Edge-connectivities for spanning trails with prescribed edges

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Abstract

For a graph G, a vertex-edge alternating sequence $v_0, e_1, v_1, e_2, \cdots, e_{k-1}, v_{k-1}, e_k, v_k$ such that all the e_i's are distinct and $e_i = v_{i-1}v_i$ for all i is called a trail. For $u, v \in V(G)$, a (u, v)-trail of G is a trail in G whose origin is u and whose terminus is v. A (u, v) trail is called a close trail if $u = v$. A trail H is called a spanning trail of a graph G if $V(H) = V(G)$. Let $X \subseteq E(G)$ and $Y \subseteq E(G)$ with $X \cap Y = \emptyset$. In this paper, we study the minimum edge-connectivity of a graph G such that for any $u, v \in V(G)$ (including $u = v$), G has a spanning (u, v)-trail H such that $X \subseteq E(H)$ and $Y \cap E(H) = \emptyset$.

1. Introduction

We follow the notation of Bondy and Murty [1], except that graphs have no loops. For a graph G, a trail is a vertex-edge alternating sequence $v_0, e_1, v_1, e_2, \cdots, e_{k-1}, v_{k-1}, e_k, v_k$ such that all the e_i's are distinct and $e_i = v_{i-1}v_i$ for all i $(1 \leq i \leq k)$. Let e', $e'' \in E(G)$. A trail in G is called an (e', e'')-trail if its first edge is e' and its last edge is e''. For $u, v \in V(G)$, a (u, v)-trail of G is a trail in G whose origin is u and whose terminus is v. A trail H is called a *spanning trail* if $V(H) = V(G)$. If $u = v$, then a

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(u, v)-trail in G is a closed trail, which is also called a *Eulerian subgraph* of G. A graph is called *supereulerian* if it has a spanning Eulerian subgraph.

Many researches have been done for the existence of spanning Eulerian trails in a graph under various conditions (see [5] and [6]). In this paper, we study the following problem.

> For a graph G and an integer $r \geq 0$, let X and Y be two edge disjoint subsets of $E(G)$ with $|X| + |Y| \leq r$. Find the minimum edge-connectivity for G such that for any $u, v \in V(G)$ (or $e', e'' \in E(G)$), G has a spanning (u, v)-trail (or (e', e'')-trail) H such that $X \subseteq E(H)$ and $Y \cap E(H) = \emptyset$.

There are many 3-edge-connected graphs such as the Petersen graph, and any 3-connected cubic graph that does not have a proper 3-edge-coloring is not supereulerian. Then the minimum edge-connectivity for a graph to assure the existence of a spanning Eulerian subgraph is at least four. Some special cases of the problem above were studied by several researchers ([2], [7], [8], [10], [12]).

Theorem 1.1 (Catlin [2]). If G is 4-edge-connected, then for any $u, v \in V(G)$ there is a spanning Eulerian (u, v) trail in G.

Zhan [12] proved the following.

Theorem 1.2 (Zhan [12]). If G is a 4-edge-connected graph, then for any edges e_1, $e_2 \in E(G)$ there is a spanning (e_1, e_2)-trail in G.

For the case when $Y = \emptyset$, Lai [10] proved the following result.

Theorem 1.3 (Lai [10]). Let $r \geq 0$ be an integer. For a graph G, let $X \subseteq E(G)$ with $|X| \leq r$. Then G has a spanning Eulerian subgraph H such that $X \subseteq E(H)$ if and only if G is $f(r)$-edge-connected, where $f(r)$ is defined by

$$f(r) = \begin{cases} 4, & 0 \leq r \leq 2, \\ r + 1, & r \geq 3 \text{ and } r \text{ is odd}, \\ r, & r \geq 4 \text{ and } r \text{ is even}. \end{cases}$$

In [7], the authors extended the results in [10] and solved the problem for the case when $Y = \emptyset$. Following closely the method of [7], we extend that result for $Y \neq \emptyset$. In the next section, we will present Catlin's reduction method and some preliminary results. Our main results are in Sections 3 and 4.

2. Catlin's reduction method and Preliminary results

In [2], Catlin defined collapsible graphs. For a graph G, let $O(G)$ be the set of odd degree vertices of G. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, G has a spanning connected subgraph H_R such that $O(H_R) = R$. We regard an empty set as an even subset and K_1 as a collapsible graph. Therefore, if G is a collapsible graph, then G has a spanning eulerian subgraph H_R as $R = \emptyset$, and G has a spanning (u, v)-trail H_{R_1} for any u and v in $V(G)$ as $R_1 = \{u, v\}$. In [2], Catlin proved the following.

Collapsible Partition Theorem (Catlin [2]). *Every graph G has a unique collection of vertex disjoint maximal collapsible subgraphs H_1, H_2, $\cdots$, H_c such that $V(G) = V(H_1) \cup V(H_2) \cup \cdots \cup V(H_c)$.*

Let H be a connected subgraph of G. The contraction G/H is obtained from G by contracting each edge of H and deleting the resulting loops. Let H_1, H_2, $\cdots$, H_c be the set of vertex disjoint maximal collapsible subgraphs of G. The *reduction* of G is obtained from G by contracting each H_i into a vertex v_i for all i $(1 \leq i \leq c)$, and is denoted by G'. Each H_i is called a preimage of v_i in G, and v_i is called the contraction image of H_i in G'. A vertex v in G' is called a *trivial contraction* if its preimage in G is K_1. A graph G is reduced if G is the reduction of some graph. Let $F(G)$ be the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees.

Theorem 2.1 (Catlin [2]). Let G be a graph, and let G' be the reduction of G. Each of the following holds.
(a) G is supereulerian if and only if G' is supereulerian.
(b) G is collapsible if and only if $G' \cong K_1$

It is well known that a $2k$-edge-connected graph has k edge-disjoint spanning trees (Kundu [9], and Polesskii [11]). Catlin [2] proved that if G has two edge-disjoint spanning trees, then G is collapsible. Thus, if G is 4-edge-connected, then G is collapsible.

In [3], Catlin proved the following.

Theorem 2.2 (Catlin [3]). Let G be a graph and let $r \geq 1$ be an integer. Then G is r-edge-connected if and only if for any $Y \subseteq E(G)$ with $|Y| \leq \lfloor (r+1)/2 \rfloor$, $G - Y$ has $\lfloor r/2 \rfloor$ edge-disjoint spanning trees.

The following theorems will be needed in our proofs.

Theorem 2.3 (Catlin et al. [4]). Let G be a connected graph. If $F(G) \leq 2$, then either G is collapsible, or the reduction of G is in $\{K_2, K_{2,t}\}$ $(t \geq 1)$.

Let e be an edge in G. Edge e is *subdivided* when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by that path of length 2 is called subdividing e. Let G be a graph and let $X \subseteq E(G)$. Let G_X be the graph obtained from G by subdividing each edge in X. Then $V(G_X) = V(G) \cup \{v(e) | \; e \in X\}$. For a graph G, let $X \subseteq E(G)$ and $Y \subseteq E(G)$ with $X \cap Y = \emptyset$. Define $(G-Y)_X$ as a graph obtained from G by removing all the edges in Y and subdividing each edge in X.

We need the following lemma, which was proved in [7].

Lemma 2.4 (Chen et al. [7]). Let G be a connected graph. Then each of the following holds:

(a) Let $k \geq 2$ be an integer. If G has k edge-disjoint spanning trees, then for any $X \subseteq E(G)$ with $|X| \leq 2k-2$, $F(G_X) \leq 2$.

(b) Let $X = X_1 \cup X_2$ with $X_1 \cap X_2 = \emptyset$. Then $F(G_X) \leq F((G - X_1)_{X_2})$.

Combining Theorem 2.2 and Lemma 2.4 we have the following.

Lemma 2.5. Let G be a connected graph and let $r \geq 1$ be an integer. Let X and Y be two disjoint subsets of $E(G)$. If G is r-edge-connected, $|Y| \leq \lfloor (r+1)/2 \rfloor$ and $|X| \leq 2\lfloor r/2 \rfloor - 2$, then $F((G-Y)_X) \leq 2$.

Proof. By Theorem 2.2, $G - Y$ has $\lfloor r/2 \rfloor$ edge-disjoint spanning trees. Then by Lemma 2.4, $F((G-Y)_X) \leq 2$. The lemma is proved. □.

3. A Main Result on $(G - Y)_X$

Let $r > 2$ be an integer. For a graph G, let X and Y be two disjoint subsets of $E(G)$ such that

$$|Y| \leq \lfloor (r+1)/2 \rfloor \text{ and } |X \cup Y| \leq r + \lfloor r/2 \rfloor - 2. \tag{1}$$

If $|X \cup Y| \le 2\lfloor r/2 \rfloor - 2$, define $X_0 = X$ and $Y_0 = Y$. If $|X \cup Y| > 2\lfloor r/2 \rfloor - 2$, then since $|Y| \le \lfloor (r+1)/2 \rfloor$, we can choose Y_0 in such a way that Y_0 contains all the edges in Y and some edges in X (if $|Y| < \lfloor (r+1)/2 \rfloor$), such that $|Y_0| = \lfloor (r+1)/2 \rfloor$. Then define $X_0 = (X \cup Y) - Y_0$. And so $X_0 \subseteq X$ and $|X_0| = |X \cup Y| - |Y_0| \le r + \lfloor r/2 \rfloor - 2 - \lfloor (r+1)/2 \rfloor = 2\lfloor r/2 \rfloor - 2$. Thus, for any disjoint subsets X and Y satisfying (1) above, we have X_0 and Y_0 of $E(G)$ such that

$$X_0 \subseteq X, Y \subseteq Y_0,\ X_0 \cap Y_0 = \emptyset,\ |Y_0| \le \lfloor (r+1)/2 \rfloor \text{ and } |X_0| \le 2\lfloor r/2 \rfloor - 2. \quad (2)$$

Lemma 3.0. Let G be a graph and let X, Y, X_0 and Y_0 be subsets of $E(G)$ defined in (1) and (2). Then

$$F((G-Y)_X) \le F((G-Y_0)_{X_0}). \quad (3)$$

Proof. Let $X_1 = X - X_0$. Then $Y_0 = Y \cup X_1$, $X_0 = X - X_1$ and so $X_0 \cap X_1 = \emptyset$. Let $G_1 = G - Y$. Since $X \cap Y = \emptyset$, X_1 and X_0 are subsets of $E(G-Y) = E(G_1)$. By Lemma 2.4, $F((G-Y)_X) \le F(((G-Y)-X_1)_{X_0})$. Since $Y_0 = Y \cup X_1$, $G - Y_0 = (G-Y) - X_1$. Hence, $F((G-Y)_X) \le F((G-Y_0)_{X_0})$. The lemma is proved. □

Theorem 3.1. Let $r \ge 4$ be an integer. Let G be an r-edge-connected graph and let $X \subseteq E(G)$ and $Y \subseteq E(G)$ with $X \cap Y = \emptyset$, $|Y| \le \lfloor (r+1)/2 \rfloor$ and $|X| + |Y| \le r + \lfloor r/2 \rfloor - 2$. Then one of the following holds.

(a) $(G-Y)_X$ is collapsible, or

(b) $\kappa'(G) \le |X| + |Y|$ and $(G-Y)_X$ can be contracted to $K_{2,t}$, i.e. the reduction of $(G-Y)_X$ is $K_{2,t}$, and

 (b1) $\kappa'(G-Y) \le t \le |X|$ if $\kappa'(G-Y) \ge 3$ or $r \ge 6$;

 (b2) $\kappa'(G-Y) \le t < |X| + |Y|$ if $\kappa'(G-Y) = 2$ (then $r = 4$ or 5).

Proof. Let X_0 and Y_0 be the two edge subsets of $E(G)$ defined above. By Lemma 3.0, $F((G-Y)_X) \le F((G-Y_0)_{X_0})$. Since $|Y_0| \le \lfloor (r+1)/2 \rfloor$, by Theorem 2.2, $(G - Y_0)$ has $\lfloor r/2 \rfloor$-edge-disjointed spanning trees. By the definition of X_0 and Y_0, $|X_0| \le 2\lfloor r/2 \rfloor - 2$. Then by Lemma 3.0 and Lemma 2.5, $F((G-Y)_X) \le F((G-Y_0)_{X_0}) \le 2$. By Theorem 2.3, either $(G-Y)_X$ is collapsible or $(G-Y)'_X \in \{K_2, K_{2,t}\}$. Assume that $(G-Y)_X$

is not collapsible. Then $(G-Y)'_X \in \{K_2, K_{2,t}\}$. We will show that the statement (b) holds.

Since G is r-edge-connected, $r \geq 4$ and $|Y| \leq \lfloor (r+1)/2 \rfloor$,

$$\kappa'(G-Y) \geq \kappa'(G) - |Y| \geq r - \lfloor (r+1)/2 \rfloor \geq \lfloor r/2 \rfloor \geq 2. \qquad (4)$$

Thus, $(G-Y)'_X$ is 2-edge-connected. Therefore, $(G-Y)'_X = K_{2,t}$ $(t \geq 2)$.

Let $E((G-Y)'_X) = E(K_{2,t}) = \{uw_1, uw_2, \cdots, uw_t, vw_1, vw_2, \cdots, vw_t\}$ where w_i $(1 \leq i \leq t)$ is a degree two vertex in $(G-Y)'_X$. Let $E' = \{vw_1, vw_2, \cdots, vw_t\}$. Then E' is an edge-cut of $(G-Y)'_X$.

If $\kappa'(G-Y) \geq 3$, then each w_i is a vertex obtained by subdividing an edge in X. Therefore, $|E'| \leq |X|$. Let E_X be the edge subset of X in which the edges are subdivided to obtain the edges in E'. Since E' is an edge-cut of $(G-Y)'_X$, E_X is an edge-cut of $(G-Y)$, and so X is an edge-cut of $G-Y$. Hence, $|X| \geq |E_X| = |E'| = t \geq \kappa'(G-Y)$. Therefore $X \cup Y$ is an edge-cut of G and so $\kappa'(G) \leq |X \cup Y|$. The statement holds if $\kappa'(G-Y) \geq 3$. If $r \geq 6$, since G is r-edge-connected and $|Y| \leq \lfloor (r+1)/2 \rfloor$, $\kappa'(G-Y) \geq 3$. Thus the statement (b1) holds if $r \geq 6$.

Next we consider the case if $\kappa'(G-Y) = 2$.

Claim 1. If w_i is not a vertex obtained by subdividing an edge in X, then there are at least $r-2$ edges in Y adjacent to some vertices in the preimage of w_i.

Proof of Claim 1: It follows from that G is r-edge-connected and $r \geq 4$.

Claim 2. At most one edge in E' is not from subdividing the edges in X.

Proof of Claim 2: Since $\kappa'(G-Y) = 2$, the equalities in (4) hold. So $r = 4$ or 5 and $|Y| = \lfloor (r+1)/2 \rfloor = 2$ or 3. Thus we have

$$2 \leq |Y| \leq 3. \qquad (5)$$

Since G is either 4 or 5 edge-connected, by Claim 1 after removing 2 or 3 edges in Y from G, at most one vertex in $\{w_i\}$ $(1 \leq i \leq t)$ is not from subdividing edges in X. Claim 2 is proved.

Thus, by Claim 2, $|E'| - 1 \leq |X|$, and so by (5),

$$2 = \kappa'(G-Y) \leq t = |E'| \leq |X| + 1 < |X| + |Y|. \qquad (6)$$

To complete the proof of statement (b2), we still need to show $|X|+|Y| \geq \kappa'(G) = r$.

By way of contradiction, suppose that $|X|+|Y| < r$. By (6), $2 \leq t = |E'| < |X|+|Y| < r$. Thus, $t = |E'| = 2$ if $r = 4$ and $2 \leq t = |E'| \leq 3$ if $r = 5$. Therefore, $(G-Y)'_X = K_{2,t} \in \{K_{2,2}, K_{2,3}\}$, and $E' = \{vw_i\}$ $(1 \leq i \leq t)$ is corresponding to an edge-cut with size t in $G-Y$ that separates the pre images of u and v in $G-Y$.

If $r = 4$, then G is 4-edge-connected, $(G-Y)'_X = K_{2,2}$. Since $|X|+|Y| < r = 4$, $|X| < 4-|Y| < 2$. By Claim 2, $|X| \geq 1$, and so $|X| = 1$ and $|Y| = 2$. Therefore, at least one vertex in $\{w_1, w_2\}$, say w_1, is not a vertex obtained by subdividing an edge in X. Therefore, by Claim 1, the two edges in Y must be adjacent to some vertices in the preimage of w_1. Therefore, at least one of the preimage of u or v in G is connected by at most three edges to the rest of the graph G. Thus, $\kappa'(G) \leq 3$, contrary to that G is 4-edge-connected.

If $r = 5$, then G is 5-edge-connected. Since $|X|+|Y| < r = 5$ and by Claim 2 and (5), $|X| \geq 1$, $2 \leq |Y| \leq 3$. Note that $(G-Y)'_X = K_{2,t} \in \{K_{2,2}, K_{2,3}\}$. By Claim 2, at least one vertex in $\{w_1, .., w_t\}$ ($t = 2$ or 3), say w_1, is not a vertex obtained by subdividing an edge in X. By Claim 1 and $r-2 = 3$ and $|Y| \leq 3$, Y should have 3 edges and the 3 edges in Y are adjacent to some vertices in the preimage of w_1. Therefore, no matter $(G-Y)'_X = K_{2,2}$ or $K_{2,3}$, at least one of the preimage of u or v in G is connected by at most four edges to the rest of the graph G. Thus, G is at most 4-edge-connected, contrary to that G is 5-edge-connected. Thus $|X|+|Y| \geq r = \kappa'(G)$. Theorem 3.1 is proved. □

From the proof of Theorem 3.1, we have

Corollary 3.2. Let $r \geq 4$ be an integer. Let G be an r-edge-connected graph and let $X \subseteq E(G)$ and $Y \subseteq E(G)$ with $X \cap Y = \emptyset$, $|Y| \leq \lfloor (r+1)/2 \rfloor$ and $|X|+|Y| \leq r + \lfloor r/2 \rfloor - 2$. If $\kappa'(G-Y) \geq 3$, then one of the following holds:

(i) $(G-Y)_X$ is collapsible, or

(ii) $(G-Y)_X$ can be contracted to $K_{2,t}$ in such a way that each degree two vertex in $K_{2,t}$ is a trivial contraction obtained in $(G-Y)$ by subdividing the edges in X, and $(r-|Y|) \leq t \leq |X|$.

Proof. Corollary 3.2 follows from the proof of Theorem 3.1 and the fact that $\kappa'(G-Y) \geq \kappa'(G) - |Y| \geq r - |Y|$. □

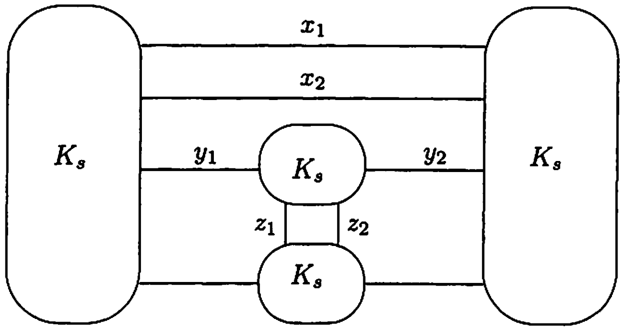

Figure 1

Let G be the 4-edge-connected graph shown in Figure 1 where $s \geq 5$. Let $X = \{x_1, x_2\}$ and $Y_1 = \{z_1, z_2\}$. Then the reduction of $(G - Y_1)_X$ is $K_{2,t} = K_{2,4}$. This shows that $t \leq |X| + |Y| = r = 4$ is the best possible in Theorem 3.1. Let $X = \{x_1, x_2\}$ and $Y_2 = \{y_1, y_2\}$. Then $\kappa'(G - Y_2) = 2$. The reduction of $(G - Y_2)_X$ is $K_{2,3}$ in which one degree two vertex is not a trivial contraction. Thus, $\kappa'(G - Y_2) \geq 3$ is necessary in Corollary 3.2. This graph G has no spanning Eulerian subgraph H with $X \subseteq E(H)$ and $Y_2 \cap E(H) = \emptyset$.

4. Spanning Eulerian Trails

Let G be a graph and let $X \subseteq E(G)$ and $Y \subseteq E(G)$ with $X \cap Y = \emptyset$ and $|X| + |Y| \leq r$. In this section, we present the result on the minimum edge-connectivity of G such that G has a spanning Eulerian subgraph or spanning (u,v) -trail (or (e_1, e_2)-trail) H for any $u, v \in V(G)$ (or any $e_1, e_2 \in E(G)$) such that $X \subseteq E(H)$ and $Y \cap E(H) = \emptyset$.

The following property of an Eulerian graph will be needed:
Eulerian property. A connected graph G is Eulerian if and only if the cardinality of every minimum edge-cut of G is even.

Theorem 4.1. Let $r \geq 3$. For a graph G, let $X \subseteq E(G)$ and $Y \subseteq E(G)$ which satisfy the following

$$X \cap Y = \emptyset,\ |Y| \leq \lfloor (r+1)/2 \rfloor,\ |X \cup Y| = |X| + |Y| \leq r. \qquad (7)$$

Then each of the following holds:

(a) For any X and Y satisfying (7) G has a spanning Eulerian subgraph H such that $X \subseteq E(H)$ and $Y \cap E(H) = \emptyset$ if and only if G is $(r+1)$-edge-connected.

(b) For any X and Y satisfying (7) and for any u and v in $V(G)$ G has a spanning (u, v)-trail T such that $X \subseteq E(T)$ and $Y \cap E(T) = \emptyset$ if and only if G is $(r+1)$-edge-connected.

Proof. We prove the necessary condition first. Suppose that $\kappa'(G) = r$. Let E_0 be an edge-cut of G with $|E_0| = r$. Let H_1 and H_2 be two components of $G - E_0$. If r is even, choose an edge e in E_0 and let $Y = \{e\}$, and let $X = E_0 - Y$. If r is odd, then let $Y = \emptyset$ and $X = E_0$. Then $|X| + |Y| = r$ and $|X|$ is odd. If G has a spanning Eulerian subgraph H such that $X \subseteq E(H)$ and $Y \cap E(H) = \emptyset$, then H has an odd minimum edge cut X which separates induced subgraphs $H[V(H_1)]$ and $H[V(H_2)]$ in H, contrary to the Eulerian property. This shows that G is at least $(r+1)$-edge-connected.

Next, we will prove the sufficient condition.

Since G is $(r+1)$-edge-connected and $r \geq 3$, $\lfloor (r+1)/2 \rfloor \geq 2$. Then X and Y satisfying (7) will have $|Y| \leq \lfloor (r+1)/2 \rfloor \leq \lfloor (r+2)/2 \rfloor$ and $|X|+|Y| \leq r \leq (r+1) + \lfloor (r+1)/2 \rfloor - 2$, which satisfies Theorem 3.1. Therefore, since $\kappa'(G) \geq r+1$ and $|X| + |Y| \leq r$, by Theorem 3.1, $(G-Y)_X$ is collapsible. Since $V(G) = V(G-Y) \subseteq V((G-Y)_X)$ and by the collapsibility of $(G-Y)_X$, $(G-Y)_X$ has a spanning Eulerian subgraph H_s and a spanning (u, v)-trail T_s for any $u, v \in V(G)$. Then each degree two vertex in $(G-Y)_X$ must be in H_s and in T_s. Obviously, $Y \cap E(H_s) = Y \cap E(T_s) = \emptyset$. Let H (or T) be the graph obtained from H_s (or T_s) by replacing each path of length two in $(G-Y)_X$ by its corresponding edge in X. Therefore, G has a spanning Eulerian subgraph H and a (u, v) trail T such that $X \subseteq E(H)$ and $X \subseteq E(T)$, and $Y \cap E(H) = Y \cap E(T) = \emptyset$. The theorem is proved. □

If we only consider the existence of spanning Eulerian subgraph, then when $r \geq 4$ and $r - |Y|$ is even, the edge-connectivity of graph G can be reduced to r instead of $r+1$ in Theorem 4.1(a).

Theorem 4.2. Let $r \geq 4$. For a graph G, let $X \subseteq E(G)$ and $Y \subseteq E(G)$ such that X and Y satisfy (7), $r - |Y|$ is even and $\kappa'(G-Y) \geq 3$. Then G has a spanning Eulerian subgraph H such that $X \subseteq E(H)$ and $Y \cap E(H) = \emptyset$

for any such X and Y if and only if G is r-edge-connected.

Proof. We prove the necessary condition first. Suppose that G is $(r-1)$-edge connected. Let E_0 be an edge-cut of G with $|E_0| = r-1$. Let H_1 and H_2 be the two components of $G - E_0$. If $r \geq 4$ is even, choose $Y = \emptyset$. Then $\kappa'(G-Y) = \kappa'(G) \geq r-1 \geq 3$. If $r \geq 4$ is odd, then $r \geq 5$. Choose an edge e in E_0 and let $Y = \{e\}$. Then $\kappa'(G-Y) \geq \kappa'(G) - 1 = r-2 \geq 3$. Let $X = E_0 - Y$. Then $|X| + |Y| = |E_0| = r-1$ and $r - |Y|$ is even. Thus, X and Y are two subsets of $E(G)$ that satisfy all the requirements in Theorem 4.2. However, if G has a spanning Eulerian subgraph H such that $X \subset E(H)$ and $Y \cap E(H) = \emptyset$, then H has an odd minimum edge cut X, contrary to the Eulerian property. Thus, G is at least r-edge-connected.

Next, we will show the sufficient condition. Without loss of generality, we only need to prove the statement for the case $|X|+|Y| = r$. By Corollary 3.2, either $(G-Y)_X$ is collapsible or the reduction of $(G-Y)_X$ is $(G-Y)'_X = K_{2,t}$ where $r - |Y| \leq t \leq |X|$. Since $|X| + |Y| = r$ and $r - |Y|$ is even, $t = |X| = r - |Y|$ is even. Therefore, $K_{2,t}$ is an Eulerian graph. By Theorem 2.1, $G - Y$ has spanning Eulerian subgraph. Thus, $G - Y$ has a spanning Eulerian subgraph containing all the vertices of degree two in $(G-Y)_X$, and so $G - Y$ has a spanning Eulerian subgraph containing all the edges in X. The theorem is proved. □

The graph of Figure 1 shows that when G is 4-edge-connected, the condition $\kappa'(G-Y) \geq 3$ in Theorem 4.2 is necessary. This theorem also implies that if G is 4-edge-connected, then for any $X \subseteq E(G)$ and $Y \subseteq E(G)$ with $X \cap Y = \emptyset$, $|Y| \leq 2$, $\kappa'(G-Y) \geq 3$ and $|X \cup Y| \leq 4$, G has a spanning Eulerian subgraph H such that $X \subseteq E(H)$ and $Y \cap E(H) = \emptyset$. Let G be the graph defined in Figure 2 below with $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, where each H_i ($i = 1, 2, 3$ or 4) is a complete graph K_s ($s \geq 5$). Obviously, G is 4-edge-connected and $G - Y$ is 3-edge-connected. However, the reduction of $(G-Y)_X$ is not a $K_{2,t}$ graph, and has no spanning Eulerian subgraph containing all the edges in X. Thus, $|X \cup Y| \leq 4$ is the best possible in Theorem 4.1 and Theorem 4.2 for the case $r = 4$. We can also show that $|Y| \leq \lfloor (r+1)/2 \rfloor$ is necessary for the case $r = 4$ or 5 in Theorem 3.1 from this graph by adding an edge between H_1 and H_2 (and an edge between H_3 and H_4 for case $r = 5$).

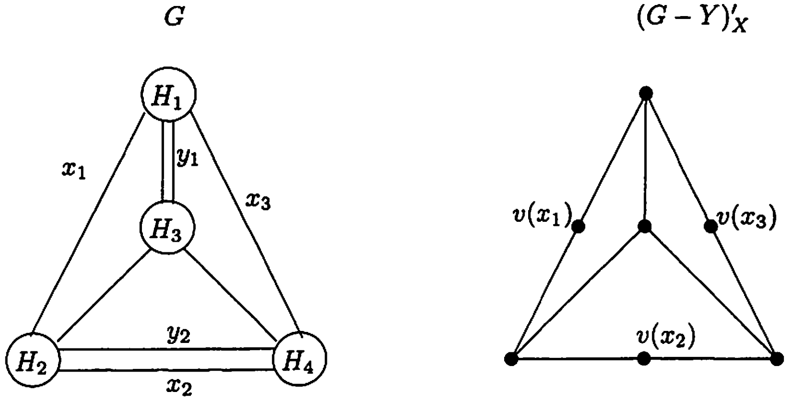

Figure 2

Next we consider the edge-connectivity for spanning (e_1, e_2)-trails with prescribed edges.

Lemma 4.3. Let G be a graph and let $e_1, e_2 \in E(G)$ and let $X \subseteq E(G)$. Let $X_1 = X \cup \{e_1, e_2\}$. Let $v(e_1)$ and $v(e_2)$ be the two vertices subdividing e_1 and e_2, respectively. Then if G_{X_1} is collapsible or has a spanning $(v(e_1), v(e_2))$-trail, then G has a spanning (e_1, e_2)-trail containing X.

Proof. It follows from the definitions of collapsibility and G_{X_1}. □

The following lemma was proved in [7].

Lemma 4.4 (Chen et al.[7]). Let G be a 3-edge-connected graph. Let $X \subseteq E(G)$ and let e', $e'' \in E(G)$. Let $X_1 = X \cup \{e', e''\}$. Suppose that $G'_{X_1} = K_{2,t}$ where $t \geq 3$. If $t > |X|$, then G has a spanning (e', e'')-trail H such that $X \subseteq E(H)$.

Using Theorem 3.1, we prove the following result on (e_1, e_2)-trails analogous to Theorem 4.1 which extends Theorem 1.3 [12].

Theorem 4.5. Let $r \geq 3$. For a graph G, let X and Y be the subsets of $E(G)$ such that

$$X \cap Y = \emptyset,\ |Y| \leq \lfloor (r+1)/2 \rfloor,\ \kappa'(G-Y) \geq 3 \text{ and } |X| + |Y| \leq r-1. \quad (8)$$

If G is an $(r+1)$-edge-connected graph then G has a spanning (e_1, e_2)-trail H in G for any e_1, $e_2 \in E(G) - (X \cup Y)$ such that $X \subseteq E(H)$ and $Y \cap E(H) = \emptyset$.

Proof. Let $X_1 = X \cup \{e_1, e_2\}$. Let $(G-Y)_{X_1}$ be the graph obtained from $G-Y$ by subdividing each edge in X_1. Since $r \geq 3$, $\lfloor (r+1)/2 \rfloor \geq 2$. Then $|X_1 \cup Y| \leq |X \cup Y| + 2 \leq r+1 \leq (r+1) + \lfloor (r+1)/2 \rfloor - 2$. By Theorem 3.1, either $(G-Y)_{X_1}$ is collapsible or $(G-Y)_{X_1}$ is contractible to $K_{2,t}$ with $t \geq r$. If $(G-Y)_{X_1}$ is collapsible, then by Lemma 4.3, $G-Y$ has a spanning (e_1, e_2)-trail containing X. If $(G-Y)_{X_1}$ is contractible to $K_{2,t}$ with $t \geq 4$, since $t \geq r > |X|$, by Lemma 4.4, $G-Y$ has a spanning (e_1, e_2)-trail H containing the edges in X. □

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