An implicit σ_3 type condition for heavy cycles in weighted graphs*

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Abstract

In 2003, Li introduced a concept called implicit weighted degree, denoted by $id^w(v)$ for a vertex v in a weighted graph. In this paper, we prove that: Let G be a 2-connected weighted graph which satisfies the following conditions: (a) The implicit weighted degree sum of any three independent vertices is at least m; (b) For each induced claw, each induced modified claw and each induced P_4 of G, all of its edges have the same weight. Then G contains either a hamiltonian cycle or a cycle of weight at least $2m/3$.

Keywords: Weighted graph; hamiltonian cycles; heavy cycles; implicit weighted degree

1 Introduction

Throughout this paper, we consider only finite undirected and simple graphs. For notation and terminology not defined here can be found in [2].

Let $G = (V(G), E(G))$ be a graph and H be a subgraph of G. For a vertex $u \in V(G)$, the neighborhood of u in H is denoted by $N_H(u) = \{v \in V(H) : uv \in E(G)\}$ and the degree of u in H is denoted by $d_H(u) = |N_H(u)|$. If $G = H$, we always use $N(u)$ and $d(u)$ in place of $N_G(u)$ and $d_G(u)$, respectively. We use $N_2(u)$ to denote the set of vertices in G which are at distance 2 from u. And we call $N_2(u)$ the 2-neighborhood of u.

In study of hamiltonian problem, Zhu, Li and Deng found that some vertices may have small degrees, but the vertices around them have large degrees, such as the vertices in their neighborhoods and 2-neighborhoods. And we hope to use some large degree vertices to replace small degree vertices in the right position considered in the proofs, so that we may

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construct a longer cycle. This idea leads to the concept of implicit degrees of vertices [8].

Definition 1 *([8]) Let v be a vertex of a graph G. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, then set $k = d(v) - 1$, $M_2 = \max\{d(u) : u \in N_2(v)\}$ and $m_2 = \min\{d(u) : u \in N_2(v)\}$. Suppose $d_1 \leq d_2 \leq d_3 \leq \ldots \leq d_k \leq d_{k+1} \leq \ldots$ be the degree sequence of vertices of $N(v) \cup N_2(v)$. Then the implicit degree $id(v)$ of v, is defined as*

$$id(v) = \begin{cases} \max\{d(v), m_2\}, & \text{if } d_k < m_2; \\ \max\{d(v), d_{k+1}\}, & \text{if } d_k \geq m_2 \text{ and } d_{k+1} > M_2; \\ \max\{d(v), d_k\}, & \text{if } d_k \geq m_2 \text{ and } d_{k+1} \leq M_2. \end{cases}$$

If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then $id(v) = d(v)$.

Clearly, $id(v) \geq d(v)$ for each vertex v from the definition of implicit degree.

A graph G is called a weighted graph if each edge e is assigned a non-negative number $w(e)$, and $w(e)$ is called the weight of e. Clearly, an unweighted graph can be regarded as a weighted graph in which each edge is assigned a weight 1. The weight of a subgraph H of G and the weighted degree of a vertex v in G are defined as

$$w(H) = \sum_{e \in E(H)} w(e) \quad \text{and} \quad d^w(v) = \sum_{u \in N(v)} w(uv), \text{ respectively.}$$

A path P in a weighted graph G is called a heaviest longest path if P is a longest path of G, and $w(P)$ is maximum among all longest paths in G.

In [7], Li extended Definition 1 into weighted graphs as follows:

Definition 2 *([7]) Let v be a vertex of a weighted graph G. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, then set $k = d(v) - 1$, $m_2^w = \min\{d^w(u) : u \in N_2(v)\}$ and $M_2^w = \max\{d^w(u) : u \in N_2(v)\}$. Suppose $d_1^w \leq d_2^w \leq \cdots \leq d_{k+1}^w \leq \cdots$ be the weighted degree sequence of vertices of $N(v) \cup N_2(v)$. Then the implicit weighted degree $id^w(v)$ of v is defined as*

$$id^w(v) = \begin{cases} \max\{d^w(v), m_2^w\}, & \text{if } d_k^w < m_2^w; \\ \max\{d^w(v), d_{k+1}^w\}, & \text{if } d_k^w \geq m_2^w \text{ and } d_{k+1}^w > M_2^w; \\ \max\{d^w(v), d_k^w\}, & \text{otherwise.} \end{cases}$$

If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then $id^w(v) = d^w(v)$.

Clearly, $id^w(v) \geq d^w(v)$ for every vertex v.

Let $\alpha(G)$ be the independent number of a graph G. For a positive integer $k \leq \alpha(G)$, we define $\sigma_k(G) = \min\{d(x_1) + d(x_2) + \ldots + d(x_k) :$

$x_1, x_2, \ldots, x_k$ are k independent vertices in $G\}$ and $i\sigma_k(G) = \min\{id(x_1) + id(x_2) + \ldots + id(x_k) : x_1, x_2, \ldots, x_k$ are k independent vertices in $G\}$. For a weighted graph G, let $\sigma_k^w(G) = \min\{d^w(x_1) + d^w(x_2) + \ldots + d^w(x_k) : x_1, x_2, \ldots, x_k$ are k independent vertices in $G\}$ and $i\sigma_k^w(G) = \min\{id^w(x_1) + id^w(x_2) + \ldots + id^w(x_k) : x_1, x_2, \ldots, x_k$ are k independent vertices in $G\}$. If $k > \alpha(G)$, then they are all equal to $+\infty$.

We call the graph $K_{1,3}$ a claw, $K_{1,3} + e$ (e is an edge between two nonadjacent vertices in $K_{1,3}$) a modified claw. A path with l vertices is denoted by P_l.

A graph G is called hamiltonian if it has a hamiltonian cycle, *i.e.* a cycle that contains all vertices of G. There are many results about the existence of heavy cycles in graphs in terms of the weighted degree sum of independent vertices. The following two theorems are well-known.

Theorem 1 *([1]) Let G be a 2-connected weighted graph with $\sigma_2^w(G) \geq m$, then G contains either a hamiltonian cycle or a cycle of weight at least m.*

Theorem 2 *([3]) Let G be a 2-connected weighted graph which satisfies the following conditions:*
(1) $\sigma_3^w(G) \geq m$;
(2) For each induced claw and each induced modified claw of G, all of its edges have the same weight.
Then G contains either a hamiltonian cycle or a cycle of weight at least $2m/3$.

Now, we have the folowing question: Can $\sigma_3^w(G) \geq m$ in Theorem 2 be replaced by $i\sigma_3^w(G) \geq m$? If the answer is affirmative, then we can give a generalization of Theorem 2. In this paper, we give a partial answer to this problem.

Theorem 3 *Let G be a 2-connected weighted graph which satisfies the following conditions:*
(i) $i\sigma_3^w(G) \geq m$;
(ii) For each induced claw, each induced modified claw and each P_4 of G, all of its edges have the same weight.
Then G contains either a hamiltonian cycle or a cycle of weight at least $2m/3$.

We postpone the proof of Theorem 3. Here we give a graph G with $i\sigma_3^w(G) > \sigma_3^w(G)$. It is said that we can get a heavier cycle by using Theorem 3 than using Theorem 2.

Example 1 *Let G be a graph as Figure 1, where K_m is a complete graph and x, z are adjacent to every vertex of each K_m, $N(y) = \{x, z\}$. We* *assign weight 2 to each edge of* G*. It is easy to verify that* $d^w(x) = d^w(z) = 4m+2, d^w(y) = 4$ *and* $d^w(u) = 2(m+1)$ *for any* $u \in V(2K_m)$*. And by the definition of implicit weighted degree, we can get that* $id^w(x) = id^w(z) = 4m+2, id^w(y) = 2(m+1)$ *and* $id^w(u) = 2(m+1)$ *for any* $u \in V(2K_m)$*. Then* $\sigma_3^w(G) = 4m+8$ *and* $i\sigma_3^w(G) = 6(m+1)$*.*

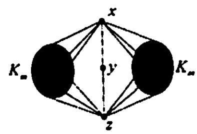

Figure 1

2 Proof of Theorem 3

Our proof of Theorem 3 is based on the following three lemmas.

Lemma 1 *([1]) Let* G *ba a non-hamiltonian 2-connected weighted graph and* $P = x_1x_2\ldots x_p$ *be a heaviest longest path of* G*. Then there is a cycle* C *in* G *with* $w(C) \geq d^w(x_1) + d^w(x_p)$*.*

Lemma 2 *([5]) Let* G *be a* k*-connected graph with at least three vertices. If* $k \geq \alpha(G)$*, then* G *is hamiltonian.*

Lemma 3 *([6]) Let* G *be a 2-connected graph such that* $i\sigma_3(G) \geq c$*, then* G *contains either a hamiltonian cycle or a cycle of length at least* $2c/3$*.*

Lemma 4 *Let* G *be a non-hamiltonian 2-connected weighted graph that satisfies the conditions of Theorem 3. Suppose there exist two edges* e_1 *and* e_2 *such that* $w(e_1) \neq w(e_2)$*, then there exists a heaviest longest path* $P = x_1x_2\ldots x_p$ *with* $d^w(x_1) + d^w(x_p) \geq 2m/3$*.*

We postpone the proof of Lemma 4 in next section.

Proof of Theorem 3 Let G be a weighted graph satisfying the conditions of Theorem 3. If $\alpha(G) \leq 2$, then G is hamiltonian by Lemma 2.

Hence we assume $\alpha(G) \geq 3$. If all edges of G have the same weight t. When $t = 0$, there is nothing to say. Suppose $t \neq 0$. By Definitions 1 and 2, we have $id^w(v) = t(id(v))$ for every $v \in V(G)$. Hence, $i\sigma_3(G) = i\sigma_3^w(G)/t = m/t$. Then, by Lemma 3, G contains either a hamiltonian cycle or a cycle C of length at least $2m/3t$. If G is not hamiltonian, then $w(C) = t \times |E(C)| \geq t \times (2m/3t) = 2m/3$.

Suppose there exist two edges e_1 and e_2 such that $w(e_1) \neq w(e_2)$ and G is non-hamiltonian. Then there is a heaviest longest path $P = x_1x_2\ldots x_p$

with $d^w(x_1) + d^w(x_p) \geq 2m/3$ by Lemma 4. Therefore, there is a cycle C in G with $w(C) \geq d^w(x_1) + d^w(x_p) \geq 2m/3$ by Lemma 1. The proof of Theorem 3 is complete.

3 Proof of Lemma 4

Let $P = x_1x_2\ldots x_p$ be a path of a graph G, define $N(x_1)^- = \{x_i : x_{i+1}x_1 \in E(G)\}$. To prove Lemma 4, we need the following two lemmas.

Lemma 5 *([7]) Let G be a 2-connected weighted graph and $P = x_1x_2\ldots x_p$ be a longest path of G. If $d^w(x_1) < id^w(x_1)$ and $x_1x_p \notin E(G)$, then either (1) there is some $x_j \in N(x_1)^-$ such that $d^w(x_j) \geq id^w(x_1)$; or (2) $N(x_1) = \{x_2, x_3, \ldots, x_{d(x_1)+1}\}$, $d^w(x_j) < id^w(x_1)$ for any $x_j \in N(x_1)^-$ and $id^w(x_1) = \min\{d^w(v) : v \in N_2(x_1)\}$.*

Lemma 6 *([4]) Let G be a 2-connected weighted graph satisfying condition (ii) of Theorem 3. If there are two edges with different weights, then each pair of vertices of G are at distance at most 2.*

Proof of Lemma 4 Let G be graph satisfying the conditions of Lemma 4.

Claim 1. There exists a heaviest longest path $P = x_1x_2\ldots x_p$ such that $id^w(x_1) + id^w(x_p) \geq 2m/3$.

Proof. Suppose to the contrary that $id^w(x_1) + id^w(x_p) < 2m/3$ for any heaviest longest path $P = x_1x_2\ldots x_p$. We choose a heaviest longest path $P = x_1x_2\ldots x_p$ such that $id^w(x_1) + id^w(x_p)$ is as large as possible.

Assume without loss of generality that $id^w(x_1) < m/3$. Since G is non-hamiltonian, $x_1x_p \notin E(G)$. From the choice of P, we can get that $N(x_1) \cup N(x_p) \subseteq V(P)$ and there is no cycle of length p. Since G is 2-connected, x_1 is adjacent to at least one vertex other than x_2. So $3 \leq k \leq p-1$, where $k = \max\{i : x_1x_i \in E(G)\}$.

Case 1. $N(x_1) = \{x_2, x_3, \ldots, x_k\}$.

Since $G - x_k$ is connected and P is a longest path of G, there must exist an edge x_rx_s with $r < k < s$. We choose such an edge x_rx_s such that
(I) s is as large as possible;
(II) r is as large as possible, subject to (I).

Case 1.1. $s \geq k+2$.

Claim 1.1. $w(x_1x_{r+1}) = w(x_rx_{r+1})$.

Proof. If $r < k-1$, by the choice of x_k and x_r, we have $x_1x_s \notin E(G)$ and $x_{r+1}x_s \notin E(G)$. So $\{x_r, x_1, x_{r+1}, x_s\}$ induces a modified claw. Then $w(x_1x_{r+1}) = w(x_rx_{r+1})$.

Suppose $r = k-1$. By Lemma 6, we have $d(x_1, x_p) = 2$. So $x_k x_p \in E(G)$. Now $\{x_k, x_1, x_{k-1}, x_p\}$ induces a modified claw. So $w(x_1 x_k) = w(x_{k-1}x_k)$. □

Claim 1.2. $w(x_{s-1}x_s) = w(x_r x_s)$.

Proof. If $x_r x_{s-1} \in E(G)$, we have $x_1 x_{s-1} \notin E(G)$ and $x_1 x_s \notin E(G)$ by the choice of x_k. Then $\{x_r, x_1, x_{s-1}, x_s\}$ induces a modified claw. So $w(x_{s-1}x_s) = w(x_r x_s)$.

Suppose $x_r x_{s-1} \notin E(G)$. Since $x_r x_{s+1} \notin E(G)$ by the choice of x_s, $\{x_s, x_r, x_{s-1}, x_{s+1}\}$ induces a claw or a modified claw. Thus $w(x_{s-1}x_s) = w(x_r x_s)$. □

It follows from Claims 1.1 and 1.2 that $x_{s-1}x_{s-2}\ldots x_{r+1}x_1x_2\ldots x_r x_s x_{s+1}\ldots x_p$ is a heaviest longest path different from P. Since G has no cycle of length p, $x_{s-1}x_p \notin E(G)$. Therefore, $\{x_1, x_{s-1}, x_p\}$ is an independent set of G. Then $id^w(x_{s-1}) + id^w(x_p) \geq m - id^w(x_1) > 2m/3$, contrary to the choice of P.

Case 1.2. $s = k+1$.

Since $G - x_{k+1}$ is connected, there must exist an edge $x_k x_t$ with $k+1 < t < p$ by the choice of x_r and x_s. Choose such an edge $x_k x_t$ such that t is as large as possible.

Claim 1.3. $w(x_1 x_{r+1}) = w(x_r x_{r+1})$.

Proof. If $r < k-1$, by the same proof as in Claim 1.1, we get $w(x_1 x_{r+1}) = w(x_r x_{r+1})$. Suppose $r = k-1$. By the choice of x_k and x_s, we have $x_1 x_t \notin E(G)$ and $x_{k-1}x_t \notin E(G)$. Then $\{x_k, x_1, x_{k-1}, x_t\}$ induces a modified claw. So $w(x_1 x_k) = w(x_{k-1}x_k)$. □

Claim 1.4. $w(x_r x_{k+1}) = w(x_k x_{k+1})$.

Proof. Since G has no cycle of length p, $x_r x_p \notin E(G)$ and $x_k x_p \notin E(G)$. By Lemma 6, we have $d(x_r, x_p) = 2$. So $x_{k+1}x_p \in E(G)$. Then $\{x_{k+1}, x_r, x_k, x_p\}$ induces a claw or a modified claw. Thus $w(x_r x_{k+1}) = w(x_k x_{k+1})$. □

Claim 1.5. $w(x_{t-1}x_t) = w(x_k x_t)$.

Proof. If $x_k x_{t-1} \notin E(G)$, then $x_k x_{t+1} \notin E(G)$ by the choice of x_t. Thus $\{x_t, x_k, x_{t-1}, x_k\}$ induces a claw or a modified claw. So $w(x_{t-1}x_t) = w(x_k x_t)$.

Suppose $x_k x_{t-1} \in E(G)$. By the choice of x_k, $x_1 x_{t-1} \notin E(G)$ and $x_1 x_t \notin E(G)$. Then $\{x_k, x_1, x_{t-1}, x_t\}$ induces a modified claw. So $w(x_{t-1}x_t) = w(x_k x_t)$. □

It follows from Claims 1.3, 1.4 and 1.5 that $x_{t-1}x_{t-2}\ldots x_{k+1}x_r x_{r-1}\ldots x_1 x_{r+1}x_{r+2}\ldots x_k x_t x_{t+1}\ldots x_p$ is a heaviest longest path of G different from P. Since G has no cycle of length p, $x_{t-1}x_p \notin E(G)$. Therefore, $\{x_1, x_{t-1}, x_p\}$ is an independent set of G. Then $id^w(x_{t-1}) + id^w(x_p) \geq m - id^w(x_1) > 2m/3$, contrary to the choice of P.

Case 2. $N(x_1) \neq \{x_2, x_3, \dots, x_k\}$.

Choose $x_r \notin N(x_1)$ with $2 < r < k$ such that r is as large as possible. Clearly, $x_1x_i \in E(G)$ for each i with $r < i \leq k$. Let j be the smallest index such that $j > r$ and $x_j \notin N(x_1) \cap N(x_r)$. Since $x_{r+1} \in N(x_1) \cap N(x_r)$ and $x_{k+1} \notin N(x_1) \cap N(x_r)$, we have $r + 2 \leq j \leq k + 1$. We have the following claim:

Claim 2.1. $w(x_1x_{r+1}) = w(x_rx_{r+1})$.

Proof. If $r < k - 1$, since G has no cycle of length p, $x_{r+1}x_p \notin E(G)$. By Lemma 2, we know $d(x_{r+1}, x_p) = 2$. So there exists a vertex x_s such that $x_s \in N(x_{r+1}) \cap N(x_p)$. Now $\{x_1, x_{r+1}, x_s, x_p\}$ induces a P_4 or a modified claw, which implies that $w(x_1x_{r+1}) = w(x_sx_p)$. At the same time, $\{x_r, x_{r+1}, x_s, x_p\}$ induces a P_4 or a modified claw, which implies that $w(x_rx_{r+1}) = w(x_sx_p)$. Therefore, $w(x_1x_{r+1}) = w(x_rx_{r+1})$.

Suppose $r = k - 1$. If $x_kx_p \in E(G)$, then $\{x_k, x_1, x_{k-1}, x_p\}$ induces a claw. So $w(x_1x_k) = w(x_{k-1}x_k)$. Suppose $x_kx_p \notin E(G)$. By similar arguments as $r < k - 1$, we get $w(x_1x_{r+1}) = w(x_rx_{r+1})$. □

By Claim 2.1, $x_rx_{r-1} \dots x_1x_{r+1}x_{r+2} \dots x_p$ be a heaviest longest path of G different from P. Since G has no cycle of length p, $x_rx_p \notin E(G)$. Therefore, $\{x_1, x_r, x_p\}$ is an independent set of G. Then $id^w(x_r) + id^w(x_p) \geq m - id^w(x_1) > 2m/3$, contrary to the choice of P. Now we complete the proof of Claim 1. □

By Claim 1, we can get a heaviest longest path $P = x_1x_2 \dots x_p$ in G such that $id^w(x_1) + id^w(x_p) \geq 2m/3$. We choose such a heaviest longest path $P = x_1x_2 \dots x_p$ in G such that $d^w(x_1) + d^w(x_p)$ is as large as possible and suppose $d^w(x_1) + d^w(x_p) < 2m/3$. We assume without loss of generality that $d^w(x_1) < id^w(x_1)$.

Case 2.1. There is some $x_j \in N(x_1)^- \setminus \{x_1\}$ such that $d^w(x_j) \geq id^w(x_1)$.

Claim 2.2. $w(x_1x_{j+1}) = w(x_jx_{j+1})$.

Proof. If $x_{j+1}x_p \in E(G)$, since G has no cycle of length p, $\{x_{j+1}, x_1, x_j, x_p\}$ induces a claw or a modified claw. So $w(x_1x_{j+1}) = w(x_jx_{j+1})$.

Suppose $x_{j+1}x_p \notin E(G)$. By similar arguments as in Claim 2.1, we can get that $w(x_1x_{j+1}) = w(x_jx_{j+1})$. □

It follows from Claim 2.2 that $x_jx_{j-1} \dots x_1x_{j+1}x_{j+2} \dots x_p$ be a heaviest longest path different from P. But $d^w(x_j) + d^w(x_p) \geq id^w(x_1) + d^w(x_p) > d^w(x_1) + d^w(x_p)$, contrary to the choice of P.

Case 2.2. $d^w(x_j) < id^w(x_1)$ for each $x_j \in N(x_1)^- \setminus \{x_1\}$.

By Lemma 5, $N(x_1) = \{x_2, x_3, \dots, x_k\}$ and $id^w(x_1) = \min\{d^w(u) : u \in N_2(x_1)\}$, where k is defined as in Claim 1. With similar arguments as in Case 1, we choose an edge v_rv_s as in Case 1 and have the following claim.

Claim 2.3. $w(x_1x_{r+1}) = w(x_rx_{r+1})$. □

Claim 2.4. $s > k + 1$.

Proof. Suppose $s = k+1$. By Lemma 2, $d(x_1, x_p) = 2$. Since G has no cycle of length p, $x_k x_p \in E(G)$. Now $x_1 x_2 \ldots x_r x_{k+1} x_{k+2} \ldots x_p x_k x_{k-1} \ldots x_{r+1}$ x_1 is a cycle of length p, a contradiction. □

Claim 2.5. $w(x_{s-1}x_s) = w(x_r x_s)$.

Proof. If $x_r x_{s-1} \in E(G)$, then $\{x_r, x_1, x_{s-1}, x_s\}$ induces a modified claw, which implies that $w(x_{s-1}x_s) = w(x_r x_s)$.

Suppose $x_r x_{s-1} \notin E(G)$. Then $\{x_s, x_r, x_{s-1}, x_{s+1}\}$ induces a claw or a modified claw. So $w(x_{s-1}x_s) = w(x_r x_s)$. □

It follows from Claims 2.3 and 2.5 that $x_{s-1}x_{s-2} \ldots x_{r+1}x_1 x_2 \ldots x_r x_s$ $x_{s+1} \ldots x_p$ is a heaviest longest path. By the choice of x_k, we have $x_1 x_{s-1} \notin E(G)$. So $d(x_1, x_{s-1}) = 2$. Thus $d^w(x_{s-1}) \geq id^w(x_1)$. Therefore, $d^w(x_{s-1}) + d^w(x_p) \geq id^w(x_1) + d^w(x_p) > d^w(x_1) + d^w(x_p)$, contrary to the choice of P. The proof of Lemma 4 is completed. □

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