{aligned} T(2n) &\geq Tk(G'-X') \\ &> |X'| + \tau(G'-X') \\ &\geq (n+1) + \lceil 2nc - \lfloor \frac{n}{2} \rfloor - 3 \rceil \\ &\geq (n+1) + (2nc - \lfloor \frac{n}{2} \rfloor - 3) \end{aligned}$$

Thus

$$n < \lfloor \frac{n}{2} \rfloor + 2$$

and this is contradiction for $n > 2$. Therefore $\tau(G'-X') = \lceil 2nc - \lfloor \frac{n}{2} \rfloor - 2 \rceil$.
We may assume that $X' \cap K_{1,n} = \{s\}$, otherwise suppose $D = K_{1,n} - s$, then

$$\begin{aligned} Tk(G'-(X'-D)) &\geq Tk(G'-X') \\ &> |X'| + \tau(G'-X') \\ &> |X'-D| + \tau(G'-(X'-D)) \end{aligned}$$

and we could use $X'-D$ instead of X'. Therefore $k(G'-X') = 2n$ and $X' \subset V(G) \cup \{s\}$.
Thus

$$\begin{aligned} Tk(G'-X') = T(2n) &> |X'| + \lceil 2nc - \lfloor \frac{n}{2} \rfloor - 2 \rceil \\ &\geq |X'| + (2nc - \lfloor \frac{n}{2} \rfloor - 2) \end{aligned}$$

Then we have

$$|X'| < 2nc - 2nc + \lfloor\frac{n}{2}\rfloor + 2 = \lfloor\frac{n}{2}\rfloor + 2$$

and since $|X'|$ is an integer number, we have $|X'| \leq \lfloor\frac{n}{2}\rfloor + 1$.
Let $X = X' - s$, then $|X| \leq \lfloor\frac{n}{2}\rfloor$, and $X \subset V(G)$. We have shown that no two vertices $v_i, v_j \in V(G)$ are connected by a path in $G' - X'$. Thus $G - X$ contains at least $\lceil\frac{n}{2}\rceil$ components. Choosing one vertex in each component of $G - X$ yields a set of at least $\lceil\frac{n}{2}\rceil$ independent vertices in G.

A decision problem C is NP-complete if C is in NP, and every problem in NP is reducible to C in polynomial time. When a decision version of a combinatorial optimization problem is proved to belong to the class of NP-complete problems, then the optimization version is NP-hard. By theorems 1 and 2 we proved that it is NP-complete to solve decision problem of T-tenacious graphs for any fixed positive rational number T, and therefore finding tenacity of a graph is NP-hard.

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