Fibonacci $(p,\ r)$-cubes which are partial cubes*

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Abstract

Fibonacci $(p,\ r)$-cube is an interconnection topology, which unifies a wide range of connection topologies, such as the hypercube, classical Fibonacci cube, postal network, etc. It is known that classical Fibonacci cubes are partial cubes. In this paper we show that a Fibonacci $(p,\ r)$-cube is partial cube if and only if either $p = 1$, or $p \geq 2$ and $r \leq p + 1$. Furthermore, we show that for Fibonacci $(p,\ r)$-cubes, almost-median graphs, semi-median graphs and partial cubes are all equivalent.

Key words: Fibonacci (p, r)-cube; partial cube; median graph; almost-median graph; semi-median graph

1 Introduction

Hypercubes are an important class of graphs. Set $B = \{0, 1\}$. The n-dimensional hypercube Q_n is the graph defined on the vertex set $\mathcal{B}_n = \{b_1b_2 \ldots b_n | b_i \in B, i = 1, \ldots, n\}$, two vertices $\alpha = a_1a_2 \ldots a_n$ and $\beta = b_1b_2 \ldots b_n$ being adjacent if $a_i \neq b_i$ holds for exactly one $i \in \{1, \ldots, n\}$.

Inspired by classical Fibonacci sequence of numbers, Hsu [3] introduced the Fibonacci cube as follows, which has similar properties as hypercube. For $n \geq 1$ set $\mathcal{F}_n = \{b_1b_2 \ldots b_n |\ b_i \in B, b_ib_{i+1} = 0, i = 1, \ldots, n-1\}$. Fibonacci cube Γ_n is the graph defined on the vertex set $\mathcal{F}_n$ and two vertices are adjacent if they differ in exactly one coordinate. Structural and enumerative properties of the Fibonacci cubes were given in [13]. It is showed that Fibonacci cubes are precisely the resonance graphs of zigzag hexagonal chains in [11]. Plane bipartite graphs whose resonance graphs are exactly

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Fibonacci cubes were characterized in [17]. For more results on application and structure of Fibonacci cubes, see [7] for a survey.

Egiazarian and Astola [2] introduced a wide generalization of Fibonacci cubes, which were primarily studied as interconnection networks. Let n, p and r be positive integers with $n \geq p$, r. Then a Fibonacci (p, r)-string of length n is a binary string of length n in which there are at most r consecutive 1s and at least p 0s between two substrings composed of (at most r) consecutive 1s. Let $\mathcal{F}_n^{(p,r)}$ denote the set of Fibonacci (p, r)-strings of length n. The Fibonacci (p, r)-cube $\Gamma_n^{(p,r)}$ is the graph whose vertex set is $\mathcal{F}_n^{(p,r)}$ and two vertices are adjacent if they differ in exactly one coordinate. Note that $\Gamma_n^{(1,n)} \cong Q_n$ and $\Gamma_n^{(1,1)} \cong \Gamma_n$. $\Gamma_5^{(3,2)}$, $\Gamma_4^{(2,3)}$ and $\Gamma_4^{(2,4)}$ are showed in Fig. 1. In [15], Fibonacci (p, r)-cubes which can be the resonance graphs of perfect matchings of plane (bipartite) graphs were determined.

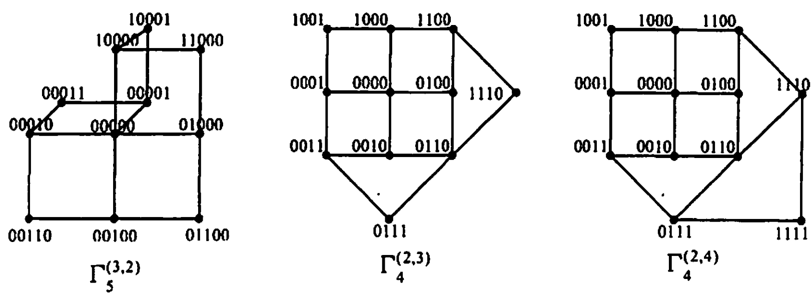

Fig. 1. Fibonacci (p, r)-cubes $\Gamma_5^{(3,2)}$, $\Gamma_4^{(2,3)}$ and $\Gamma_4^{(2,4)}$.

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For $X \subseteq V(G)$, let $\langle X \rangle$ denote the subgraph of G induced by X. Then $\Gamma_n = \langle \mathcal{F}_n \rangle$ and $\Gamma_n^{(p,r)} = \langle \mathcal{F}_n^{(p,r)} \rangle$ are the induced subgraphs of Q_n.

The distance $d_G(\alpha, \beta)$ between vertices α and β of a graph G is the length of a shortest α, β-path. Given two binary strings α and β of $\mathcal{B}_n$, their Hamming distance $H(\alpha, \beta)$ is the number of coordinates in which they differ. As we have seen in [6], $d_{Q_n}(\alpha, \beta) = H(\alpha, \beta)$ for any two vertices α and β of Q_n. So for any two vertices α, β of a subgraph G of Q_n, we have the inequality

$$d_G(\alpha, \beta) \geq H(\alpha, \beta). \tag{1.1}$$

Furthermore, since hypercube is bipartite, for any two vertices α, β of a subgraph G of Q_n, if $d_G(\alpha, \beta) \neq H(\alpha, \beta)$, then

$$d_G(\alpha, \beta) \geq H(\alpha, \beta) + 2. \tag{1.2}$$

For a subgraph H of graph G, if $d_H(\alpha,\beta) = d_G(\alpha,\beta)$ for all $\alpha,\beta \in V(H)$, then we say H is an isometric subgraph of G. More generally, if H and G are arbitrary graphs, then a mapping $f : V(H) \to V(G)$ is an isometric embedding if $d_H(u,v) = d_G(f(u), f(v))$ holds for any $u, v \in V(H)$. A partial cube is a connected graph that admits an isometric embedding into a hypercube. Partial cubes constitute a large class of graphs with many applications and includes, for example, benzenoid graphs. There have several applications of partial cubes related to communication and chemical theory, for instance, the Wiener index of a partial cube can be computed easily [8].

It is easy to check that $\Gamma_5^{(3,2)}$ and $\Gamma_4^{(2,3)}$ are partial cubes, but $\Gamma_4^{(2,4)}$ is not a partial cube. So it is natural to raise a question of determining which Fibonacci (p,r)-cubes are partial cubes. This question is a fundamental question. The reason is that in those cubes that admit isometric embedding one can design a very simple local routing. This question is solved completely in this paper. We obtain the following main result, which will be proved in Section 2.

Theorem 1.1. *$\Gamma_n^{(p,r)}$ is a partial cube if and only if either $p = 1$, or $p \geq 2$ and $r \leq p+1$.*

Further, we have the following corollary.

Corollary 1.2. *$\Gamma_n^{(p,r)}$ is a partial cube if and only if it is an isometric subgraph of Q_n.*

Following Djoković [1] the isometric dimension, $\dim_I(G)$, of a partial cube G is the smallest integer n such that G can be isometrically embedded into Q_n. Since the maximum degree of $\Gamma_n^{(p,r)}$ is n (such a vertex is 0^n), if $\Gamma_n^{(p,r)}$ is a partial cube, then

$$\dim_I(\Gamma_n^{(p,r)}) \geq n \tag{1.3}$$

By Corollary 1.2 and Ineq. (1.3), we obtain the isometric dimension of $\Gamma_n^{(p,r)}$ which is a partial cube.

Corollary 1.3. *Let $\Gamma_n^{(p,r)}$ be a partial cube. Then $\dim_I(\Gamma_n^{(p,r)}) = n$.*

A median of a triple of vertices α, β, ω of a graph G is a vertex ν that lies on a shortest α,β-path, on a shortest β,ω-path and on a shortest α,ω-path. A graph is a median graph if every triple of its vertices has a unique median. It was proved by Mulder [12] that a median graph can be obtained from a one-vertex graph by an expansion procedure and from this characterization some nice properties are derived. For more about the structure, characterization and application of median graphs, see [9]. It is easy to check that $\Gamma_5^{(3,2)}$ is a median graph, but $\Gamma_4^{(2,3)}$ and $\Gamma_4^{(2,4)}$ is not a median graph. Ou and Zhang determined which Fibonacci (p,r)-cubes are median graphs in the following theorem.

Theorem 1.4 ([14]). *$\Gamma_n^{(p,r)}$ is a median graph if and only if either $r \leq p$ and $r \leq 2$, or $p = 1$ and $r = n$.*

We give a new proof of Theorem 1.4 in the final section of this paper.

In the theory of partial cubes and median graphs, the following sets play an important role. For a connected graph G with $\alpha\beta \in E$, let

$W_{\alpha\beta} = \{\omega \in V \mid d_G(\alpha, \omega) < d_G(\beta, \omega)\}$;

$U_{\alpha\beta} = \{\omega \in W_{\alpha\beta} \mid \omega$ has a neighbor in $W_{\beta\alpha}\}$; and

$F_{\alpha\beta} = \{e \in E \mid e$ is an edge between $W_{\alpha\beta}$ and $W_{\beta\alpha}\}$.

A bipartite graph is a semi-median graph if it is a partial cube in which for any edge $\alpha\beta$, $\langle U_{\alpha\beta}\rangle$ is connected. Similarly, a bipartite graph G is an almost-median graph if it is a partial cube and for any edge $\alpha\beta$, $\langle U_{\alpha\beta}\rangle$ is isometric in G. Those graphs with some structural properties appeared in [5]. A characterization of almost-median graph was given in [10]. The following theorem proved in Section 3 determines which Fibonacci (p, r)-cubes are almost-median graphs.

Theorem 1.5. *$\Gamma_n^{(p,r)}$ is an almost-median graph if and only if it is a partial cube.*

Since an almost-median graph must be a semi-median graph, the following result holds.

Corollary 1.6. *$\Gamma_n^{(p,r)}$ is a semi-median graph if and only if it is a partial cube.*

Those results show that for Fibonacci (p, r)-cubes, partial cubes, semi-median graphs and almost-median graphs are all equivalent. But for general graphs it is not like this situation [5].

2 Proofs of Theorem 1.1 and related corollaries

A subgraph H of G is convex if for any $\alpha, \beta \in V(H)$, every shortest α, β-path in G lies entirely in H. The following theorem due to Djoković puts forth a fundamental characterization of partial cube.

Theorem 2.1 ([1]). *A graph G is a partial cube if and only if G is bipartite, and $\langle W_{\alpha\beta}\rangle$ and $\langle W_{\beta\alpha}\rangle$ are convex subgraphs of G for every $\alpha\beta \in E(G)$.*

Note that $\Gamma_n^{(1,r)} \cong Q_n(1^{r+1})$, where $Q(1^{r+1})$ is the graph obtained from Q_n by removing all vertices that contain 1^{r+1} as a substring [4].

Lemma 2.2 ([4]). *Let $r \geq 1$. Then $\Gamma_n^{(1,r)}$ is an isometric subgraph of Q_n.*

Let $\alpha = a_1a_2\ldots a_n$ and $\beta = b_1b_2\ldots b_n$ be two binary strings of length n. Let $\mu = a_ia_{i+1}\ldots a_{i+t}$ and $\nu = b_jb_{j+1}\ldots b_{j+t}$ be substrings of length $t+1$ of α and β, respectively. If $i = j$, then we say μ and ν appearing in the same coordinates of α and β. For instance, if $\alpha = 1100110001$ and $\beta = 0010011111$, then $\mu = 10001$ and $\nu = 11111$ appear in the same coordinates of α and β.

Lemma 2.3. *Let $\alpha = a_1a_2\dots a_n$ and $\beta = b_1b_2\dots b_n \in \mathcal{F}_n^{(p,r)}$, $p \geq 2$. Then $d_{\Gamma_n^{(p,r)}}(\alpha,\beta) = H(\alpha,\beta)$ if and only if there exist neither substring 10^t1 and 11^t1, nor 11^t1 and 10^t1 appearing in the same coordinates of α and β for any t with $p \leq t \leq r-2$.*

Proof. We prove the necessity by contradiction. Assume that $d_{\Gamma_n^{(p,r)}}(\alpha,\beta) = H(\alpha,\beta)$ but there exist substrings 10^t1 and 11^t1, or 11^t1 and 10^t1 appearing in the same coordinates of α and β. Let $H(\alpha,\beta) = s$ and $a_{i_j} = b_{i_j}$ for $j = 1,2,\dots,n-s$. Since $p \geq 2$, in any shortest path of $\Gamma_n^{(p,r)}$ connecting α and β there must exist at least one vertex obtained from some vertex by changing some a_{i_j} to $1 - a_{i_j}$ and there also must exist another vertex obtained from some vertex by changing $1 - a_{i_j}$ to a_{i_j} in order to go to β. Hence $d_{\Gamma_n^{(p,r)}}(\alpha,\beta) \neq H(\alpha,\beta)$, a contradiction.

We prove the sufficiency by induction on s, where $s = H(\alpha,\beta)$. It is obvious for the basic case $s = 1$. For $s \geq 2$ suppose it holds for $s-1$. Let $a_{i_j} \neq b_{i_j}$ for $j = 1,\dots,s$. Without loss of generality we may assume that $a_{i_1} = 1$ and $b_{i_1} = 0$.

If $i_1 = 1$, $a_{i_1-1} = 0$, or $a_{i_1+1} = 0$, let α' be the binary string obtained from α by changing a_{i_1} from 1 to 0. Then α' is a Fibonacci (p,r)-string adjacent to α in $\Gamma_n^{(p,r)}$ and $H(\alpha',\beta) = s-1$. So $d_{\Gamma_n^{(p,r)}}(\alpha',\beta) = H(\alpha',\beta) = s-1$ by the induction hypothesis, and there exists a path of length s connecting α and β in $\Gamma_n^{(p,r)}$. Thus $s = d_{\Gamma_n^{(p,r)}}(\alpha,\beta)$ by Ineq. (1.1).

If $a_{i_1-1} = 1$ and $a_{i_1+1} = 1$, then $b_{i_1-1} = 1$ and $b_{i_1+1} = 0$. By the condition that there exists neither substrings 10^t1 and 11^t1, nor 11^t1 and 10^t1 appearing in the same coordinates of α and β, so there must exist k $(r-1 \geq k \geq 1)$ such that $a_{i_1+1} = \cdots = a_{i_1+k} = 1$ and $b_{i_1+1} = \cdots = b_{i_1+k} = 0$, but $a_{i_1+k+1} = 0$ and $b_{i_1+k+1} = 0$ or $b_{i_1+k+1} = 1$, or $i_1 + k = n$. Then there is a Fibonacci (p,r)-cube string α' adjacent to α obtained from α by changing a_{i_1+k} from 1 to 0 and $H(\alpha',\beta) = s-1$. By the induction hypothesis, $d_{\Gamma_n^{(p,r)}}(\alpha',\beta) = s-1 = H(\alpha',\beta)$. Then there exists a path of length s connecting α and β. So $d_{\Gamma_n^{(p,r)}}(\alpha,\beta) \leq s$. Hence $d_{\Gamma_n^{(p,r)}}(\alpha,\beta) = s$ by Ineq. (1.1). The result follows. □

Corollary 2.4. *Let $p \geq 2$ and $r \leq p+1$. Then $\Gamma_n^{(p,r)}$ is an isometric subgraph of Q_n.*

Proof. Suppose that there exist substring 10^t1 and 11^t1 appearing in the same coordinates of some two strings α and β in $\mathcal{F}_n^{(p,r)}$. Then $t \geq p$ and $t+2 \leq r$, that is, $p \leq t \leq r-2$, a contradiction to $r \leq p+1$. So $\Gamma_n^{(p,r)}$ is an isometric subgraph of Q_n by Lemma 2.3. □

Lemma 2.5. *Let $p \geq 2$ and $r \geq p+2$. Then $\Gamma_n^{(p,r)}$ is not an isometric subgraph of Q_n. Furthermore, it is not a partial cube.*

Proof. We choose $2p+6$ vertices of $\Gamma_n^{(p,r)}$ as follows:
$\alpha = 0^{n-p-2}01^p0$, $\beta = 0^{n-p-2}11^p0$, $\gamma = 0^{n-p-2}10^p1$, $\delta = 0^{n-p-2}11^p1$,
$\chi_i = 0^{n-p-2}0^{p+2-i}1^i$ and $\eta_i = 0^{n-p-2}1^i0^{p+2-i}$ for $i = 1, 2, \ldots, p+1$.

Note that $\eta_{p+1} = \beta$, $H(\gamma, \delta) = p$ and $\alpha\beta$ is an edge of $\Gamma_n^{(p,r)}$. By Lemma 2.3 and direct calculation we have

$d_{\Gamma_n^{(p,r)}}(\alpha, \gamma) = H(\alpha, \gamma) = p+2$, $d_{\Gamma_n^{(p,r)}}(\beta, \gamma) = H(\beta, \gamma) = p+1$,
$d_{\Gamma_n^{(p,r)}}(\alpha, \delta) = H(\alpha, \delta) = 2$, $d_{\Gamma_n^{(p,r)}}(\beta, \delta) = H(\beta, \delta) = 1$,
$d_{\Gamma_n^{(p,r)}}(\alpha, \chi_i) = H(\alpha, \chi_i) = p+2-i$, $d_{\Gamma_n^{(p,r)}}(\beta, \chi_i) = H(\beta, \chi_i) = p+3-i$,
$d_{\Gamma_n^{(p,r)}}(\alpha, \eta_i) = H(\alpha, \eta_i) = p+2-i$ and $d_{\Gamma_n^{(p,r)}}(\beta, \eta_i) = H(\beta, \eta_i) = p+1-i$ for $i = 1, 2, \ldots, p+1$. So $\gamma, \delta, \eta_i \in W_{\beta\alpha}$, $i = 1, 2, \ldots, p+1$ and $\chi_i \in W_{\alpha\beta}$, $i = 1, 2, \ldots, p+1$.

By Lemma 2.3 and Ineq. (1.2), $d_{\Gamma_n^{(p,r)}}(\gamma, \delta) \geq H(\gamma, \delta) + 2$ since there exist 10^p1 and 11^p1 appearing in the same coordinates of γ and δ. So $\Gamma_n^{(p,r)}$ is not an isometric subgraph of Q_n. Obviously, $P_1 = \gamma\chi_1\chi_2 \cdots \chi_p\chi_{p+1}\delta$ and $P_2 = \gamma\eta_1\eta_2 \ldots \eta_p\eta_{p+1}\delta$ both are paths of length $p+2$ connecting γ and δ, so $d_{\Gamma_n^{(p,r)}}(\gamma, \delta) \leq p+2$. Hence $d_{\Gamma_n^{(p,r)}}(\gamma, \delta) = p+2$. So P_1 and P_2 are two shortest path between γ and δ in $\Gamma_n^{(p,r)}$. Since the path P_1 does not lie entirely in $\langle W_{\beta\alpha} \rangle$ because $\chi_i \in W_{\alpha\beta}$ for $i = 1, 2, \ldots, p+1$, $\langle W_{\beta\alpha} \rangle$ is not a convex subgraph of $\Gamma_n^{(p,r)}$. Thus $\Gamma_n^{(p,r)}$ is not a partial cube for $p \geq 2$ and $n \geq r \geq p+2$ by Theorem 2.1. The result follows. □

Proof of Theorem 1.1. By Lemma 2.2 and Corollary 2.4, $\Gamma_n^{(p,r)}$ is a partial cube if $p = 1$, or $p \geq 2$ and $r \leq p+1$. By Lemma 2.5, $\Gamma_n^{(p,r)}$ is not a partial cube for $p \geq 2$ and $r \geq p+2$. So the result follows. □

Proof of Corollary 1.2. The sufficiency is obvious. Now we prove the necessity. By Theorem 1.1, if $\Gamma_n^{(p,r)}$ is a partial cube, then $p = 1$, or $p \geq 2$ and $r \leq p+1$. By Lemma 2.2 and Corollary 2.4, $\Gamma_n^{(p,r)}$ is an isometric subgraph of Q_n if it satisfies one of these two conditions. □

3 Proof of Theorem 1.5

For an isometric subgraph G of Q_n, suppose $\mu\nu$ is an edge of G such that μ, ν differ in k^{th} coordinate. By Theorem 4 of [16], the following lemma is immediate.

Lemma 3.1. *Let $\eta\varepsilon$ be any edge of G. Then $\eta\varepsilon \in F_{\mu\nu}$ if and only if η and ε differ in exactly the k^{th} coordinate.*

Proof of Theorem 1.5. By the definition of almost-graph the necessity is obvious.

So we now consider the sufficiency. Suppose $\Gamma_n^{(p,r)}$ is a partial cube. By Corollary 1.2, $\Gamma_n^{(p,r)}$ is an isometric subgraph of Q_n. Let $\mu\nu$ be any edge of $\Gamma_n^{(p,r)}$ and μ and ν differ in only i_k^{th} coordinate. Let $\alpha = a_1a_2\dots a_n$, and β be any two vertices of $\langle U_{\mu\nu}\rangle$. Let $s := d_{\Gamma_n^{(p,r)}}(\alpha,\beta) = H(\alpha,\beta)$. It suffices to show that $d_{\langle U_{\mu\nu}\rangle}(\alpha,\beta) = s$. We prove this by induction on s.

For the case $s = 1$ the result is obvious. For $s \geq 2$ suppose it is true for distance $s-1$. Let $\beta' = b_1b_2\dots b_n$ and α' be the neighbors of β and α in $\langle U_{\nu\mu}\rangle$, respectively. By Lemma 3.1, $H(\beta',\alpha) = s+1$. Let $a_{i_j} \neq b_{i_j}$ for $j = 1,2,\dots,s+1$ and $i_{j_1} < i_{j_2}$ if $j_1 < j_2$. Then β and β', and α and α' differ exactly in the i_k^{th} coordinate and $H(\alpha',\beta') = s$ by Lemma 3.1. Let γ be the neighbor of β on some shortest α,β-path and γ, β differ in the i_j^{th} coordinate. Then $\gamma \in W_{\mu\nu}$. Without loss of generality, suppose $i_j < i_k$. Let γ' be the string such that γ' and γ differ only in i_k^{th} position. If $\gamma' \in V(\Gamma_n^{(p,r)})$, then $\gamma\gamma' \in F_{\mu\nu}$ by Lemma 3.1. Since $d_{\Gamma_n^{(p,r)}}(\alpha,\gamma) = H(\alpha,\gamma) = s-1$, by the induction hypothesis there exists an α,β-path of length s lying entirely in $\langle U_{\mu\nu}\rangle$.

Now assume $\gamma' \notin V(\Gamma_n^{(p,r)})$. First we consider the case $p = 1$. Since γ' and γ differ only in i_k^{th} position we have $b_{i_k} = 1$. By $d_{\Gamma_n^{(p,r)}}(\beta',\alpha') = H(\beta',\alpha') = s$, there must exist a vertex ε' on some shortest α',β'-path such that ε' and β' differ in only i_m^{th} coordinate, where $i_m \neq i_j, i_k$. Let ε be the string obtained from ε' by changing b_{i_k} from 1 to 0. Obviously $\varepsilon \in V(\Gamma_n^{(p,r)})$ since $p = 1$. Then $\varepsilon \in V(\Gamma_n^{(p,r)})$ and $\varepsilon\varepsilon' \in F_{\mu\nu}$ by Lemma 3.1, and $d_{\langle U_{\mu\nu}\rangle}(\alpha,\varepsilon) = H(\alpha,\varepsilon) = s-1$ by the induction hypothesis. So there exists an α,β-path of length s lying entirely in $\langle U_{\mu\nu}\rangle$.

Now we turn to the case $p \geq 2$. Note that $r \leq p+1$ by Theorem 1.1. There are five cases: (1) $i_k = n$; (2) $i_k < n$ and $b_{i_k+1} = 0$; (3) $i_k < n$, $b_{i_k+1} = 1$, $b_{i_k-1} = 1$; (4) $i_k < n$, $b_{i_k+1} = 1$, $b_{i_k-1} = 0$ and $i_j < i_k - 1$; and (5) $i_k < n$, $b_{i_k+1} = 1$, $b_{i_k-1} = 0$ and $i_j = i_k - 1$. Case (3) is impossible. In fact, if $b_{i_k} = 0$, then there exist 101 in β'; if $b_{i_k} = 1$, then there exist 101 in β. That is a contradiction. If (1), (2), or (4) holds, then $b_{i_k} = 1$ since $\gamma' \notin V(\Gamma_n^{(p,r)})$ and $\gamma \in V(\Gamma_n^{(p,r)})$. If (5) holds, then $b_{i_k} = 0$. Otherwise there exist a substring 101 in γ, a contradiction.

Now we suppose $b_{i_k} = 1$. Based on the above discussion of (1),(2) and (4), we distinguish three subcase.

Case 1. $i_k < n$, $b_{i_k+1} = 1$, $b_{i_k-1} = 0$ and $i_j < i_k - 1$, or $i_k = n$ and $b_{i_k-1} = 0$.

Obviously $r \geq 2$ if $i_k < n$. Since β' and γ' differ in only i_j^{th} coordinate, $b_{i_j} = 0$, $i_k = i_j + p$ and $b_{i_j+1} = \cdots = b_{i_k-1} = 0$. Let t_1 be the number of consecutive 1s in front of b_{i_j} in γ' and t_2 be the number of consecutive 1s behind b_{i_j} in γ'. Then $1 \le t_1 \le r$ and $1 \le t_2 \le r-1$ if $i_k < n$, or $t_2 = 0$ if $i_k = n$.

Since $H(\alpha', \beta') = s$, there must exist $\eta' \in V(\Gamma_n^{(p,r)})$ on some shortest α', β'-path such that η' and β' differ in i_m^{th} ($i_m \ne i_j, i_k$) position. Let η be the string obtained from η' by changing b_{i_k} to $1 - b_{i_k}$. If $i_k = n$, then $\eta \in V(\Gamma_n^{(p,r)})$ by $b_{i_k} = 1$. If $i_k < n$ and $t_1 \ge 2$, then $i_m > i_k$ or $i_m < i_j$, so $\eta \in V(\Gamma_n^{(p,r)})$ by $b_{i_k - 1} = 0$ and $b_{i_k} = 1$. If $\eta \notin V(\Gamma_n^{(p,r)})$, then $i_k < n$, $t_1 = 1$ and $i_m = i_k - 1$. We distinguish two subcases.

Subcase 1.1. $p = 2$.

Obviously $j+1 = m$. If $k = s+1$ or $i_{k+1} > i_k + t_2$, then there exist 1^{3+t_2} in α' from the i_j^{th} coordinate to the $(i_k + t_2)^{th}$ coordinate, so $r \ge 3 + t_2$, a contradiction to $r \le p+1$. Hence $k \le s$ and $i_{k+1} \le i_k + t_2$. If $t_2 = 1$, then $i_{k+1} = i_k + 1$. If $t_2 \ge 2$, we claim that $i_k + t_2 \in \{i_{k+1}, \dots, i_{s+1}\}$. Otherwise there at most $t_2 - 1$ ($t_2 - 1 \le r - 2 \le p - 1$) 0s between the i_k^{th} and $(i_k + t_2)^{th}$ coordinates of α', a contradiction. Let δ' be the string obtained from β' by changing $b_{i_k + t_2}$ to $1 - b_{i_k + t_2}$, obviously $\delta' \in V(\Gamma_n^{(p,r)})$ by $i_k + t_2 = n$ or $b_{i_k + t_2 + 1} = 0$. Let δ be the string obtained from δ' by changing b_{i_k} to $1 - b_{i_k}$. Then $\delta \in V(\Gamma_n^{(p,r)})$ by $b_{i_k - 1} = 0$ and $\delta \in U_{\mu\nu}$ by Lemma 3.1. So there exists an α, β-path of length s lying entirely in $\langle U_{\mu\nu} \rangle$ by the induction hypothesis.

Subcase 1.2. $p > 2$.

First we claim that $j + 1 \ne m$. Otherwise there exists 0^{p-2} between the i_j^{th} and i_m^{th} coordinates of α', a contradiction. Furthermore, $i_j + t = i_{j+t}$ and $b_{i_j + t} = 0$ for $t = 1, \dots, p-2$. By a similar discussion as in Subcase 1.1, we can prove that $k \le s$ and $i_k + t_2 \in \{i_{k+1}, \dots, i_{s+1}\}$. Then we find a vertex δ' obtained from β' by changing $b_{i_k + t_2}$ to $1 - b_{i_k + t_2}$ and a vertex δ obtained from δ' by changing i_k^{th} coordinate from 1 to 0. Then $\delta\delta' \in F_{\mu\nu}$ by Lemma 3.1. So there exists an α, β-path of length s lying entirely in $\langle U_{\mu\nu} \rangle$ by the induction hypothesis.

Case 2. $i_k < n$, $b_{i_k + 1} = 0$ and $b_{i_k - 1} = 1$, or $i_k = n$ and $b_{i_k - 1} = 1$.

Since γ' and γ differ in only i_k^{th} coordinate and β' and γ' differ in only i_j^{th} coordinate, $i_k = i_j + r$, $b_{i_j + 1} = \cdots = b_{i_k - 1} = 1$ and $b_{i_j} = 0$. By $H(\alpha', \beta') = s$, there must exist a vertex ε' on some shortest α', β'-path, where ε' and β' differ in only i_m^{th} ($i_m \ne i_j, i_k$) position. Obviously if $i_k < n$, then $i_m \ne i_k + 1$, otherwise there exists a substring 1^{r+1} in β', a contradiction. So by $b_{i_k + 1} = 0$ or $i_k = n$, $\varepsilon \in V(\Gamma_n^{(p,r)})$, where ε and ε' differ in only i_k^{th} position. Then $\varepsilon\varepsilon' \in F_{\mu\nu}$ by Lemma 3.1, and $d_{\langle U_{\mu\nu} \rangle}(\varepsilon, \alpha) = H(\varepsilon, \alpha) = s - 1$ by the induction hypothesis. Hence there exists an α, β-path of length s lying entirely in $\langle U_{\mu\nu} \rangle$.

Case 3. $i_k < n$, $b_{i_k - 1} = 0$ and $b_{i_k + 1} = 0$.

Since $d_{\Gamma_n^{(p,r)}}(\alpha', \beta') = H(\alpha', \beta') = s$, there must exist $\varepsilon' \in V(\Gamma_n^{(p,r)})$ on some shortest α', β'-path, where ε' and β' differ in only $i_m^{th}(i_m \neq i_j, i_k)$ position. Obviously by $b_{i_k-1} = 0$ and $b_{i_k+1} = 0$, one of the $(i_k - 1)^{th}$ and $(i_k + 1)^{th}$ coordinate of ε' must be 0. Let ε be obtained from ε' by changing b_{i_k} from 1 to 0. Then $\varepsilon \in V(\Gamma_n^{(p,r)})$. Thus $\varepsilon\varepsilon' \in F_{\mu\nu}$ by Lemma 3.1 and $d_{\langle U_{\mu\nu}\rangle} H(\varepsilon, \alpha) = H(\varepsilon, \alpha) = s - 1$ by the induction hypothesis. Hence there exists an α, β-path of length s lying entirely in $\langle U_{\mu\nu}\rangle$.

Finally, suppose $b_{i_k} = 0$. Then (5) holds. We claim $i_k + 1 = i_{k+1}$. If not, then there exist a substring 101 in α', a contradiction. Let δ' be the string obtained from β' by changing b_{i_k+1} to $1 - b_{i_k+1}$. Then $\delta' \in V(\Gamma_n^{(p,r)})$ by $b_{i_k} = 0$. Let δ be the string obtained from δ' by changing b_{i_k} to $1 - b_{i_k}$. If $\delta \in V(\Gamma_n^{(p,r)})$, then the proof is completed. Now assume $\delta \notin V(\Gamma_n^{(p,r)})$. Note that $\delta \notin V(\Gamma_n^{(p,r)})$ if and only if there exist 10^t1 in δ from the i_k^{th} to $(i_k+t+1)^{th}$ coordinates, where $1 \leq t \leq p-1$. Since γ and δ only differ in i_j^{th} and $(i_k+1)^{th}$ positions, there exist $10^{t-1}1$ in γ from $(i_k+1)^{th}$ to $(i_k+t+1)^{th}$ coordinates if $t \geq 2$, a contradiction. So $t = 1$. Then by a similar argument as in Subcase 1.1, $k + 1 \leq s$ and $i_k + t + z \in \{i_{k+2}, \ldots, i_{s+1}\}$, where z is the number of consecutive 1s starting at the $(i_k + t + 1)^{th}$ coordinate. Let ε' be the string obtained from β' by changing b_{i_k+t+z} to $1 - b_{i_k+t+z}$ and ε obtained from ε' by changing b_{i_k} to $1 - b_{i_k}$. Then $\varepsilon\varepsilon' \in U_{\mu\nu}$ by Lemma 3.1. So there exists an α, β-path of length s lying entirely in $\langle U_{\mu\nu}\rangle$ by the induction hypothesis. The result follows. □

4 Proof of Theorem 1.4

The following lemma is a characterization of a median graph via almost-median graph given by Imrich and Klavžar.

Lemma 4.1 ([5]). *A graph is a median graph if and only if it is almost-median and contains no Q_3^- as a convex subgraph, where Q_3^- stands for the graph obtained from the hypercube Q_3 by removing one of its vertices.*

By Lemma 4.1 and Theorem 1.5, a Fibonacci (p, r)-cube is a median graph if and only if it is a partial cube and contains no Q_3^- as a convex subgraph.

Lemma 4.2. *Let $p = 1$ and $n > r \geq 2$, or $p \geq 2$ and $r \geq 3$. Then $\Gamma_n^{(p,r)}$ contains Q_3^- as a convex subgraph.*

Proof. For $p = 1$ and $n > r \geq 2$, let $X = \{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \lambda\} \subseteq V(\Gamma_n^{(p,r)})$, where $\alpha = 0^{n-r-1}1^{r-2}100$, $\beta = 0^{n-r-1}1^{r-2}101$, $\gamma = 0^{n-r-1}1^{r-2}000$, $\delta = 0^{n-r-1}1^{r-2}010$, $\epsilon = 0^{n-r-1}1^{r-2}001$, $\eta = 0^{n-r-1}1^{r-2}011$ and $\lambda = 0^{n-r-1}1^{r-2}110$. Obviously, $\langle X\rangle \cong Q_3^-$. If $\langle X\rangle$ is not a convex subgraph of $\Gamma_n^{(p,r)}$, then $\mu = 0^{n-r-1}1^{r-2}111 \in V(\Gamma_n^{(p,r)})$, which contains a substring 1^{r+1}, a contradiction.

For $p \geq 2$ and $r \geq 3$, let $X' = \{\alpha', \beta', \gamma', \delta', \epsilon', \eta', \lambda'\} \subseteq V(\Gamma_n^{(p,r)})$, where $\alpha' = 0^{n-3}000$, $\beta' = 0^{n-3}100$, $\gamma' = 0^{n-3}010$, $\delta' = 0^{n-3}001$, $\epsilon' = 0^{n-3}110$, $\eta' = 0^{n-3}011$ and $\lambda' = 0^{n-3}111$. Obviously, $\langle X' \rangle \cong Q_3^-$. If $\langle X' \rangle$ is not a convex subgraph of $\Gamma_n^{(p,r)}$, then $\mu' = 0^{n-3}101 \in V(\Gamma_n^{(p,r)})$, which contains a substring 101, a contradiction. The result holds. □

Lemma 4.3. *Let $p \geq r$ and $r \leq 2$. Then the following statements are equivalent.*

(i) There exists a vertex $\alpha = a_1 a_2 \ldots a_n \in \Gamma_n^{(p,r)}$ such that $a_{i_1} = a_{i_2} = a_{i_3} = 1$ for some $i_1, i_2, i_3 \in \{1, 2, \ldots, n\}$;

(ii) There exist vertices $\alpha_1, \alpha_2, \alpha_3 \in \Gamma_n^{(p,r)}$, where α_j is obtained by changing the i_j^{th} coordinate of some $\alpha \in B_n$ from 1 to 0, $j = 1, 2, 3$; and

(iii) Q_3 is an induced subgraph of $\Gamma_n^{(p,r)}$.

Proof. Obviously (iii) implies (ii).

Suppose (i) holds. If $i_1 = 1$, then $\alpha_1 \in \Gamma_n^{(p,r)}$. If $i_3 = n$, then $\alpha_3 \in \Gamma_n^{(p,r)}$. If $i_1 > 1$ and $i_3 < n$, then at least one of $a_{i_j - 1}$ and $a_{i_j + 1}$ is 0 by $r \leq 2$. So α_j can be obtained by changing the i_j^{th} coordinate of α from 1 to 0 and $\alpha_j \in \Gamma_n^{(p,r)}$, $j = 1, 2, 3$. So (i) implies (ii). Furthermore, strings α_4, α_5, α_6 and $\alpha_7 \in \Gamma_n^{(p,r)}$, where α_4, α_5 is obtained by changing the i_3^{th} coordinate of α_1, α_2 from 1 to 0, respectively, and α_6, α_7 is obtained by changing the i_1^{th} coordinate of α_2, α_5 from 1 to 0, respectively. Let $X = \{\alpha_j | j = 1, 2, \ldots, 7\}$. Then $\langle X \rangle \cong Q_3$. So (i) implies (iii).

Now we prove that (ii) implies (i). First we consider the case $r = 1$. Obviously, $i_1 + p < i_2$ and $i_2 + p < i_3$ since $r = 1$. Then $a_{i_2 - t} = 0$ by $\alpha_3 \in \Gamma_n^{(p,r)}$, and $a_{i_2 + t} = 0$ by $\alpha_1 \in \Gamma_n^{(p,r)}$, $t = 1, \ldots, p$. So α can be obtained by changing the i_2^{th} coordinate of α_2 from 0 to 1 and $\alpha \in \Gamma_n^{(p,r)}$.

Now we consider the case $r = 2$.

If $i_2 = i_1 + 1$, then $i_2 + p < i_3$. Otherwise $\alpha_2 \notin \Gamma_n^{(p,r)}$. By $\alpha_1 \in \Gamma_n^{(p,r)}$, $a_{i_2 + t} = 0$, $t = 1, \ldots, p$. By $\alpha_3 \in \Gamma_n^{(p,r)}$, $i_1 = 1$, or $1 < i_1 \leq p$ and $a_j = 0, j = 1, \ldots, i_1 - 1$, or $i_1 \geq p + 1$ and $a_{i_1 - t} = 0, t = 1, \ldots, p$. Thus α can be obtained by changing i_1^{th} coordinate of α_1 from 0 to 1 and so $\alpha \in \Gamma_n^{(p,r)}$.

If $i_3 = i_2 + 1$, then $i_1 + p < i_2$, by a similar argument as above we have $\alpha \in \Gamma_n^{(p,r)}$.

Now suppose $i_1 + p < i_2$ and $i_2 + p < i_3$. There are three cases to be considered. If $a_{i_2 + 1} = 1$, then $a_{i_2 - t} = 0$ by $\alpha_3 \in \Gamma_n^{(p,r)}$, $t = 1, \ldots, p$. If $a_{i_2 - 1} = 1$, then $a_{i_2 + t} = 0$ by $\alpha_1 \in \Gamma_n^{(p,r)}$, $t = 1, \ldots, p$. If $a_{i_2 - 1} = a_{i_2 + 1} = 0$, then by $\alpha_1 \in \Gamma_n^{(p,r)}$ and $\alpha_3 \in \Gamma_n^{(p,r)}$, $a_{i_2 - t} = 0$ and $a_{i_2 + t} = 0$, $t = 1, \ldots, p$. In the above three cases, α can be obtained by changing i_2^{th} coordinate of α_2 from 0 to 1 and $\alpha \in \Gamma_n^{(p,r)}$. The result follows. □

Corollary 4.4. *Let $p \geq r$ and $r \leq 2$. Then $\Gamma_n^{(p,r)}$ contains no Q_3^- as a convex subgraph.*

Proof. Let $X = \{\alpha, \beta, \lambda, \eta, \zeta, \gamma, \delta\} \subseteq V(\Gamma_n^{(p,r)})$ and $\langle X \rangle \cong Q_3^-$, where $\alpha = a_1 a_2 \ldots a_n$, $d_{Q_3^-}(\alpha, \beta) = d_{Q_3^-}(\alpha, \lambda) = 1$, $d_{Q_3^-}(\alpha, \eta) = d_{Q_3^-}(\alpha, \zeta) = d_{Q_3^-}(\alpha, \gamma) = 2$ and $d_{Q_3^-}(\alpha, \delta) = 3$ (see Fig. 2). Suppose α and δ differ in exactly i_1^{th}, i_2^{th} and i_3^{th} coordinates. There are three cases to be considered.

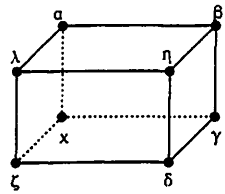

Fig. 2. An illustration to the proof of Corollary 4.4.

Case 1. β and α differ in i_1^{th} coordinate, λ and α differ in i_2^{th} coordinate. In this case η and γ can be obtained from β by changing a_{i_2} and a_{i_3} to $1-a_{i_2}$ and $1-a_{i_3}$, respectively, and ζ can be obtained from λ by changing a_{i_3} to $1-a_{i_3}$. Let χ be the string obtained from α by changing a_{i_3} to $1-a_{i_3}$, which can be obtained from ζ by changing $1-a_{i_2}$ to a_{i_2} and also can be obtained from γ by changing $1-a_{i_1}$ to a_{i_1}. We only need to show that $\chi \in V(\Gamma_n^{(p,r)})$. If not, we claim that $a_{i_3} = 0$, $a_{i_2} = 1$ and $a_{i_1} = 1$. In fact, by $\chi \notin V(\Gamma_n^{(p,r)})$, $\alpha \in V(\Gamma_n^{(p,r)})$ and $r \leq 2$ we have $a_{i_3} = 0$; by $\zeta \in V(\Gamma_n^{(p,r)})$, $r \leq 2$, and $a_{i_2-1} = 0$ or $a_{i_2+1} = 0$ we have $a_{i_2} = 1$; by $\chi \notin V(\Gamma_n^{(p,r)})$, $\gamma \in V(\Gamma_n^{(p,r)})$ and $r \leq 2$, we have $a_{i_1} = 1$. Hence the i_1^{th}, i_2^{th} and i_3^{th} coordinate of α are 1, 1, and 0, respectively, the i_1^{th}, i_2^{th} and i_3^{th} coordinate of γ are 0, 1, and 1, respectively, and the i_1^{th}, i_2^{th} and i_3^{th} coordinate of ζ are 1, 0, and 1, respectively. By Lemma 4.3, $\chi \in V(\Gamma_n^{(p,r)})$, a contradiction. Thus $\chi \in V(\Gamma_n^{(p,r)})$. So $\langle X \rangle \cong Q_3^-$ is not a convex subgraph of $\Gamma_n^{(p,r)}$.

Case 2. β and α differ in i_1^{th} coordinate, λ and α differ in i_3^{th} coordinate. In this case η and γ can be obtained from β by changing a_{i_3} and a_{i_2} to $1-a_{i_3}$ and $1-a_{i_2}$, respectively, and ζ can be obtained from λ by changing a_{i_2} to $1-a_{i_2}$. By a similar discussion as in Case 1, $\chi \in V(\Gamma_n^{(p,r)})$, where χ is obtained from α by changing a_{i_2} to $1-a_{i_2}$. Hence $\langle X \rangle \cong Q_3^-$ is not a convex subgraph of $\Gamma_n^{(p,r)}$.

Case 3. β and α differ in i_2^{th} coordinate, λ and α differ in i_3^{th} coordinate. In this case η and γ can be obtained from β by changing a_{i_3} and a_{i_1} to $1-a_{i_3}$ and $1-a_{i_1}$, respectively, and ζ can be obtained from λ by changing

a_{i_1} to $1 - a_{i_1}$. Let χ be the binary string obtained from α by changing a_{i_1} to $1 - a_{i_1}$. By a similar discussion as in Case 1, $\chi \in V(\Gamma_n^{(p,r)})$. Hence $\langle X \rangle \cong Q_3^-$ is not a convex subgraph of $\Gamma_n^{(p,r)}$. □

Proof of Theorem 1.4. By Lemma 4.2, $\Gamma_n^{(p,r)}$ is not a median graph for $p = 1$ and $n > r \geq 2$, or $p \geq 2$ and $r \geq 3$. By Theorem 1.5 and Corollary 4.4, $\Gamma_n^{(p,r)}$ is a median graph for $p \geq r$ and $r \leq 2$. Finally, $\Gamma_n^{(1,n)} \cong Q_n$ of course is a median graph. The result follows. □

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