On the Surface Areas of the Alternating Group Graph and the Split-Star Graph

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Abstract

An important invariant of an interconnection network is its *surface area*, the number of nodes at distance i from a node. We derive explicit formulas, via two different approaches: direct counting and generating function, for the surface areas of the alternating group graph and the split-star graph, two Cayley graphs that have been proposed to interconnect processors in a parallel computer.

1 Introduction

Given a node u in a graph G, a question one may ask is *how many nodes are at distance* i *from* u for $i \in [1, D(G)]$, where $D(G)$ is the diameter of G. In other words, we want to calculate the quantity of $|\{v|d_G(u,v) = i\}|$. This quantity is referred to, in the literature, as the "Whitney numbers of the second kind of the poset" [19], or the "surface area of a node with radius i" [12]. We choose to use the latter term in this paper. This quantity is especially well defined for node symmetric graphs, which constitute the majority of the interconnection networks with mesh being one of the few exceptions, as the surface area for any node in a node symmetric graph G equals that for any other node in G. We can thus discuss the *surface area of such a graph* G.

The surface area of a graph can find several applications in network performance evaluation, e.g., in computing various bounds for the problem of k-neighborhood broadcasting [8] in interconnection networks, and in deriving the "transmission of a graph" [23], defined as the sum of all the distances in a graph G, an indicator for the speed of an average communication. This notion of transmission is also suggested to achieve the generalized Moore bound, an important concept in extremal graph theory.

As a result, this surface area problem has been studied for a variety of graphs, including the star graph [19, 18, 20, 12, 29, 26, 27, 4], the mesh [24], the (n, k)-star graph [28, 5], the k-ary n-cube [25], the rotator graph [6], and the WK-Recursive and swapped network [13].

The surface areas for some of the node symmetric networks are straightforward. For example, the surface area of the n-dimensional binary hypercube network, with radius $i \in [0, n]$, is $\binom{n}{i}$. It is also relatively easy to see the surface area of the complete transition network [11, 17] of n dimensions with radius i is $s(n, n-i), i \in [0, n-1]$, where $s(-,-)$ refers to the *signless Stirling numbers of the first kind* [21, §4.3]; while that of the adjacent transposition network, i.e., the *bubble-sort graph* [16], is $I_{n,i}, i \in [2, n]$, the number of permutations with n symbols and i inversions (An explicit formula for $I_{n,i}$ is given in [2, Eq. 2.5].). Unfortunately, such "easy solutions" are the exception rather than the norm. The solution of this surface area problem is often quite involved and challenging as reported in the aforementioned research.

Another issue is that, out of all the results obtained so far, only few of them are given in the ideal closed-form, namely, a finite sum of standard and basic operations, including that of power and factorials, and the rest in the form of either a recurrence or an *explicit form,* where the number of summands is bounded by a polynomial of the involved parameter(s). For example, for the star graph, several different but equivalent recurrences are given in, e.g., [19, 20, 29]; and several different explicit formulas in [19, 26, 12, 4, 27].

Besides satisfying an academic curiosity via its derivation, an explicit formula often leads to a simpler and more space efficient solution as compared with a recurrence since the maximum size of the problem that we can solve via a recurrence is bounded from above by the size of the system stack, while an explicit formula only needs a constant amount of space. Moreover, an explicit formula can sometimes be further simplified to a closed-form expression (For numerous examples of such simplification, readers are referred to [10, §5].).

In this paper, we provide explicit formulas of the surface areas for the natural and rich classes of the alternating group graph [14] and the split-star graph [3] via two different approaches, namely, the direct counting and generating functions.

The rest of this paper proceeds as follows: Sections 2 is on the direct counting approach. Specifically, we discuss a general process of deriving the surface areas of a node symmetric graph of permutation groups in Section 2.1. We then present, in Section 2.2, some of the basic notions related to the alternating group graph and an explicit formula of its surface area using the general scheme as discussed in Section 2.1. The generating function treatment of the problem for the alternating group graph is given in Section 3. Section 4 discusses the surface area of the split-star and concludes this paper.

2 An approach based on direct counting

2.1 A general process of deriving surface areas

Let $\langle n\rangle$ denote $\{1, 2, \ldots, n\}$, $n \geq 2$, S_n the collection of all the $n!$ permutations of $\langle n\rangle$, and let e denote the *identity permutation* $12\cdots n$. It is well known that every permutation $v(\neq e)$ of S_n can be expressed as a product of disjoint cycles of length at least 2; which is unique except for the order of these cycles. We refer to such a factorization as the *cycle structure* of v, denoted as $C(v)$.

Moreover, any cycle, $C = (c_1, c_2, \ldots, c_q), q \geq 2$, can be factorized into a product of $q-1$ transpositions: $(c_1, c_q) \circ (c_1, c_{q-1}) \circ \cdots \circ (c_1, c_2)$, which puts every symbol $c_j \in C, j \in [1, q]$, to its original position in e. It turns out that the above product is a shortest one of this nature [7, Lemma 1]. Here compositions are taken from right to left as usual. Hence, the concatenation of all such shortest products, associated with the cycle structure of v, leads to a *minimum transition string that changes* v *to* e.

For example, let v be 647251893, then $C(v) = (16)(24)(3789)$ and the following gives a shortest transition that changes v to e.

$$647251893 \overset{(1,6)}{\rightarrow} 147256893 \overset{(2,4)}{\rightarrow} 127456893 \overset{(3,7)}{\rightarrow}$$
$$128456793 \overset{(3,8)}{\rightarrow} 129456783 \overset{(3,9)}{\rightarrow} 123456789.$$

On the other hand, the specific structure of G, a graph defined on S_n, often places certain restriction on the nature of the transpositions permitted in G, which clearly defines a shortest path between v and e in G. If we can somehow derive $d_G(C(v), e)$, a *distance formula in terms of the cycle structure of* v in G, we can readily derive an explicit formula for the surface area of G of radius i, by enumerating all the cycle structures that satisfy the property that $d_G(C(v), e) = i$.

In other words, for $n \geq 2, i \in [1, D(G)]$, let $B_G(n, i)$ stand for the surface area of G with radius i, then

$$B_G(n, i) = |\{v | v \in G \text{ and } d_G(C(v), e) = i\}|.$$

As both the alternating group graph and the split-star graph have $n!/2$ vertices, any naive approach would have to examine each vertex, thus requiring $\Omega(n!)$ steps to find the surface area. This is clearly not acceptable. The ideal solution is to have a closed-form formula for $B_G(n, i)$. However, empirical study suggests that it is unlikely to have a closed-form expression for $B_G(n, i)$ for these two graphs. So the next best hope is to obtain an explicit formula. In this paper, we also include (signless) Stirling numbers of the first kind as a basic operation due to the known explicit formula for these numbers. By following such a direct counting approach, we have derived the respective surface areas of the star graph [27] and the (n, k)-star graph [28]. We now derive the surface area of the alternating group graph.

2.2 The surface area of the alternating group graph

For $i \in [3, n]$, let $g_i^+ = (1, 2, i)$, $g_i^- = (1, i, 2)$ and let $\Omega = \{g_i^+ | i \in [3, n]\} \cup \{g_i^- | i \in [3, n]\}$, $AG_n(V, E)$, the *alternating group graph of dimension* n, AG_n for short, is defined in [14] as follows: V is the collection of all the even permutations in $\mathcal{S}_n$; and for all $u, v \in V$, $(u, v) \in E$ iff for some $g \in \Omega, v = g \circ u$.

When compared with a star graph [1] of the same dimension, an alternating graph has half the nodes, but nearly twice the degree. It is both node and edge symmetric. There is also a Hamiltonian path between any pair of nodes in such a graph, and all the cycles with length between 3 and $n!/2$ can be embedded in AG_n with dilation 1.

Let $v \in AG_n$, an even permutation in $\mathcal{S}_n$, and let $b(v)$ and $g(v)$ stand for the number of symbols and that of the cycles in $C(v)$, respectively. The following distance formula, i.e., the length of the shortest path between v and e in AG_n, in terms of $C(v)$, is given in [14, Lemma 3.2] (A trivial cycle is a cycle of size one):

$$d_{AG_n}(C(v),e) = \begin{cases} 1.\ b(v)+g(v)-3 & \text{if 1 and 2 belong to the same cycle in } C(v); \\ 2.\ b(v)+g(v)-4 & \text{if 1 and 2 belong to distinct non-trivial cycles in } C(v); \\ 3.\ b(v)+g(v)-2 & \text{if exactly one symbol of 1 and 2 belongs to a trivial cycle in } C(v); \\ 4.\ b(v)+g(v) & \text{if both 1 and 2 belong to trivial cycles in } C(v). \end{cases} \quad (1)$$

For example, $1342 = (234)$, i.e., it contains three symbols, one cycle, and its cycle structure falls into Case 3. Thus, $d((234), e) = 3 + 1 - 2 = 2$. Indeed, one shortest path between 1342 and e consists of two edges:

$$1234 \overset{(1,2,4)}{\rightarrow} 4132 \overset{(1,3,2)}{\rightarrow} 1342.$$

In the rest of this paper, we use b and g in place of $b(v)$ and $g(v)$, when the context is clear.

The diameter of the alternating group graph is given as $\lfloor \frac{3n}{2} - 3 \rfloor$ by maximizing the above distance formula.

We first characterize all the even permutations in $\mathcal{S}_n$. Let $v \in \mathcal{S}_n$, and let $C_i = (d_1, d_2, \dots, d_{q(i)})$ be a cycle in $C(v)$, the following product is clearly a shortest factorization of C_i :

$$C_i = (d_1, d_{q(i)}) \circ \cdots \circ (d_1, d_3) \circ (d_1, d_2).$$

Since all the cycles in $C(v)$ are disjoint, we have the following formula for the length of a shortest representation of a permutation $v \in \mathcal{S}_n$ as a product of transpositions:

$$\sum_{i \in [1, g(v)]} (q(i) - 1) = b - g.$$

Since the parity of the length of such transposition products is an invariant [22, Corollary 3.7], we have the following result:

Lemma 2.1 *Let $v \in \mathcal{S}_n$, and let b and g be the number of symbols and that of the cycles in $C(v)$, its cycle structure, then v is even iff $b - g$ is even.*

Let $B_{AG}(n, i)$ be the number of nodes at distance i from e in AG_n. Since the four cases as given in Eq. 1 are mutually exclusive, we have that

$$B_{AG}(n, i) = B_1(n, i) + B_2(n, i) + B_3(n, i) + B_4(n, i),$$

where, $B_k(n, i)$ stands for the number of cycle structures falling into Case k, $k \in [1, 4]$, of Eq. 1.

For the case when both 1 and 2 belong to the same cycle in $C(v)$, there are exactly $\binom{n-2}{b-2}$ ways of choosing b symbols out of n symbols to construct $g(\geq 1)$ cycles. Obviously, $b \leq n$.

Let C_1 be the cycle where both 1 and 2 occur, containing $c_1(\geq 2)$ symbols. To construct C_1, we only need to select $c_1 - 2$ symbols out of $b-2$ symbols, in $\binom{b-2}{c_1-2}$ ways, thus $c_1 \leq b$. Furthermore, with any such a chosen set of c_1 symbols, we can construct $(c_1 - 1)!$ distinct cycles.

We now have $b - c_1$ symbols left to construct the remaining $g - 1$ cycles, each containing at least two symbols.

This *number of factorizing n symbols into i cycles, each of which contains at least two symbols,* denoted as $\mathbf{d}(n,i)$, is discussed in [21, §4.4]. Based on Eq. 4.18 [21]: for $n \geq 2i \geq 1$,

$$\mathbf{d}(n,i) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}s(n-j,i-j).$$

Recall that $s(-,-)$ stands for the signless Stirling numbers of the first kind, which can be represented by an explicit formula themselves [12, Eqs. 5 and 6].

Since $v \neq e$, $g \geq 1$. On the other hand, since each cycle contains at least two symbols, $2g \leq b$. As, for this case, $b = i - g + 3$, $2g \leq i - g + 3$, i.e., $3g \leq i + 3$. Thus, $g \leq \lfloor \frac{i+3}{3} \rfloor$, by [10, Eq. 3.7(d)]. Finally, $2 \leq c_1 \leq b = i - g + 3$.

To summarize,

$$\begin{aligned} B_1(n,i) &= \sum_{b,g,c_1}[b-g \text{ is even}]\binom{n-2}{b-2}\binom{b-2}{c_1-2}(c_1-1)!\mathbf{d}(b-c_1,g-1) \\ &= \sum_{g=1}^{\lfloor \frac{i+3}{3}\rfloor}\sum_{c_1=2}^{i-g+3}[i \text{ is odd}]\binom{n-2}{i-g+1}\binom{i-g+1}{c_1-2} \\ &\qquad (c_1-1)!\mathbf{d}(i-g-c_1+3,g-1). \end{aligned}$$

where $[i \text{ is odd}]$, in *Iverson's convention* [15, §2.1], is given as follows:

$$[i \text{ is odd}] = \begin{cases} 1, & \text{if } i \text{ is odd,} \\ 0, & \text{otherwise;} \end{cases}$$

since $b - g$ is even iff $i - 2g + 3$ is even iff i is odd.

Similar argument can be used to compute $B_2(n,i)$, $B_3(n,i)$, and $B_4(n,i)$, leading to the following theorem:

Theorem 2.1

$$B_{AG}(n,i) = B_1(n,i) + B_2(n,i) + B_3(n,i) + B_4(n,i), \tag{2}$$

where

$$B_1(n,i) = \sum_{g=1}^{\lfloor\frac{i+3}{3}\rfloor}\sum_{c_1=2}^{i-g+3}[i \text{ is odd}]\binom{n-2}{i-g+1}\binom{i-g+1}{c_1-2}(c_1-1)!\mathbf{d}(i-g-c_1+3,g-1);$$

$$B_2(n,i) = \sum_{g=1}^{\lfloor\frac{i+4}{3}\rfloor}\sum_{c_1=2}^{i-g+2}\sum_{c_2=2}^{i-g+4-c_1}[i \text{ is even}]\binom{n-2}{i-g+2}\binom{i-g+2}{c_1-1}\binom{i-g+3-c_1}{c_2-1}(c_1-1)!(c_2-1)!\mathbf{d}(i-g+4-c_1-c_2,g-2);$$

$$B_3(n,i) = 2\sum_{g=1}^{\lfloor\frac{i+2}{3}\rfloor}[i \text{ is even}]\binom{n-2}{i-g+1}\mathbf{d}(i-g+2,g);$$

and

$$B_4(n,i) = \sum_{g=1}^{\lfloor\frac{i}{3}\rfloor}[i \text{ is even}]\binom{n-2}{i-g}\mathbf{d}(i-g,g).$$

We remark that Theorem 2.1 is an explicit formula with $O(n^4)$ summands where Stirling numbers are treated as basic operations. Moreover, the bottleneck is in computing $B_2(n,i)$.

3 A generating function approach

3.1 A general framework

Generating functions are a powerful tool to deal with sequences of numbers and play an important role in enumerating combinatorial structures. In this section, we apply the generating function approach to the problem of finding an explicit formula for the surface area of the alternating group graph.

From Eq. 1, we see that the distance from node v to the identity in an alternating group graph depends only on the cycle structure of v and the positions of the symbols 1 and 2. The number of permutations with a specific cycle structure can be captured in the following exponential generating function: $e^{x+x^2/2+x^3/3+\cdots}$ where the term x^i/i means that we allow an i-cycle, a cycle of length i. (See 3.3.5 in [9] or 3.5 in [30].) This generating function can be used to count the number of permutations with respect to a specific cycle structure. A popular example is the number of derangements on n symbols. Since a derangement is a permutation with no fixed points, the exponential generating function is $e^{x^2/2+x^3/3+\cdots}$, that is, dropping x in the exponent. We will adjust this generating function to obtain the following propositions that will be crucial in our calculation.

Proposition 3.1 *Let $D_{m,r}$ be the coefficient of $x^m y^r/m!$ in*

$$\frac{e^{x-xy^2}}{2}\left(\frac{1}{(1-xy)^y}+(1+xy)^y\right).$$

Then $D_{m,r}$ is the number of even permutations of length m (that is, on m symbols) with the property that the number of non-trivial cycles in the cycle structure plus the number of symbols in the non-trivial cycles is r.

Let $F_{m,r}$ be the coefficient of $x^m y^r/m!$ in

$$\frac{e^{x-xy^2}}{2}\left(\frac{1}{(1-xy)^y}-(1+xy)^y\right).$$

Then $F_{m,r}$ is the number of odd permutations of length m with the property that the number of non-trivial cycles in the cycle structure plus the number of symbols in the non-trivial cycles is r.

Proof: Let x mark a symbol, y mark a symbol in a non-trivial cycle, z mark a non-trivial cycle and w mark an even cycle in the cycle structure of a permutation. Then the generating function (exponential in x, ordinary in y, z and w, it is ordinary in y as the labelling part has been taken care of by the exponential marker x) is

$$f(x,y,z,w)=e^{x+zwy^2x^2/2+zy^3x^3/3+zwy^4x^4/4\cdots}.$$

We will only prove the first claim as the proof for the second one is similar.

Note that

$$\begin{aligned}&x+zwy^2x^2/2+zy^3x^3/3+zwy^4x^4/4+\cdots=\\&x-xyz+\frac{z}{2}\ln\frac{1}{1-xy}-\frac{z}{2}\ln\frac{1}{1+xy}+\frac{zw}{2}\ln\frac{1}{1-xy}+\frac{zw}{2}\ln\frac{1}{1+xy}.\end{aligned}$$

Since we are only interested in even permutations, we want even numbers of even cycles in the cycle structure. So we only want even powers of w.

This can be accomplished by computing $(f(x,y,z,w)+f(x,y,z,-w))/2$. Once we have this, the marker w no longer serves its purpose and we set $w=1$. Hence we have $g(x,y,z)=(f(x,y,z,1)+f(x,y,z,-1))/2$. Now the coefficient of $\frac{x^m}{m!}y^p z^q$ of $g(x,y,z)$ is the number of permutations of length m whose cycle structure has exactly p symbols in q non-trivial cycles. Note that if we replace z by y, the coefficient of $\frac{x^m}{m!}y^r$ in $g(x,y,y)$ is the number of permutations of length m with the desired property. □

Proposition 3.2 *Let $h^{[1]}(x,y)$ be a function obtained by first differentiating*

$$\frac{e^{x-xyz}}{2}\left(\frac{1}{(1-xy)^z}+(1+xy)^z\right)$$

with respect to y, then multiplying by y, and then letting $z=y$, and let $E^{[1]}_{m,r}$ be the coefficient of $x^m y^r/m!$ in $h^{[1]}(x,y)$. Then

$$E^{[1]}_{m,r}=\sum_{p+q=r} pC_{m,p,q},$$

where $C_{m,p,q}$ is the number of even permutations of length m whose cycle structure has exactly p symbols in q non-trivial cycles.

Proof: From the proof of Proposition 3.1, we have

$$g(x,y,z)=\frac{e^{x-xyz}}{2}\left(\frac{1}{(1-xy)^z}+(1+xy)^z\right).$$

So the coefficient of $\frac{x^m}{m!}y^p z^q$ in $g(x,y,z)$ is the number of permutations of length m whose cycle structure has exactly p symbols in q non-trivial cycles. Differentiating with respect to y maps y^p to py^{p-1} and multiplying by y maps the result to py^p. Now the result follows as in the proof of Proposition 3.1 □

Proposition 3.3 *Let $h^{[2]}(x,y)$ be a function obtained by first differentiating*

$$\frac{e^{x-xyz}}{2}\left(\frac{1}{(1-xy)^z}+(1+xy)^z\right)$$

with respect to y, then multiplying by y^2, then differentiating with respect to y, and then letting $z=y$, and let $E^{[2]}_{m,r}$ be the coefficient of $x^m y^r/m!$ in $h^{[2]}(x,y)$. Then

$$E^{[2]}_{m,r}=\sum_{p+q=r} p(p+1)C_{m,p,q}.$$

Proof: Similar to the proof of Proposition 3.2. □

We will use Propositions 3.1, 3.2, and 3.3 to compute $B_{AG}(n,i)$. We start with $B_4(n,i)$ as it is the easiest. We want the number of even permutations, v, of length n, where each of symbols 1 and 2 belongs to a trivial cycle in $C(v)$ and that $b(v)+g(v)=i$. Since each of 1 and 2 belongs to a trivial cycle, we may exclude them. Therefore we are really looking for an even v' on $n-2$ symbols with $b(v')+g(v')=i$. Hence $B_4(n,i)=D_{n-2,i}$.

We now consider $B_3(n,i)$. We want the number of even permutations, v, of length n, where exactly one of symbols 1 and 2 belongs to a trivial cycle in $C(v)$ and that $b(v)+g(v)=i+2$. We may assume that 1 belongs to a trivial cycle whereas 2 belongs to a non-trivial cycle in $C(v)$ and multiply the resulting answer by 2. We first count the number of such even permutations with 1 in a trivial cycle and with no restrictions on the symbol 2. Then we are really looking for an even v' on $n-1$ symbols with $b(v')+g(v')=i+2$. Hence this number is $D_{n-1,i+2}$. But we have overcounted those with each of 1 and 2 belonging to a trivial cycle and there are $D_{n-2,i+2}$ such even permutations. Hence $B_3(n,i)=2D_{n-1,i+2}-2D_{n-2,i+2}$.

We now consider $B_2(n,i)$. We want the number of even permutations of length n, v, where the symbols 1 and 2 belong to distinct non-trivial cycle in $C(v)$ and that $b(v)+g(v)=i+4$. We consider 4 cases.

1. *The symbol 1 belongs to a cycle of length 2 and the symbol 2 belongs to a cycle of length 2.* There are $(n-2)(n-3)$ ways to choose a companion for 1 and a companion for 2. We note that there is only one way to form a cycle with two symbols. Now we delete these four symbols and we are really looking for v' on $n-4$ symbols with $(b(v')+4)+(g(v')+2)=i+4$. We note that the $+2$ comes from the two non-trivial cycles containing the symbol 1 and the symbol 2 respectively, and the $+4$ comes from the 4 symbols that we need to count. Since we have preselected two 2-cycles, v' is even. So the answer is $(n-2)(n-3)D_{n-4,i-2}$.

2. *The symbol 1 belongs to a cycle of length at least 3 and the symbol 2 belongs to a cycle of length 2.* There are $(n-2)$ ways to choose such a companion for 2. Now we delete the symbols 1, 2 and the companion for 2. Again there is only one way to form a cycle with two symbols. So we are really looking for v' on $n-3$ symbols with $(b(v')+3)+(g(v')+1)=i+4$. The question is should v' be even or odd which we will answer later. We note that the $+1$ comes from the non-trivial cycle containing the symbol 2, and the $+3$ comes from the 3 symbols (1, 2 and the companion of 2) that we need to count. So one may think that the answer is $(n-2)D_{n-3,i}$. But this is incorrect as we need to insert 1 into one of the $g(v')$ non-trivial cycles, and

there are $b(v')$ ways to do it. So the number we want is $(n-2)E^{[1]}_{n-3,i}$. This of course assumes that v' is even. We will now justify this. Since we have preselected a 2-cycle, one may think that v' should be odd. However, inserting the symbol 1 into the non-trivial cycles will change the parity of v'. Hence v' should be even.

3. *The symbol 1 belongs to a cycle of length 2 and the symbol 2 belongs to a cycle of length at least 3.* The number we want is $(n-2)E^{[1]}_{n-3,i}$.

4. *The symbols 1 and 2 belong to different cycles of length at least 3.* We withhold the symbols 1 and 2. So we look for v' on $n-2$ symbols with $(b(v')+2)+(g(v')+0) = i+4$. We note that the $+2$ comes from the 2 symbols (the symbols 1 and 2) that we need to count. The question is whether v' should be even or odd. For each v', we need to insert the symbols 1 and 2 into two distinct non-trivial cycles. We will first relax the "distinct" requirement. Then there are $b(v')$ ways to insert 1 and then there are $b(v')+1$ ways to insert 2. Two such insertions preserve the parity of the permutation. So v' should be even and the required number is $E^{[2]}_{n-2,i+2}$. But we have relaxed the "distinct" requirement. So we have to subtract those configurations with 1 and 2 inserted in the same non-trivial cycle in the cycle structure of v'. To accomplish this, we restart the process. We want 1 and 2 to be in a cycle of length at least 4. (Note that we indeed want 4 and not 3.) We withhold the symbols 1 and 2. For each $2 \le j \le n-2$, we consider the case in which we have j symbols together with 1 and 2 to form a cycle of length $j+2$. There are $(j+1)!$ ways to rearrange these $j+2$ symbols. Now we look for v' on $n-j-2$ symbols with $(b(v')+j+2)+(g(v')+1) = i+4$. We note that the $+1$ comes from the non-trivial cycle containing the symbols 1 and 2, and the $+(j+2)$ comes from the $j+2$ symbols that we need to count. If $j+2$ is even, then v' should be odd. If $j+2$ is odd, then v' should be even. So this number is $D_{n-j-2,i-j+1}$ if $j+2$ is odd, and $F_{n-j-2,i-j+1}$ if $j+2$ is even. So the number we want is $E^{[2]}_{n-2,i+2} - \sum_{j=2}^{n-2} \binom{n-2}{j}(j+1)!DF_{n-j-2,i-j+1}$ where $DF_{n-j-2,i-j+1} = D_{n-j-2,i-j+1}$ if j is odd and $DF_{n-j-2,i-j+1} = F_{n-j-2,i-j+1}$ if j is even.

So the desired number is $(n-2)(n-3)D_{n-4,i-2} + 2(n-2)E^{[1]}_{n-3,i} + E^{[2]}_{n-2,i+2} - \sum_{j=2}^{n-2} \binom{n-2}{j}(j+1)!DF_{n-j-2,i-j+1}$.

We finally consider $B_1(n,i)$. We want the number of even permutations v of length n, where the symbols 1 and 2 belong to the same cycle in $C(v)$ and that $b(v)+g(v) = i+3$. We withhold the symbols 1 and 2. For each $0 \le j \le n-2$, we consider the case in which we have j symbols together with 1 and 2 to form a cycle of length $j+2$. There are $(j+1)!$ ways to rearrange these $j+2$ symbols. Now we look for v' on $n-j-2$ symbols with $(b(v')+j+2)+(g(v')+1) = i+3$. We note that the $+1$ comes from the non-trivial cycle containing the symbols 1 and 2, and the $+(j+2)$ comes from the $j+2$ symbols that we need to count. Again v' is either even or odd depending on the parity of $j+2$. This number is $D_{n-j-2,i-j}$ if $j+2$ is odd and $F_{n-j-2,i-j}$ if $j+2$ is even. So the number we want is $\sum_{j=0}^{n-2}\binom{n-2}{j}(j+1)!DF_{n-j-2,i-j}$ where $DF_{n-j-2,i-j}$ is $D_{n-j-2,i-j}$ if j is odd and $F_{n-j-2,i-j}$ if j is even.

Finally, we have

$$\begin{aligned} B_{AG}(n,i) &= D_{n-2,i}+2D_{n-1,i+2}-2D_{n-2,i+2}+(n-2)(n-3)D_{n-4,i-2} && (3)\\ &\quad +2(n-2)E^{[1]}_{n-3,i}+E^{[2]}_{n-2,i+2} && (4)\\ &\quad -\sum_{j=2,\text{odd}}^{n-2}\binom{n-2}{j}(j+1)!D_{n-j-2,i-j+1} \\ &\quad -\sum_{j=2,\text{even}}^{n-2}\binom{n-2}{j}(j+1)!F_{n-j-2,i-j+1} && (5)\\ &\quad +\sum_{j=0,\text{odd}}^{n-2}\binom{n-2}{j}(j+1)!D_{n-j-2,i-j} \\ &\quad +\sum_{j=0,\text{even}}^{n-2}\binom{n-2}{j}(j+1)!F_{n-j-2,i-j}. && (6) \end{aligned}$$

3.2 Computing $D_{m,r}$, $F_{m,r}$, $E^{[1]}_{m,r}$ and $E^{[2]}_{m,r}$

We start with the well-known generating function for the signless Stirling numbers of the first kind $s(p,k)$,

$$\frac{1}{(1-t)^w} = \sum_{p\geq k\geq 0} s(p,k)w^k\frac{t^p}{p!}. \tag{7}$$

We set $w=y$ and $t=xy$ to obtain

$$\frac{1}{(1-xy)^y} = \sum_{p\geq k\geq 0} s(p,k)y^{p+k}\frac{x^p}{p!} \tag{8}$$

and setting $w=-y$ and $t=-xy$, we get

$$(1+xy)^y = \frac{1}{(1+xy)^{-y}} = \sum_{p\geq k\geq 0}(-1)^{p+k}s(p,k)y^{p+k}\frac{x^p}{p!}. \tag{9}$$

We remark that $(1+t)^w$ is the generating function for the signed Stirling numbers of the first kind. So Eq. (9) is related to it.

Throughout this discussion, we use the standard notation of $[x^a y^b]f(x,y)$ to denote the coefficient of $x^a y^b$ in $f(x,y)$. Now

$$D_{m,r} = m![x^m y^r]\frac{e^{x-xy^2}}{2}\left(\frac{1}{(1-xy)^y} + (1+xy)^y\right).$$

By using Eqs. 8 and 9, we have

$$D_{m,r} = m![x^m y^r]\frac{e^{x-xy^2}}{2}\sum_{p\geq k\geq 0}(1+(-1)^{p+k})s(p,k)y^{p+k}\frac{x^p}{p!}.$$

Therefore

$$D_{m,r} = \frac{m!}{2}\sum_{p\geq k\geq 0}\frac{(1+(-1)^{p+k})}{p!}s(p,k)[x^{m-p}y^{r-p-k}]e^{x-xy^2}.$$

But

$$e^{x-xy^2} = e^{x(1-y^2)} = \sum_{j\geq 0}\sum_{l=0}^{j}\frac{x^j}{j!}\binom{j}{l}(-1)^l y^{2l}.$$

So

$$D_{m,r} = \frac{m!}{2}\sum_{p\geq k\geq 0}\frac{(1+(-1)^{p+k})}{p!}s(p,k)\frac{1}{(m-p)!}\binom{m-p}{(r-p-k)/2}(-1)^{(r-p-k)/2}.$$

An almost identical derivation gives

$$F_{m,r} = \frac{m!}{2}\sum_{p\geq k\geq 0}\frac{(1-(-1)^{p+k})}{p!}s(p,k)\frac{1}{(m-p)!}\binom{m-p}{(r-p-k)/2}(-1)^{(r-p-k)/2}.$$

We remark that we now have an explicit formula corresponding to (3,5,6) in the expression of $B_{AG}(n,i)$. Since $D_{m,r}$ contains $O(m^2)$ summands, (3) contains $O(n^2)$ summands and (5,6) contain $O(n^3)$ summands. The remaining work is to find an explicit formula for the expression in (4) and to bound the number of summands. To obtain $E^{[1]}_{m,r}$, we compute

$$h^{[1]}(x,y) = \frac{xy^2 e^{x-xy^2}}{2}\left(\frac{1}{(1-xy)^{y+1}} + (1+xy)^{y-1} - \frac{1}{(1-xy)^y} - (1+xy)^y\right).$$

To obtain $E^{[2]}_{m,r}$, we compute

$$h^{[2]}(x,y) = \frac{xy^2 e^{x-xy^2}}{2}\left(\frac{xy^2+xy}{(1-xy)^{y+2}} + (xy^2-xy)(1+xy)^{y-2}\right.$$

$$+\frac{2-2xy^2}{(1-xy)^{y+1}}+(2-2xy^2)(1+xy)^{y-1}$$
$$+\frac{xy^2-2}{(1-xy)^y}+(xy^2-2)(1+xy)^y\Big).$$

We can find explicit formulas for $E^{[1]}_{m,r}$ and $E^{[2]}_{m,r}$ by obtaining formal power series expansion of $\frac{1}{(1-xy)^{y+\beta}}$ and $(1+xy)^{y-\beta}$ where $\beta=0,1$. This can be done as follows: For positive integer β, we use Eq. 7 with $w=y+\beta$ and $t=xy$ for $\frac{1}{(1-xy)^{y+\beta}}$ and $w=-y+\beta$ and $t=-xy$ in Eq. 7 for $(1+xy)^{y-\beta}$ together with the binomial theorem to expand $(y+\beta)^k$ in the first case and $(y-\beta)^k$ in the second case. The resulting formulas for $E^{[1]}_{m,r}$ and $E^{[2]}_{m,r}$ will have $O(m^3)$ summands which implies the total number of summands in the explicit formula for $B_{AG}(n,i)$ is $O(n^3)$. However, an alternate way will result in $O(m^2)$ summands in the explicit formulas for $E^{[1]}_{m,r}$ and $E^{[2]}_{m,r}$ which we now present. Notice that this will not reduce the overall complexity as the bottleneck of $O(n^3)$ summands is in (5,6). Differentiating Eq. 8 and Eq. 9 with respect to x gives

$$\frac{y^2}{(1-xy)^{y+1}}=\sum_{p\geq k\geq 0}s(p,k)py^{p+k}\frac{x^{p-1}}{p!}=\sum_{1\leq p\geq k\geq 0}s(p,k)y^{p+k}\frac{x^{p-1}}{(p-1)!} \quad (10)$$

and

$$y^2(1+xy)^{y-1} = \sum_{p\geq k\geq 0}(-1)^{p+k}ps(p,k)y^{p+k}\frac{x^{p-1}}{p!}$$
$$= \sum_{1\leq p\geq k\geq 0}(-1)^{p+k}s(p,k)y^{p+k}\frac{x^{p-1}}{(p-1)!}. \quad (11)$$

Therefore,

$$\frac{1}{(1-xy)^{y+1}}=\sum_{1\leq p\geq k\geq 0}s(p,k)y^{p+k-2}\frac{x^{p-1}}{(p-1)!} \quad (12)$$

and

$$(1+xy)^{y-1}=\sum_{1\leq p\geq k\geq 0}(-1)^{p+k}s(p,k)y^{p+k-2}\frac{x^{p-1}}{(p-1)!}. \quad (13)$$

Now we are ready.

$$E^{[1]}_{m,r} = m![x^my^r]\frac{xy^2e^{x-xy^2}}{2}\left(\frac{1}{(1-xy)^{y+1}}+(1+xy)^{y-1}\right)$$
$$-m![x^my^r]\frac{xy^2e^{x-xy^2}}{2}\left(\frac{1}{(1-xy)^{y}}+(1+xy)^{y}\right).$$

The second term is

$$
\begin{aligned}
& m![x^{m-1}y^{r-2}]\frac{e^{x-xy^2}}{2}\left(\frac{1}{(1-xy)^y}+(1+xy)^y\right) = \\
& \frac{m!}{(m-1)!}(m-1)![x^{m-1}y^{r-2}]\frac{e^{x-xy^2}}{2}\left(\frac{1}{(1-xy)^y}+(1+xy)^y\right) \\
= \;& mD_{m-1,r-2}.
\end{aligned}
$$

For the first term, we use Eq. 12 and Eq. 13 to get

$$
\frac{m!}{2}\sum_{1\le p\ge k\ge 0}\frac{(1+(-1)^{p+k})}{(p-1)!}s(p,k)[x^{m-p}y^{r-p-k}]e^{x-xy^2}.
$$

So all that remains is to extract the coefficient of e^{x-xy^2} which is given earlier. Hence

$$
\begin{aligned}
E^{[1]}_{m,r} &= \frac{m!}{2}\sum_{1\le p\ge k\ge 0}\frac{(1+(-1)^{p+k})}{(p-1)!}s(p,k)\frac{1}{(m-p)!}\binom{m-p}{(r-p-k)/2}(-1)^{(r-p-k)/2} \\
& -mD_{m-1,r-2}.
\end{aligned}
$$

We now consider $E^{[2]}_{m,r}$ by getting the power series expansion for $h^{[2]}(x,y)$. Differentiating Eq. 12 and Eq. 13 with respect to x gives

$$
\begin{aligned}
\frac{y(y+1)}{(1-xy)^{y+2}} &= \sum_{1\le p\ge k\ge 0} s(p,k)(p-1)y^{p+k-2}\frac{x^{p-2}}{(p-1)!} \\
&= \sum_{2\le p\ge k\ge 0} s(p,k)y^{p+k-2}\frac{x^{p-2}}{(p-2)!} \qquad (14)
\end{aligned}
$$

and

$$
\begin{aligned}
y(y-1)(1+xy)^{y-2} &= \sum_{1\le p\ge k\ge 0}(-1)^{p+k}s(p,k)(p-1)y^{p+k-2}\frac{x^{p-2}}{(p-1)!} \\
&= \sum_{2\le p\ge k\ge 0}(-1)^{p+k}s(p,k)y^{p+k-2}\frac{x^{p-2}}{(p-2)!}. \qquad (15)
\end{aligned}
$$

Recall that $E^{[2]}_{m,r} = m![x^m y^r]h^{[2]}(x,y)$ where

$$
\begin{aligned}
h^{[2]}(x,y) &= \frac{xy^2e^{x-xy^2}}{2}\left(\frac{xy(y+1)}{(1-xy)^{y+2}} + xy(y-1)(1+xy)^{y-2}\right. \\
&\quad +\frac{2-2xy^2}{(1-xy)^{y+1}} + (2-2xy^2)(1+xy)^{y-1} \\
&\quad \left. +\frac{xy^2-2}{(1-xy)^y} + (xy^2-2)(1+xy)^y\right).
\end{aligned}
$$

The first term is

$$T_1 = \frac{m!}{2} \sum_{2 \le p \ge k \ge 0} \frac{(1+(-1)^{p+k})}{(p-2)!} s(p,k) \frac{1}{(m-p)!} \binom{m-p}{(r-p-k)/2} (-1)^{(r-p-k)/2}$$

by using Eq. 14 and Eq. 15. For the second term, we use Eq. 12 and Eq. 13 to get

$$2\frac{m!}{2}[x^{m-1}y^{r-2}]e^{x-xy^2}\left(\frac{1}{(1-xy)^{y+1}} + (1+xy)^{y-1}\right)$$
$$-2\frac{m!}{2}[x^{m-2}y^{r-4}]e^{x-xy^2}\left(\frac{1}{(1-xy)^{y+1}} + (1+xy)^{y-1}\right).$$

Hence, the second term is

$$\begin{aligned} T_2 &= 2\frac{m!}{2} \sum_{1 \le p \ge k \ge 0} \frac{(1+(-1)^{p+k})}{(p-1)!} s(p,k) \frac{1}{(m-p)!} \\ &\quad \binom{m-p}{(r-p-k)/2} (-1)^{(r-p-k)/2} \\ &\quad -2\frac{m!}{2} \sum_{1 \le p \ge k \ge 0} \frac{(1+(-1)^{p+k})}{(p-1)!} s(p,k) \frac{1}{(m-p-1)!} \\ &\quad \binom{m-p-1}{(r-p-k-2)/2} (-1)^{(r-p-k-2)/2}. \end{aligned}$$

For the third term, we have

$$\frac{m!}{2}[x^{m-2}y^{r-4}]e^{x-xy^2}\left(\frac{1}{(1-xy)^{y}} + (1+xy)^{y}\right)$$
$$-2\frac{m!}{2}[x^{m-1}y^{r-2}]e^{x-xy^2}\left(\frac{1}{(1-xy)^{y}} + (1+xy)^{y}\right).$$

Hence, the third term is

$$T_3 = m(m-1)D_{m-2,r-4} - 2mD_{m-1,r-2}.$$

So

$$E^{[2]}_{m,r} = T_1 + T_2 + T_3.$$

3.3 Summary of the generating function approach

We have established an explicit formula for $B_{AG}(n,i)$. A closer look reveals that some of the sums are of similar nature. Here we record our result in terms of these similar sums for easy reference.

Theorem 3.1 *Let*

$$\Psi^+(m,r,\theta,\epsilon,\delta) = \sum_{\theta\le p\ge k\ge 0} \frac{(1+(-1)^{p+k})}{(p-\theta)!} s(p,k)\frac{1}{(m-p+\epsilon)!} \binom{m-p+\epsilon}{(r-p-k-\delta)/2}(-1)^{(r-p-k-\delta)/2},$$

and

$$\Psi^-(m,r) = \sum_{p\ge k\ge 0} \frac{(1-(-1)^{p+k})}{p!} s(p,k)\frac{1}{(m-p)!}\binom{m-p}{(r-p-k)/2}(-1)^{(r-p-k)/2},$$

Then

$$D_{m,r} = \frac{m!}{2}\Psi^+(m,r,0,0,0),$$

$$F_{m,r} = \frac{m!}{2}\Psi^-(m,r),$$

$$E^{[1]}_{m,r} = \frac{m!}{2}\Psi^+(m,r,1,0,0) - \frac{m!}{2}\Psi^+(m-1,r-2,0,0,0),$$

and

$$E^{[2]}_{m,r} = \frac{m!}{2}\Psi^+(m,r,2,0,0) + m!(\Psi^+(m,r,1,0,0) - \Psi^+(m,r,1,-1,2)) + \frac{m!}{2}(\Psi^+(m-2,r-4,0,0,0) - 2\Psi^+(m-1,r-2,0,0,0)).$$

Moreover,

$$\begin{aligned} B_{AG}(n,i) = {} & D_{n-2,i} + 2D_{n-1,i+2} - 2D_{n-2,i+2} + (n-2)(n-3)D_{n-4,i-2} \\ & +2(n-2)E^{[1]}_{n-3,i} + E^{[2]}_{n-2,i+2} \\ & - \sum_{j=2,odd}^{n-2}\binom{n-2}{j}(j+1)!D_{n-j-2,i-j+1} \\ & - \sum_{j=2,even}^{n-2}\binom{n-2}{j}(j+1)!F_{n-j-2,i-j+1} \\ & + \sum_{j=0,odd}^{n-2}\binom{n-2}{j}(j+1)!D_{n-j-2,i-j} \\ & + \sum_{j=0,even}^{n-2}\binom{n-2}{j}(j+1)!F_{n-j-2,i-j}. \end{aligned}$$

Remarks:

1. We count the number of terms in the explicit formula given in Theorem 3.1. The formulas for Ψ^+ and Ψ^- are double sum with the number of summands bounded by $O(m^2)$. So the numbers of summands in $D_{m,r}$, $F_{m,r}$, $E^{[1]}_{m,r}$ and $E^{[2]}_{m,r}$ are $O(m^2)$. Therefore, the number of terms in $B_{AG}(n,i)$ is $O(n^3)$.

2. The bottleneck for the $O(n^3)$ analysis is the four sums of $D_{m,r}$'s and $F_{m,r}$'s, that is, those corresponding to (5) and (6). Since the generating function of $D_{m,r}$ is $\frac{e^{x-xy^2}}{2}\left(\frac{1}{(1-xy)^y}-(1+xy)^y\right)$, it seems unlikely that we can further reduce the double sum, or at least we are unable to. This is not to say that the four sums will not reduce to something simpler but rather we cannot deduce any possible reduction through cancellation. For example, these four sums can be combined so that if there are $O(m)$ expressions for $D_{m,r}-D_{m,r-1}$ and $F_{m,r}-F_{m,r-1}$, then the overall complexity will reduce to $O(n^2)$ summands.

3. The bottleneck is from the four sums and they come directly from the combinatorial setup, not the generating function. Given the seemingly difficult task of breaking the bottleneck from the generating function side, one may want to pursue a completely different combinatorial setup before applying the generating function technique. We did follow this approach to obtain two additional explicit formulas but the technique involves stepping beyond the realm of alternating group graphs and requires additional efforts. Although the new formulas are more aesthetically pleasing as they *look* simpler, the complexity did not improve since they are $O(n^3)$ and $O(n^4)$ respectively. So we will not present that approach here.

4 Conclusion

We have discussed two different approaches to deriving explicit formulas for the important parameter of surface area for the alternating group graph. The surface area of the split-star will be derived indirectly later (see below). The formulas from both Theorem 2.1 and Theorem 3.1 have been verified with computer programs for small values of n against results obtained by the breadth first search. For example, the surface area sequence for AG_5 is $(1,6,18,27,8)$. Additional data are given in Table 1. But perhaps more importantly, these two techniques are general enough that they have the potential to be used for obtaining such formulas for other networks, especially the many symmetric ones described in [16] that are based on Cayley graphs of permutation groups.

Table 1: Sample data for $B_{AG}(n, i)$

	i									
n	0	1	2	3	4	5	6	7	8	9
2	1	0	0	0	0	0	0	0	0	0
3	1	2	0	0	0	0	0	0	0	0
4	1	4	6	1	0	0	0	0	0	0
5	1	6	18	27	8	0	0	0	0	0
6	1	8	36	102	152	58	3	0	0	0
7	1	10	60	250	680	1,010	459	50	0	0
8	1	12	90	495	1,960	5,190	7,749	4,008	640	15

One may wonder why two vastly different methods are given considering that one approach (the direct counting) is much simpler than the other. The first method has the beauty of simplicity without the need of any advanced mathematical techniques. Moreover, the explicit formula obtained is much simpler than the second one. Although the generating function method requires more space, all the work in Sections 3.2 and 3.3 are just algebra. The crucial part is the encoding in Section 3.1, which is just as short as the first method. Nevertheless, the resulting formula is still more complicated than the one obtained by the first method. So why is the second method presented? Part of the answer is that the number of summands is actually smaller by a factor of n. Furthermore, researchers in this area would like to know the surface area for popular classes of interconnection networks. Up to this point, only the solutions to the (n, k)-star graph [28] (which include the simpler star graph) are published. We believe both methods deserve to be presented as it is unclear which method (or even both) can be extended to find surface areas of other large classes of interconnection networks. Moreover, the generating function approach has a higher potential to be used to carry out asymptotic analysis for the results so obtained.

We now briefly discuss the derivation of the surface area of another potential interconnection topology, the split-star, as suggested in [3]. For $n \geq 2$, a *split-star graph,* denoted as S_n^2, is defined as follows: *for any* $u, v \in S_n$, $(u, v) \in E$ *iff for some* $i, j \in [1, n]$, v *can be obtained from* u *by applying a transposition* $(1, 2)$; *or by applying a* 3-rotation, $(1, 2, k), k \in [3, n]$. We can follow either one of the two approaches discussed to derive an explicit formula for the surface area of the split-star graph. On the other hand, it turns out that a split-star graph is closely related to the alternating group graph in the sense that a split-star graph consists of two copies of alternating group graphs: one consisting of even permutations in S_n, another (an isomorphic copy of AG_n) consisting of odd permutations, together with a perfect matching between each even permutation and an odd one. Therefore, to route an even permutation to e in S_n^2, we simply stay in the alternating graph AG_n consisting of even permutations; otherwise, we first follow an edge corresponding to the $(1,2)$ transposition to go to an even permutation in the aforementioned AG_n and then continue the routing within this latter graph. Hence, for all $n \geq 2$ and $k \in [1, D(S_n^2(n))]$, where $D(S_n^2(n))$ is given as $\lfloor \frac{3n}{2} \rfloor - 2$ in [3], we have

$$B_{S_n^2}(n,i) = B_{AG}(n,i) + B_{AG}(n,i-1).$$

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