EXCLUDED-MINOR CHARACTERIZATION FOR THE CLASS OF COGRAPHIC SPLITTING MATROIDS

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Abstract

This paper is based on the splitting operation for binary matroids that was introduced by Raghunathan, Shikare, and Waphare [Discrete Math. 184 (1998), p.267-271] as a natural generalization of the corresponding operation in graphs. In this paper, we consider the problem of determining precisely which cographic matroids M have the property that the splitting operation, by every pair of elements, on M yields a cographic matroid. This problem is solved by proving that there are exactly five minor-minimal matroids that do not have this property.

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1. Introduction

Fleischner [5] defined the splitting operation for a graph with respect to a pair of adjacent edges as follows:

Let G be a connected graph and let v be a vertex of degree at least three in G. If $x = vv_1$ and $y = vv_2$ are two edges incident at v, then splitting away the pair x, y from v results in a new graph $G_{x,y}$ obtained from G by deleting the edges x and y, and adding a new vertex $v_{x,y}$ adjacent to v_1 and v_2. The transition from G to $G_{x,y}$ is called the splitting operation on G. For practical purposes, we denote the new edges $v_{x,y}v_1$ and $v_{x,y}v_2$ in $G_{x,y}$ by x and y, respectively. (See Figure 1 for an illustration).

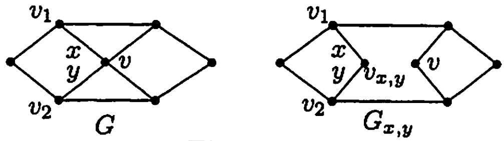


Figure 1

The splitting operation has important applications in graph theory. For example, Fleischner [5] used this operation to characterize Eulerian graphs and also gave an algorithm in terms of this operation to find all distinct Eulerian trails in an Eulerian graph. Tutte [12] characterized 3-connected graphs, and Slater [10] classified 4-connected graphs using a slight variation of this operation.

Raghunathan et al. [11] defined the splitting operation for binary matroids as follows:
Let M be a binary matroid on a set S and A be a matrix over $GF(2)$ that represents the matroid M. Consider elements x and y of M. Let $A_{x,y}$ be the matrix that is obtained by adjoining an extra row to A with this row being zero everywhere except in the columns corresponding to x and y where it takes the value 1. Let $M_{x,y}$ be the matroid represented by the matrix $A_{x,y}$. We say that $M_{x,y}$ has been obtained from M by splitting the pair of elements x and y. The matroid $M_{x,y}$ will be called the splitting matroid.

Alternatively, the splitting operation can be defined in terms of circuits of binary matroids. Let $M = (S, \mathcal{C})$ be a binary matroid on a set S together with the set $\mathcal{C}$ of circuits. Then $M_{x,y} = (S, \mathcal{C}')$ with $\mathcal{C}' = \mathcal{C}_0 \cup \mathcal{C}_1$, where
$\mathcal{C}_0 = \{C \in \mathcal{C} : x, y \in C \text{ or } x \notin C, y \notin C\}$; and
$\mathcal{C}_1 = \{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C}, x \in C_1, y \in C_2, C_1 \cap C_2 = \phi$ and $C_1 \cup C_2$ contains no member of $\mathcal{C}_0\}$.

Note that an arbitrary circuit of $M_{x,y}$ contain either both x and y or neither.

Several results concerning splitting operation have been explored in [1], [7], [8], and [9].

Let $M(G)$ and $M^*(G)$ denote the circuit matroid and the cocircuit matriod, respectively of a graph G. For undefined notation and terminology in graphs and matroids see [6].

It was shown in [11] that if x, y is a pair of adjacent edges in a graph G, then $M(G_{x,y}) = M(G)_{x,y}$. However, if x and y are non-adjacent, then $M(G)_{x,y}$ may not be graphic. Shikare and Waphare [9] characterized graphic matroids whose splitting matroids are also graphic as follows:

Theorem 1.1 (Shikare and Waphare [9]). *The splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if the circuit matroid of the corresponding graph has no minor isomorphic to the circuit matroid of any of the following four graphs.*

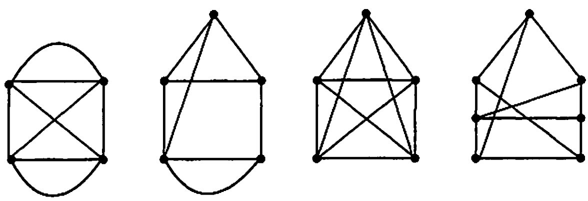
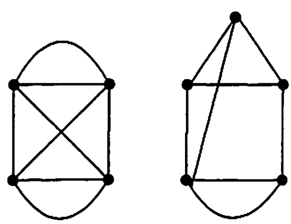


Figure 2

□

The splitting operation on a cographic matroid may not yield a cographic matroid. In the following theorem, we characterize cographic matroids M for which $M_{x,y}$ is cographic for every pair x, y of elements of M, which is the main result of this paper.

Theorem 1.2. *The splitting operation, by any pair of elements, on a cographic matroid yields a cographic matroid if and only if it has no minor isomorphic to one of the matroids $M(H_1)$ and $M(H_2)$, where the graphs H_1 and H_2 are given in Figure 3.*



H_1 H_2

Figure 3

2. The splitting operation and minors

In order to prove the main theorem, we provide some necessary results.

Lemma 2.1 (Shikare and Waphare [9]). *Let x and y be noncoloop elements of a binary matroid M and let $r(M)$ denote the rank of M. Then the following statements hold.*

(i) x *and* y *are in series in* $M_{x,y}$;

(ii) y *is a coloop in* $M_{x,y} \setminus \{x\}$ *while* x *is a coloop in* $M_{x,y} \setminus \{y\}$;

(iii) $M_{x,y} = M$ *if and only if* x *and* y *are in series in* M;

(iv) $r(M_{x,y}) = r(M) + 1$ *if and only if* x *and* y *are not in series in* M;

(v) *if* x_1, x_2 *are in series in* M, *then they are in series in* $M_{x,y}$;

(vi) $M_{x,y}/\{x\} \setminus \{y\} \cong M_{x,y}/\{y\} \setminus \{x\} \cong M_{x,y} \setminus \{x,y\} \cong M \setminus \{x,y\}$;

(vii) *if* x *is a coloop in* M, *then* x *and* y *are both coloops in* $M_{x,y}$ *and further,* $M_{x,y}/\{x,y\} \cong M_{x,y} \setminus \{x,y\} \cong M \setminus \{x,y\}$; *and*

(viii) *if* x *is a loop in* M, *then* $M_{x,y}/\{x\} \cong M \setminus \{x\}$. □

The following results are well known.

Lemma 2.2 (Oxley [6]). *A binary matroid is cographic if and only if it has no minor isomorphic to* $F_7, F_7^*, M(K_5)$, *or* $M(K_{3,3})$. □

Lemma 2.3 (Oxley [6]). *A binary matroid is graphic if and only if it has no minor isomorphic to* F_7, F_7^*, $M^*(K_5)$, *or* $M^*(K_{3,3})$. □

Notation. For convenience, let $\mathcal{F} = \{F_7,\ F_7^*,\ M(K_5),\ M(K_{3,3})\}$.

Lemma 2.4. *Let M be a cographic matroid and let $x, y \in E(M)$ such that $M_{x,y}$ is not cographic. Then there is a minor N of M such that no pair of elements of N is in series and $N_{x,y}/\{x\} \cong F$ or $N_{x,y}/\{x,y\} \cong F$ for some $F \in \mathcal{F}$.*

Proof. The proof is similar to the proof of Theorem 2.3 in [9]. □

Definition 2.5. Let M be a matroid in which no pair of elements is in series and let $F \in \mathcal{F}$. We say that M is *minimal with respect to* F if there exist two elements x and y of M such that $M_{x,y}/\{x\} \cong F$ or $M_{x,y}/\{x,y\} \cong F$.

Corollary 2.6. *Let M be a cographic matroid. For any $x, y \in E(M)$, the matroid $M_{x,y}$ is cographic if and only if M has no minor isomorphic to a minimal matroid with respect to any $F \in \mathcal{F}$.*

Proof. The proof follows from Lemma 2.1 and Lemma 2.4. □

Lemma 2.7. *Let $F \in \mathcal{F}$ and let M be a binary matroid such that either $M_{x,y}/\{x\} \cong F$ or $M_{x,y}/\{x,y\} \cong F$ for some $x, y \in E(M)$. Then the following statements hold.*

(i) M has neither loops nor coloops;

(ii) x and y cannot be parallel in M;

(iii) if x_1 and x_2 are parallel elements of M, then one of them is either x or y;

(iv) if $M_{x,y}/\{x\} \cong F$, then M has at most two pairs of parallel elements and there is no 3-circuit in M containing both x and y;

(v) if $M_{x,y}/\{x,y\} \cong F$, then M has at most one pair of parallel elements and there is no 3-circuit or 4-circuit in M containing both x and y;

(vi) if $M_{x,y}/\{x\} \cong M(K_{3,3})$ or $M_{x,y}/\{x,y\} \cong M(K_{3,3})$, then every odd circuit of M contains x or y and also, M has at most one pair of parallel elements; and

(vii) if $M_{x,y}/\{x\} \cong M(K_5)$ or $M_{x,y}/\{x,y\} \cong M(K_5)$, then every odd cocircuit of M contains x or y.

Proof. The proof follows from Lemma 2.1 and from the fact that F does not contain loops, coloops and 2-circuits. □

A matroid is *Eulerian* if its ground set can be expressed as a union of disjoint circuits of the matroid (see [3]). A matroid is *bipartite* if every circuit of it has an even number of elements. Welsh [3] proved that a binary matroid is Eulerian if and only if its dual is bipartite. It is easy to see that a binary matroid M is Eulerian if and only if the sum of columns of A is zero, where A is a matrix over $GF(2)$ that represents M. Raghunathan et al. [11] proved that a binary matroid M is Eulerian if and only if $M_{x,y}$ is

Eulerian for every pair of elements x and y.

Lemma 2.8 (Shikare and Waphare [9]). *Let M be a loopless binary matroid and $x, y \in E(M)$. Then (i) $M_{x,y}/\{x\}$ is Eulerian if and only if M is Eulerian; and (ii) if M is a circuit matroid of a graph G and x, y are non-adjacent edges of G such that $M_{x,y}/\{x, y\}$ is Eulerian then either G is Eulerian or the end vertices of x and y are precisely of odd degree.* □

Lemma 2.9 (Mills [1]). *Let M be a binary matroid and $x, y \in E(M)$. If C^* is a cocircuit of M containing both x and y with $|C^*| \geq 3$, then $C^* - \{x, y\}$ is a cocircuit of $M_{x,y}$.* □

3. The splitting of cographic matroids

In this section, we obtain the minimal matroids corresponding to the four matroids F_7, F_7^*, $M(K_{3,3})$ and $M(K_5)$, and use them to give a proof of Theorem 1.2.

Lemma 3.1. *Let M be a cographic matroid. Then M is minimal with respect to the matroid F_7 if and only if M is isomorphic to one of the two matroids $M(G_1)$ and $M(G_2)$, where G_1 and G_2 are the graphs of Figure 4.*

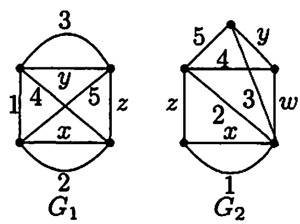


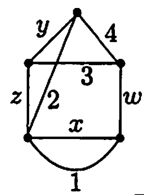
Figure 4

Proof. Suppose that M is isomorphic to $M(G_1)$ or $M(G_2)$. It follows from the matrix representation that $M(G_1)_{x,y}/\{x\} \cong F_7$ and $M(G_2)_{x,y}/\{x, y\} \cong F_7$. Further, no two elements of M are in series. Thus, M is minimal with respect to F_7.

Conversely, suppose that $M_{x,y}/\{x\} \cong F_7$ or $M_{x,y}/\{x, y\} \cong F_7$ for some

$x, y \in E(M)$. Then $|E(M)| = 8$ or 9. Suppose that M is not graphic. By Lemma 2.2 and Lemma 2.3, M has $M^*(K_{3,3})$ or $M^*(K_5)$ as a minor. Hence $|E(M)| \geq 9$. So M is isomorphic to $M^*(K_{3,3})$ and further, $M_{x,y}/\{x, y\} \cong F_7$. Therefore any two elements in M are in a 4-circuit, which is a contradiction to Lemma 2.7(v). So M is graphic. Thus, the result follows from Lemma 3.1 of [9]. □

Lemma 3.2. *Let M be a cographic matroid. Then M is minimal with respect to the matroid F_7^* if and only if M is isomorphic to the matroid $M(G_3)$, where G_3 is the graph of Figure* 5.

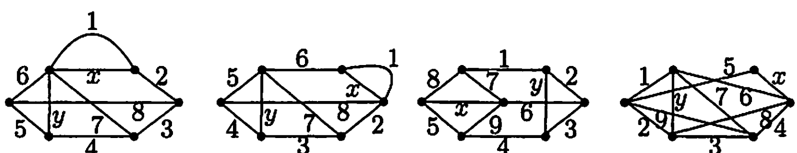


G_3 **Figure 5**

Proof. One can easily check from the matrix representation that $M(G_3)_{x,y}/\{x\} \cong F_7^*$. Therefore $M(G_3)$ is minimal with respect to F_7^*.

Conversely, suppose that either $M_{x,y}/\{x\} \cong F_7^*$ or $M_{x,y}/\{x, y\} \cong F_7^*$ for some $x, y \in E(M)$. If M is graphic, then the result follows from Lemma 3.2 of [9]. If M is not graphic then by Lemma 2.2 and Lemma 2.3, $M \cong M^*(K_{3,3})$, which is a contradiction to Lemma 2.7 (v). □

Lemma 3.3. *Let M be a cographic matroid. Then M is minimal with respect to the matroid $M(K_{3,3})$ if and only if M is isomorphic to one of the circuit matroids $M(G_4)$, $M(G_5)$, $M(G_6)$ and $M^*(G_7)$, where G_4, G_5, G_6 and G_7 are the graphs of Figure* 6.



G_4 G_5 **Figure 6** G_6 G_7

Proof. From the matrix representation, it follows that $M(G_4)_{x,y}/\{x\} \cong M(K_{3,3})$, $M(G_5)_{x,y}/\{x\} \cong M(K_{3,3})$, $M(G_6)_{x,y}/\{x,y\} \cong M(K_{3,3})$ and $M^*(G_7)_{x,y}/\{x,y\} \cong M(K_{3,3})$. Therefore $M(G_4)$, $M(G_5)$, $M(G_6)$ and $M^*(G_7)$ are minimal with respect to the matroid $M(K_{3,3})$.

Conversely, suppose that M is a minimal matroid with respect to the matroid $M(K_{3,3})$. Then there exist elements x and y of M such that $M_{x,y}/\{x\} \cong M(K_{3,3})$ or $M_{x,y}/\{x,y\} \cong M(K_{3,3})$.

Case (i). $M_{x,y}/\{x\} \cong M(K_{3,3})$.

Then $|E(M)| = 10$ and by Lemma 2.1, $r(M) = r(M_{x,y})-1 = r(M_{x,y}/\{x\}) = r(M(K_{3,3})) = 5$. We claim that M is graphic. By Lemma 2.2 and Lemma 2.3, it suffices to prove that M does not have $M^*(K_{3,3})$ and $M^*(K_5)$ as a minor. Since $r(M) = 5$, it cannot have a minor isomorphic to $M^*(K_5)$ of rank 6. Assume that M has a minor isomorphic to $M^*(K_{3,3})$. Since $r(M^*(K_{3,3})) = 4$, there exists an element q in M such that $M/q \cong M^*(K_{3,3})$. This implies that $M^* \setminus q \cong M(K_{3,3}) \cong M_{x,y}/\{x\}$. As M does not contain a pair of elements in series, the matroid M^* is simple. Hence M^* is a cycle matroid of the graph G depicted in Figure 7. One of the vertices v_1, v_2, v_3

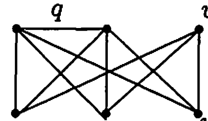


Figure 7

is not an end vertex of x and y. Let C be a cocircuit of M^* corresponding to this vertex of G. Then $C \cap \{x,y\} = \phi$ and further, C is a 3-circuit of M. Therefore C is a circuit in $M_{x,y}$. This implies that $M_{x,y}/\{x\}$ contains a circuit of size at most 3, a contradiction. We conclude that M is graphic. Let G be a graph such that $M = M(G)$. As $r(M) = 5$ and $|E(M)| = 10$, G has 6 vertices and 10 edges. Being a minimal matroid, no two elements of M are in series. Hence, by Lemma 2.7(i), G has minimum degree at least 3. Thus, degree sequence of G is (5,3,3,3,3,3) or (4,4,3,3,3,3). Further, G is planar because M is cographic. It follows from considering the circuits of $M(K_{3,3})$ and of $M(G)_{x,y}$ and Lemma 2.7 that G cannot have (i) two or more edge disjoint triangles, or (ii) a circuit of size 3 or 4 or 6 containing both x and y. By Appendix 1 of [4], every simple planar graph with 6 vertices and 10 edges contains two or more edge disjoint triangles. Therefore G is non-simple and by Lemma 2.7(vi), G has exactly one pair of parallel edges. Suppose the degree sequence of G is (5,3,3,3,3,3). Then G can be obtained from a simple graph with degree sequence (4,3,3,3,3,2) or (5,3,3,3,2,2) by adding an edge in parallel between vertices of degree 2 and degree 4, or between two vertices of degree 2, respectively. There are 3 non-isomorphic simple graphs with degree sequence (4,3,3,3,3,2) (see Appendix 1 [4]). Two of them contain disjoint triangles and hence we discard them. From the remaining graph, we obtain the graph G_4 of Figure 6. By Appendix 1 of [4], there are only 2 non-isomorphic simple graphs with degree sequence (5,3,3,3,2,2). As each of these contains disjoint triangles, the graph G cannot be obtained from any of these graphs.

Suppose that G has degree sequence (4,4,3,3,3,3). Then G can be obtained from a simple graph with degree sequences (3,3,3,3,3,3), (4,4,3,3,2,2) or (4,3,3,3,3,2) by adding an edge in parallel between two vertices of degree 3, two vertices of degree 2, or a vertex of degree 2 and a vertex of degree 3, respectively. There are 2 non-isomorphic simple graphs with degree sequence (3,3,3,3,3,3) (see Appendix 1 [4]). Both of them are discarded because one contains disjoint triangles and the other is non-planar.

Now, there are exactly 5 non-isomorphic simple graphs with degree sequence (4,4,3,3,2,2) (see Appendix 1 [4]). Out of these, two graphs are discarded because they contain disjoint triangles. In the other two graphs, vertices of degree 2 are not adjacent and hence after adding a parallel edge between two vertices of degree 2, we obtain a simple graph. Hence we discard them also. From the remaining one simple graph, we obtain the graph of Figure 8 of the degree sequence (4,4,3,3,3,3) by adding an edge in parallel.

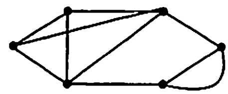


Figure 8

Since $M_{x,y}/\{x\}$ contains neither 2-circuits nor 3-circuits, every 3-circuit and 2-circuit of G must contain x or y. Hence G is not isomorphic to the graph of Figure 8.

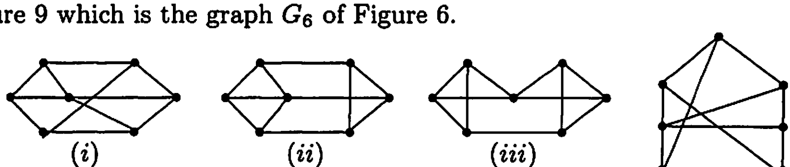
Suppose G is obtained from a simple graph of degree sequence (4,3,3,3,3,2). There are exactly 3 non-isomorphic simple graphs with degree sequence (4,3,3,3,3,2) (see Appendix 1 [4]). Out of which, two contain disjoint triangles and hence are discarded. From the remaining one, we obtain the graph G_5 of Figure 6 as a choice for G.

Case (ii). $M_{x,y}/\{x,y\} \cong M(K_{3,3})$.

Subcase (i). Suppose that M is graphic.

As $r(M(K_{3,3})) = 5$, $r(M_{x,y}) = 7$. Hence $r(M) = 6$ and $|E(M)| = 11$. Let G be a graph corresponding to M. Then G has 7 vertices, 11 edges and has minimum degree at least 3. Therefore the degree sequence of G is (4,3,3,3,3,3,3). Further, G is planar. It follows from Lemma 2.7 that G cannot have (i) more than two edge disjoint triangles, or (ii) a cycle of length other than 6 which contains both x and y, or (iii) a triangle and a 2-circuit which are edge disjoint.

Suppose that G is simple. The four graphs given in Figure 9 are the only non-isomorphic simple graphs each with degree sequence (4,3,3,3,3,3,3)(see [9]). Suppose that G is isomorphic to one of them. Observe that, in each of the graphs (i) and (iii) of Figure 9, there are no suitable x and y which belong to only a 6 cycle. Further, the graph (iv) of Figure 9 is not planar as it has $K_{3,3}$ as a minor. Therefore G is isomorphic to the graph (ii) of Figure 9 which is the graph G_6 of Figure 6.



(i) (ii) (iii) (iv)

Figure 9

Suppose that G is not simple. By Lemma 2.7, G has exactly one pair of parallel edges. Thus, G can be obtained from a simple graph with degree sequence (4,3,3,3,3,2,2) or (3,3,3,3,3,3,2) by adding an edge in parallel. However, any simple graph with degree sequence (4,3,3,3,3,2,2) can be obtained from a simple graph with degree sequence (4,3,3,3,2,1), (3,3,3,3,3,1), (3,3,3,3,2,2) or (4,3,3,2,2,2) by adding a vertex of degree 2. All the simple graphs with degree sequence (4,3,3,3,2,1) or (3,3,3,3,3,1) contain a triangle (see Appendix 1 [4]). So, after adding a vertex of degree 2 and then a parallel edge between two vertices of degree 2, we get a triangle and a 2-circuit which are edge-disjoint. Now, from the graphs with degree sequence (3,3,3,3,2,2) or (4,3,3,2,2,2), after adding a vertex of degree 2, there will be 2 vertices of degree 2 which are not adjacent. So, by putting an edge between these two vertices, we obtain a simple graph. Hence G cannot arise from the graph with degree sequence (4,3,3,3,3,2,2).

Now, by [9], the non-isomorphic multi graphs obtained from a simple graph with degree sequence (3,3,3,3,3,3,2) by adding an edge in parallel to an edge having an end vertex of degree 2, are the graphs given in Figure 10. In graphs $(i), (ii), (iii)$ and (v) of Figure 10, there is a triangle and a 2-circuit which are edge disjoint. Hence these graphs are discarded. If G is isomorphic to the graph (iv) of Figure 10, then both x and y are contained in a 2-circuit or a 5-circuit, which leads to a contradiction.

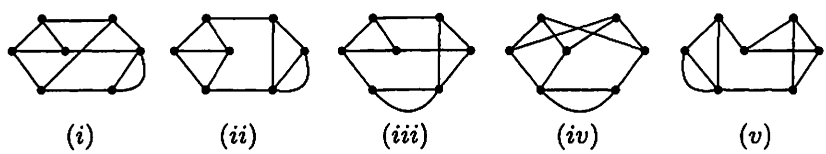


Figure 10

Subcase (ii). Suppose that M is not graphic.
Then M has a minor isomorphic to $M^*(K_{3,3})$ or $M^*(K_5)$. We claim that M does not have $M^*(K_{3,3})$ as a minor. Suppose that $M^*(K_{3,3})$ is a minor of M. Since $r(M^*(K_{3,3})) = 4$, $M/\{p,q\} \cong M^*(K_{3,3})$ for some elements p, q of M. Hence $M^* \setminus \{p,q\} \cong M(K_{3,3}) \cong M_{x,y}/\{x,y\}$. This implies that

$M^* \cong M(G)$, where G is the graph of Figure 11 or Figure 12. Since no two elements of M are in series, M^* is simple.

Suppose that G is the graph of Figure 11. Then M has two triangles corresponding to two 3-cocircuits containing edges incident at v_1 or v_2. By Lemma 2.7, each triangle in M must contain exactly one of x and y. This shows that v_i is an end vertex of either x or y for $i = 1, 2$ in G. We may assume that x is incident at v_1. Then y is not incident at v_1. If y is incident at

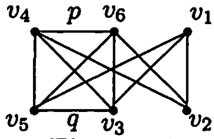


Figure 11

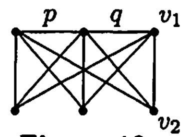
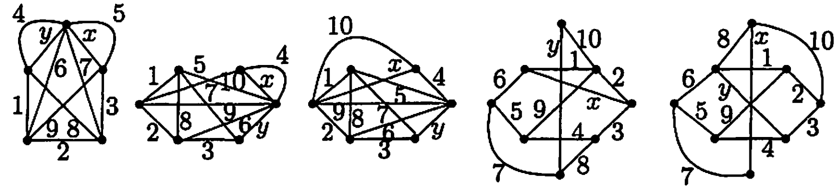


Figure 12

v_2, then x, y belong a 4-cocircuit of M^* and hence they belong to a 4-circuit of M which becomes a 2-circuit in $M_{x,y}/\{x, y\}$, a contradiction. Hence $x = v_1v_2$ and y is not adjacent to x in G. Therefore both x and y belong to a 5-cocircuit of M^* and hence belong to a 5-circuit of M, a contradiction. Suppose that G is the graph of Figure 12. Then any two elements of M^* belong to a 3-circuit or a 4-circuit. Hence any two elements in M belong to a 3-cocircuit or a 4-cocircuit. This means that $M_{x,y}/\{x, y\}$ contains a coloop or a 2-cocircuit, a contradiction. Hence M has $M^*(K_5)$ as a minor. Since $r(M) = 6$ and $|E(M)| = 11$, there exists an element q of M such that $M \setminus q \cong M^*(K_5)$. This implies that $M^*/q \cong M(K_5)$. So $M^* = M(G)$, where G is a simple graph with 6 vertices, 11 edges and has degree sequence (4,4,4,4,4,2) or (4,4,4,4,,3,3). It follows from Appendix 1 of [4] that there is only one graph with degree sequence (4,4,4,4,4,2) and G is isomorphic to this graph, which is the graph G_7 of Figure 6. Also, from Appendix 1 of [4], there are two non-isomorphic graphs with degree sequence (4,4,4,4,,3,3). As there is no choice for x, y for these two graphs, they are not isomorphic to G. □

Lemma 3.4. *Let M be a cographic matroid. Then M is minimal with respect to the matroid $M(K_5)$ if and only if M is isomorphic to one of the five matroids $M(G_8)$, $M(G_9)$, $M(G_{10})$, $M^*(G_{11})$ and $M^*(G_{12})$, where*

G_8, G_9, G_{10}, G_{11} *and* G_{12} *are the graphs of Figure* 13.



G_8 G_9 G_{10} G_{11} G_{12}

Figure 13

Proof. One can check that $M(G_8)_{x,y}/\{x\} \cong M(K_5)$, $M(G_9)_{x,y}/\{x,y\} \cong M(K_5)$, $M(G_{10})_{x,y}/\{x,y\} \cong M(K_5)$, $M^*(G_{11})_{x,y}/\{x,y\} \cong M(K_5)$, $M^*(G_{12})_{x,y}/\{x,y\} \cong M(K_5)$. Therefore $M(G_8)$, $M(G_9)$, $M(G_{10})$, $M^*(G_{11})$ and $M^*(G_{12})$ are minimal with respect to the matroid $M(K_5)$.

Conversely, suppose that M is a minimal matroid with respect to the matroid $M(K_5)$ and let x and y be the elements of M such that $M_{x,y}/\{x\} \cong M(K_5)$ or $M_{x,y}/\{x,y\} \cong M(K_5)$.

Case (i). $M_{x,y}/\{x\} \cong M(K_5)$.

Then $|E(M)| = 11$ and $r(M) = 4$. First we show that M is graphic. Suppose that M is not graphic. Then it has $M^*(K_5)$ or $M^*(K_{3,3})$ as a minor. Since $r(M^*(K_5)) = 6$, it cannot be minor of M. Hence $M^*(K_{3,3})$ is a minor of M. Since M is cographic, $M^* \cong M(G)$ for some graph G. As $r(M^*) = 11 - r(M) = 7$, G has 8 vertices and 11 edges. As $|E(M^*(K_{3,3}))| = 9$, $r(M^*(K_{3,3})) = 4$, $M \setminus \{p,q\} \cong M^*(K_{3,3})$ for some elements p,q of M. Therefore $M^*/\{p,q\} \cong M(K_{3,3})$. By Lemma 2.8($i$), M is Eulerian. This implies that G is bipartite. Further, the degree sequence of G is (3,3,3,3,3,3,2,2). Since no two elements in M are in series, G is simple. Such a simple bipartite graph can be obtained from a simple bipartite graphs with degree sequence (3,3,3,3,3,2,1) or (3,3,3,3,2,2,2) by adding a vertex of degree 2. It follows from Appendix 1 of [4] that two graphs of Figure 14 are the only non-isomorphic simple bipartite graphs with degree sequence (3,3,3,3,3,3,2,2). Clearly, in graph (i) of Figure 14, there does not exist a pair p,q of edges such that $M(G)/\{p,q\} \cong M(K_{3,3})$. If G is isomorphic to the graph (ii) of Figure 14, then p,q,r are in parallel class in M which contradicts to Lemma 2.7. This shows that M is graphic.

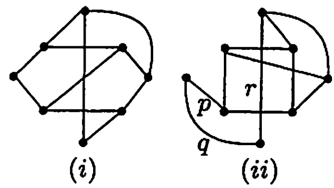


Figure 14

Suppose that $M \cong M(G)$ for some graph G. Since $r(M) = 4$ and $|E(M)| = 11$, G has 5 vertices and 11 edges. Obviously, G is non-simple and planar. Further, the minimum degree of G is at least 3. By Lemma 2.8(i), G is Eulerian. Hence the degree sequence of G is (6,4,4,4,4). By Lemma 2.7, G has at most two pairs of parallel edges. Thus, G is isomorphic to the graph G_8 of Figure 13.

Case (ii). $M_{x,y}/\{x,y\} \cong M(K_5)$.

Then $r(M) = 5$, $r(M_{x,y}) = 6$ and $|E(M)| = 12$. Suppose that M is graphic. Let G be a graph corresponding to M. Then G is a planar graph with 6 vertices and 12 edges. Further, x and y together do not belong to a 3-circuit or a 4-circuit. By Lemma 2.7, each odd cocircuit must contain exactly one of x and y. Suppose that G is simple. Then by Appendix 1 of [4], the two graphs of Figure 15 are the only simple planar graphs of 6 vertices and 12 edges. As any two edges of the graph (ii) of Figure 15 belong to a 3-circuit or a 4-circuit, G is not isomorphic to it. In graph (i) of Figure 15, except the pair x, y shown in the figure, any two edges are in a 3-circuit or a 4-circuit. If G is isomorphic to this graph, then by Lemma 2.8 (ii), $M(G)_{x,y}/\{x,y\}$ is not Eulerian, a contradiction.

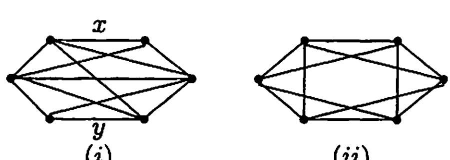


Figure 15

Suppose that G is not simple. Then by Lemma 2.7(v), G has exactly one pair of parallel edges. The graph G can be obtained from a simple graph on 6 vertices and 11 edges by adding an edge in parallel. There are 5 non-isomorphic simple planar graphs each with 6 vertices and 11 edges (see Appendix 1 [4]) as shown in Figure 16. It follows from Lemma 2.7(i), (v) and Lemma 2.8(ii) that G cannot be obtained from graphs (iii), (iv) and (v) of Figure 16. Suppose that G is obtained from graph (i) or (ii). Then G is isomorphic to one of the four graphs of Figure 17. By Lemma 2.7 (ii), (iii) and (vii), G is not isomorphic to graphs (i) and (ii) of Figure 17. Hence G is isomorphic to graphs (iii) and (iv) of Figure 17 which are the graphs G_9 and G_{10} of Figure 13.

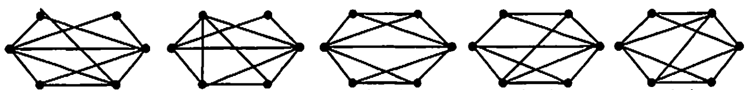


Figure 16

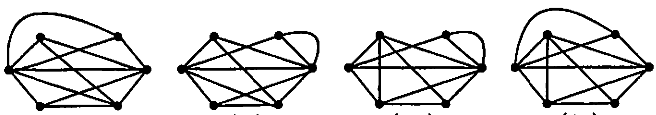
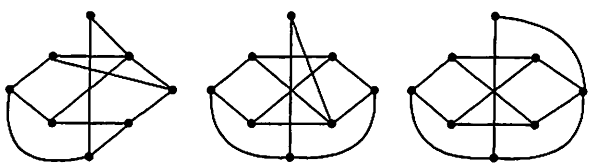


Figure 17

Suppose that M is not graphic. Then M has $M^*(K_{3,3})$ or $M^*(K_5)$ as a minor. Since $r(M^*(K_5)) = 6$ and $r(M) = 5$, M connot have $M^*(K_5)$ as a minor. So M has $M^*(K_{3,3})$ as a minor. As $r(M) = 5$ and $|E(M)| = 12$, $M\backslash \{p,q\}/\{r\} \cong M^*(K_{3,3})$ for some elements $p, q, r \in E(M)$. This implies that $M^* \setminus \{r\}/\{p,q\} \cong M(K_{3,3})$. Thus, $M^* \cong M(G)$, where G is a graph with 8 vertices and 12 edges. Since M has no 2-cocircuit, no two edges of G are parallel, and therefore G is simple. By Lemma 2.7(v), M has at most one 2-circuit and hence G has at most one vertex of degree 2. Therefore the degree sequence of G is either (4,3,3,3,3,3,3,2) or (3,3,3,3,3,3,3,3). We have $M(G) = M^* \setminus \{r\}/\{p,q\} \cong M(K_{3,3})$ for some edges $p, q, r \in E(G)$ and $M_{x,y}/\{x,y\} \cong M(K_5)$. As $K_{3,3}$ does not contain a j-circuit for $j = 1, 2, 3$, we have conditions

(i) *G does not have two or more edge disjoint triangles, or* *(ii) G does not have one triangle and one vertex of degree two contained in a 4-circuit.*

The simple graph with degree sequence (4,3,3,3,3,3,3,2) can be obtained from a simple graph with degree sequence (3,3,3,3,3,3,2) or (4,3,3,3,3,2,2) by adding a vertex of degree 2. Further, a simple graph with degree sequence (3,3,3,3,3,3,2) can be obtained from a simple graph with degree sequence (3,3,3,3,2,2) by adding a vertex of degree 2. There are 4 non-isomorphic simple graphs with degree sequence (3,3,3,3,2,2) (see Appendix 1 [4]). Any simple graph of degree sequence (4,3,3,3,3,3,3,2) obtained from these 4 graphs of degree sequence (3,3,3,3,2,2) having no disjoint triangles is isomorphic to one of the three graphs of Figure 18.

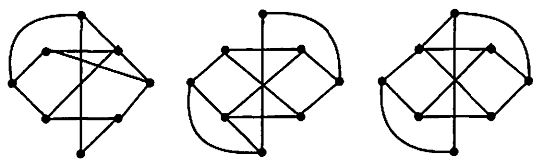


(i) *(ii)* *(iii)*

Figure 18

Suppose that a simple graph of degree sequence (4,3,3,3,3,3,3,2) is obtained from a simple graph with degree sequence (4,3,3,3,3,2,2) by adding a vertex of degree 2. Any simple graph with degree sequence (4,3,3,3,3,2,2) can be obtained from a simple graph with degree sequence (4,3,3,3,2,1), (4,3,3,2,2,2), (3,3,3,3,2,2) or (3,3,3,3,3,1) by adding a vertex of degree 2. By Appendix 1 of [4], there are 4 non-isomorphic simple graphs with degree sequence (4,3,3,3,2,1). Any simple graph of degree sequence (4,3,3,3,3,3,3,2) obtained from these 4 graphs contains disjoint triangles. Hence we discard all of them. Now, there are 4 non-isomorphic simple graphs with degree sequence (4,3,3,2,2,2) (see Appendix 1 [4]). The 2 graphs *(i)* and *(ii)* of Figure 19 are the only non-isomorphic simple graphs of degree sequence (4,3,3,3,3,3,3,2) obtained from these four graphs of degree sequence (4,3,3,2,2,2) which satisfy conditions *(i)* and *(ii)* above. It follows from Appendix 1 of [4] that there are 4 non-isomorphic simple graphs with degree sequence (3,3,3,3,2,2). Except the graph *(iii)* of Figure 19, all simple graphs of degree sequence (4,3,3,3,3,3,3,2) obtained from the graphs of degree sequence (3,3,3,3,2,2) do not satisfy conditions (*i*) or (*ii*) above. Similarly, there is only one simple graph of degree sequence (3,3,3,3,3,1) (see Appendix 1 [4]). Any simple graph of degree sequence (4,3,3,3,3,3,3,2) obtained from this graph contains disjoint triangles.

Thus, the six graphs of Figure 18 and Figure 19 are the only non-isomorphic simple graphs of degree sequence (4,3,3,3,3,3,3,2) which satisfy conditions (*i*) and (*ii*) above.

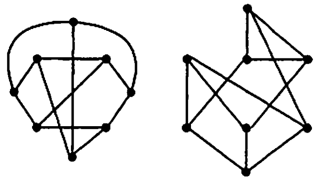


(*i*) (*ii*) (*iii*)

Figure 19

By Lemma 2.7 and Lemma 2.9, every 3-circuit and 5-circuit of G must contain exactly one of x and y. Further, no 4-circuit, 3-cocircuit, and 4-cocircuit of G contains both x and y. There is no pair of edges x, y in graphs (*ii*), (*iii*) of Figure 18 and graphs (*i*) and (*ii*) of Figure 19 which satisfy these conditions. Hence we discard these graphs. Thus, G is isomorphic to the graph (*i*) of Figure 18 or the graph (*iii*) of Figure 19 which are the graphs G_{11} and G_{12} of Figure 13.

Now, a simple graph with degree sequence (3,3,3,3,3,3,3,3) is obtained from a simple graph having degree sequence (3,3,3,3,2,2,2) by adding a vertex of degree 3 adjacent to vertices of degree 2. A simple graph with degree sequence (3,3,3,3,2,2,2) is obtained from a simple graph with degree sequence (3,3,3,3,1,1), (3,3,3,2,2,1) or (3,3,2,2,2,2) by adding two more vertices adjacent to vertices of degrees 2 and 1. By the same procedure as explained above, we construct simple graphs with degree sequence (3,3,3,3,3,3,3,3). Figure 20 shows all non-isomorphic simple graphs each of with degree sequence (3,3,3,3,3,3,3,3) which satisfy conditions (*i*) and (*ii*) on G.



(*i*) (*ii*)

Figure 20

By Lemma 2.7, G cannot have a 3-, 5-, or 7-circuit disjoint from x and y. Also, both x and y cannot belong to the same 4-cocircuit of G. Since no pair of edges (x, y) in graphs (i) and (ii) of Figure 20 satisfy these conditions, G is not isomorphic to any of these graphs. □

Now, using Lemma 3.1, 3.2, 3.3, and 3.4, we prove our main Theorem.

Proof of Theorem 1.2. Let M be a cographic matroid. On combining Corollary 2.6 and Lemma 3.1, 3.2, 3.3 and 3.4, it follows that $M_{x,y}$ is cographic for every pair $\{x, y\}$ of elements of M if and only if M has no minor isomorphic to any of the matroids $M(G_i)$, $i = 1, 2, 3, 4, 5, 6, 8, 9, 10$ and $M^*(G_j)$, $j = 7, 11, 12$, where the graphs G_i and G_j are shown in the statement of the Lemma 3.1, 3.2, 3.3 and 3.4. However, we have $M(G_3) \cong M(G_2)\backslash\{2\} \cong M(G_4)/\{5\}\backslash\{6\} \cong M(G_5)/\{3\}\backslash\{7\} \cong M(G_6)/\{3, 4\}\backslash\{9\} \cong M(G_8) \setminus \{y, 6, 7\} \cong M(G_9)/\{1\} \setminus \{2, 5, 9\} \cong M(G_{10})/\{y\} \setminus \{3, 7, 9\}$. Also $M^*(G_3) \cong M(G_7)/\{2\}\backslash\{3, y\} \cong M(G_{11})/\{7, 4, 6\}\backslash\{8\} \cong M(G_{12})/\{8, 5, 3\}\backslash \{10\}$. Therefore $M(G_3)$ is a minor of $M^*(G_7), M^*(G_{11})$ and $M^*(G_{12})$. Thus, $M_{x,y}$ is cographic if and only if M has no minor isomorphic to one of the matroids $M(G_1)$ and $M(G_3)$. But the graphs G_1 and G_3 are precisely the graphs given in the statement of the theorem. □

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