On the Spectrum of graphs with pockets

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Abstract

Let $S_{r,l}, H_v$ be simple connected graphs. Let $S_{r,l}$ be a generalized star on $rl+1$ vertices with central vertex v. Let H_v be a graph of order m, with a specified vertex v of degree $m-1$. Let $G = G[r, l, H_v]$ be the graph obtained by taking one copy of $S_{r,l}$ and one copy of H_v, and then attaching the vertex v of H_v to each vertex of $S_{r,l}$, except the central vertex of $S_{r,l}$. In this paper, we shall give the adjacency (Laplacian, signless Laplacian) spectrum of G in terms of their corresponding spectrum of $S_{r,l}$ and H_v and further extended to the adjacency(Laplacian, signless Laplacian) characteristic polynomial of the general graphs.

Key words: Adjacency matrix; Laplacian matrix; Signless Laplacian matrix; Spectrum; Corona

AMS Subject Classifications: 05C50; 15A18

1 Introduction

Throughout this article, all graphs considered are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$. The adjacency matrix of G, denoted by $A(G)$, is an $n \times n$ symmetric matrix such that $a_{ij} = 1$ if vertices v_i and v_j are adjacent and 0 otherwise. Let $d_i = d_G(v_i)$ be the degree of vertex v_i in G and $D(G) = diag(d_1, d_2, ..., d_n)$ be the diagonal matrix of vertex degrees. The Laplacian matrix and the signless Laplacian matrix of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively.

Given an $n \times n$ matrix M, denoted by

$$f(M;\lambda) = det(\lambda I_n - M)$$

the characteristic polynomial of M, where I_n is the identity matrix of size n. The adjacency eigenvalues of G, defined as $\sigma(G) = (\lambda_1(G), \lambda_2(G), ..., \lambda_n(G))$, where $\lambda_1(G) \leq \lambda_2(G) \leq ... \leq \lambda_n(G)$ are called the A-spectrum of G. Similarly, the eigenvalue of $L(G)$ and $Q(G)$, denoted by $I(G) = (\mu_1(G), \mu_2(G), ..., \mu_n(G))$ and $S(G) = (q_1(G), q_2(G), ..., q_n(G))$, respectively, are called the $L-$ spectrum and $Q-$ spectrum of G accordingly. It is well known that graph spectrum store a lot of structural information about a graph; see [3] and the references therein.

For certain families of graphs it is possible to identify a graph by looking at the spectrum. More generally, this is not possible. In some cases, the spectrum of a relatively larger graph can be described in terms of the spectrum of some smaller (and simpler) graphs using some simple graph operations. There are results that discuss the spectrum of graphs obtained by means of some operations on graphs like the disjoint union, the join of graphs, deleting or inserting an edge, the complement, etc. See the survey article by Mohar [4]. In [6], the following new graph operations are introduced.

A star on n vertices, denoted by S_n, is a tree in which there is a vertex of degree $n-1$. Let T be a tree. For a vertex w in T, a branch at w of T is a component of $T - w$. If w is an identified vertex of T of degree r, we identify the neighbors of w in T as $w_1, w_2, ..., w_r$, and we denote the branch of T resulting from the deletion of w and containing w_i by T_i, $i = 1, 2, ..., r$.

Definition 1 A generalized star is a tree T having at most one vertex of degree greater than 2. Suppose that we have a vertex w of degree r of a generalized star T, whose neighbors $w_1, w_2, ..., w_r$ are pendent vertices of their branches $T_1, T_2, ..., T_r$, respectively, and each of these branches is a path. Then we refer to w as a central vertex of T. If T_i is a path of order p_i,$i = 1, 2, ..., r$, then the generalized star T is denoted by $S[p_1, p_2, ..., p_r]$. If $p_1 = p_2 = ... = p_r = l$, then we denote the generalized star by $S_{r.l}$.

Example 1: The trees $S[3, 2, 3, 3]$ and $S_{5,3}$ in Figure 1 are generalized stars, with central vertices w and $\bar{w}$, respectively.

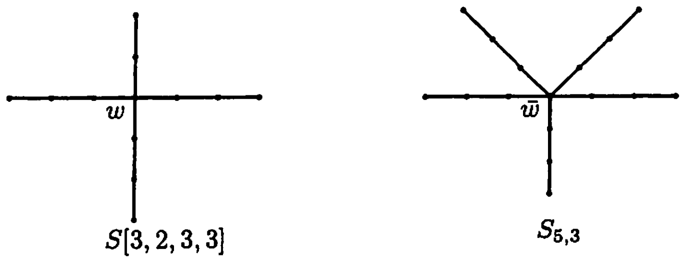

Figure 1. *Generalized stars.*

Definition 2 Let F,H be connected graphs. v be a specified vertex of H_v and $u_1, u_2, ..., u_k \in F$. Let the graph $G = G[F, u_1, u_2, ..., u_k, H_v]$ be the graph obtained by taking one copy of F and k copies of H_v, and then attaching the ith copy of H_v to the vertex u_i,$i = 1, 2, ..., k$, at the vertex v of H_v(identify u_i with the vertex v of the ith copy). Then the copies of the graph H_v that are attached to the vertices u_i,$i = 1, 2, ..., k$ are referred to as pockets, and we describe G as a graph with k pockets.

Example 2: Let F, H_v of order 4, 5, respectively, in Figure 2. The vertex v of H_v has degree 4. The $G = G[F, u, H_v]$ is shown in Figure 2.

In [6], the Laplacian spectrum of graphs with pockets were computed in terms of the Laplacian spectrum of F and H_v. Let $G = G[r, l, H_v]$ be the graph obtained by taking one copy of $S_{r,l}$ and one copy of H_v, and then attaching the vertex v of H_v to each vertex of $S_{r,l}$, except the central vertex of $S_{r,l}$. In this paper, except $2l$ eigenvalues, we describe all other eigenvalues $G[r, l, H_v]$ using the adjacency(Laplacian, signless Laplacian) spectrum of $S_{r,l}$ and H_v, respectively. In a more general case, we describe the characteristic(Laplacian, signless Laplacian) polynomial of $G[F, u_1, u_2, ..., u_k, H_v]$.

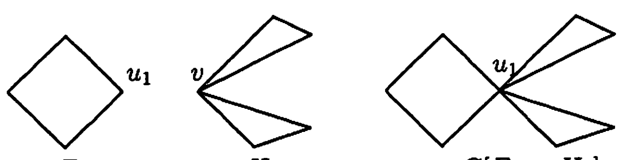

Figure 2. *Graph F and H_v, $G[F, u_1, H_v]$.*

2 Preliminaries

In this section, we determine the characteristic polynomials of graphs with the help of the *coronal* of a matrix. The M – *coronal* $T_M(\lambda)$ of an

$n \times n$ matrix M is defined [2, 7] to be the sum of the entries of the matrix $(\lambda I_n - M)^{-1}$, that is

$$T_M(\lambda) = \mathbf{1}_n{}^T(\lambda I_n - M)^{-1}\mathbf{1}_n.$$

where $\mathbf{1}_n$ denotes the column vector of dimension n with all the entries equal one.

It is well known [2, *Proposition*2] that, From Cui and Tian, [2], if M is an $n \times n$ matrix with each row sum equal to a constant t, then

$$T_M(\lambda) = \frac{n}{\lambda - t} \tag{1}$$

In particular, since for any graph G with n vertices, each row sum of $L(G)$ is equal to 0, we have$T_{L(G)}(\lambda) = \frac{n}{\lambda}$

$$T_{L(G)}(\lambda) = \frac{n}{\lambda}. \tag{2}$$

The Kronecker product of matrices $A = (a_{ij})$ and B, denoted by $A \otimes B$, is defined to be the partition matrix $(a_{ij}B)$. See [1]. In cases where each multiplication makes sense, we have

$$M_1M_2 \otimes M_3M_4 = (M_1 \otimes M_3)(M_2 \otimes M_4)$$

This implies that for nonsingular matrix M and N, $(M \otimes N)^{-1} = M^{-1} \otimes N^{-1}$, Recall also that for square matrices M and N of order k and s, respectively. $det(M \otimes N) = (detM)^k(detN)^s$.

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs on disjoint sets of m and n vertices, respectively, their union is the graph $G_1 \cup G_2$, and their join $G_1 \vee G_2$ is the graph obtained from the union $G_1 \cup G_2$ by adding all edges between a vertex of G_1 and a vertex of G_2.

Lemma 1 **([5])** Let G_1 and G_2 be graphs on disjoint sets of m, n vertices, respectively, and $G = G_1 \vee G_2$. Let $I(G_1) = (\mu_1, \mu_2, ..., \mu_m)$ and $I(G_2) = (\gamma_1, \gamma_2, ..., \gamma_n)$. Then

$$I(G) = (0, n + \mu_2, ..., n + \mu_m, m + \gamma_2, ..., m + \gamma_n, m + n).$$

2.1 The adjacency spectrum of graphs with pockets

In this section, we describe the adjacency characteristic polynomial of $G[r, l, H_v]$ and all other adjacency eigenvalues of $G[r, l, H_v]$ except $2l + 1$ eigenvalues using the adjacency spectrum of $S_{r,l}$ and H_v, respectively.

Proposition 1 Let $G = G[r, l, H_v]$ and $T_{A(H)}(\lambda)$ be the A-coronal of H. Then the characteristic polynomial of $A(G)$ is

$$f_{A(G)}(\lambda) = (f_{A(H)}(\lambda))^{rl} det(\lambda I_{rl+1} - T_{A(H)}(\lambda) \begin{pmatrix} 0 & 0^T \\ 0 & I_{rl} \end{pmatrix} - A(S_{r,l})).$$

Proof Since v is of degree $m-1$, H_v can be written as $H_v = \{v\} \vee H$, where H is the graph obtained from H_v, after deleting the vertex v and the edges incident to it, the adjacency characteristic polynomial of G is given by

$$\begin{aligned} f_{A(G)}(\lambda) &= det \begin{pmatrix} \lambda I_{rl+1} - A(S_{r,l}) & -\mathbf{1}_{m-1}^T \otimes \begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix} \\ -\mathbf{1}_{m-1} \otimes \begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}^T & \lambda I_{rl(m-1)} - A(H) \otimes I_{rl} \end{pmatrix} \\ &= det \begin{pmatrix} \lambda I_{rl+1} - A(S_{r,l}) & -\mathbf{1}_{m-1}^T \otimes \begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix} \\ -\mathbf{1}_{m-1} \otimes \begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}^T & (\lambda I_{m-1} - A(H)) \otimes I_{rl} \end{pmatrix} \\ &= det((\lambda I_{m-1} - A(H)) \otimes I_{rl}) \times det B, \end{aligned}$$

where

$$\begin{aligned} B &= \lambda I_{rl+1} - A(S_{r,l}) - (\mathbf{1}_{m-1}^T \otimes \begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}) \\ &\quad ((\lambda I_{m-1} - A(H)) \otimes I_{rl})^{-1} (\mathbf{1}_{m-1} \otimes \begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}^T) \end{aligned}$$

is the Schur complement of $((\lambda I_{m-1} - A(H)) \otimes I_{rl})$. Thus, the result follows from

$$det(\lambda I_{m-1} - A(H) \otimes I_{rl}) = (det(\lambda I_{m-1} - A(H)))^{rl}$$

and

$$\begin{aligned} det B &= det(\lambda I_{rl+1} - A(S_{r,l}) - (\mathbf{1}_{m-1}^T \otimes \begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}) \\ &\quad ((\lambda I_{m-1} - A(H)) \otimes I_{rl})^{-1} (\mathbf{1}_{m-1} \otimes \begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}^T)) \\ &= det(\lambda I_{rl+1} - A(S_{r,l}) - (\mathbf{1}_{m-1}^T (\lambda I_{m-1} - A(H))^{-1} \mathbf{1}_{m-1}) \\ &\quad \otimes (\begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix} I_{rl} \begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}^T)) \\ &= det(\lambda I_{rl+1} - T_{A(H)}(\lambda) \begin{pmatrix} 0 & 0^T \\ 0 & I_{rl} \end{pmatrix} - A(S_{r,l})). \end{aligned}$$

Hence, the adjacency characteristic polynomial of G is

$$f_{A(G)}(\lambda) = (f_{A(H)}(\lambda))^{rl} det(\lambda I_{rl+1} - T_{A(H)}(\lambda) \begin{pmatrix} 0 & 0^T \\ 0 & I_{rl} \end{pmatrix} - A(S_{r,l})).$$

The proof is completed. □

Proposition 1 enables us to compute the adjacency spectrum except $2l+1$ eigenvalues when H_v is a complete graph K_m.

Theorem 2 Let $G = G[r, l, K_m]$. Suppose that $\sigma(S_{r,l}) = (\eta_1, \eta_2, ..., \eta_{rl+1})$. Then

(i) $-1 \in \sigma(G)$ with multiplicity $(m-2)rl$,

(ii) The eigenvalues

$$\frac{(m-2+\eta_i) \pm \sqrt{(m-\eta_i)^2 + 4\eta_i}}{2}$$

with multiplicity $(r-1)$ for each eigenvalue $\eta_i (i = 2, r+2, ..., (l-1)r+2)$ of $S(r, l)$.

Proof Since K_{m-1} is $(m-2)$-regular with $m-1$ vertices, (1) implies that

$$T_{A(K_{m-1})}(\lambda) = \frac{m-1}{\lambda - (m-2)}.$$

The only pole of $T_{A(K_{m-1})}(\lambda)$ is $\lambda = m-2$. By Proposition 1, -1 is an eigenvalue of G with multiplicity of $(m-2)rl$. The remaining eigenvalues are obtained by solving $\lambda - \frac{m-1}{\lambda-(m-2)} = \eta_i$ for each $i = 2, r+2, ..., (l-1)r+2$, and this yields the eigenvalue in (ii).

Note we obtain $mrl - 2l$ eigenvalues of G. The other $(mrl+1) - (mrl - 2l) = 2l+1$ eigenvalues of G must come from the only pole $\lambda = m-2$ of $T_{A(H)}(\lambda)$. This completes the proof of theorem. □

Next, we consider a more general case. Let $G = G[F, u_1, u_2, ..., u_k, H_v]$ be the graph as defined in Definition 2. Then we describe the characteristic polynomial of G using the characteristic polynomial of F and H_v, when v has degree $m-1$.

Theorem 3 Let $G = G[F, u_1, u_2, ..., u_k, H_v]$. Then

$$f_{A(G)}(\lambda) = (f_{A(H)}(\lambda))^k det(\lambda I_n - T_{A(H)}(\lambda) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix} - A(F)).$$

Proof With a proper labeling of vertices, the adjacency matrix of G can be written as

$$A(G)=\begin{pmatrix} A(F) & \begin{pmatrix} I_k\otimes\mathbf{1}_{m-1}^T \\ 0 \end{pmatrix} \\ \begin{pmatrix} I_k\otimes\mathbf{1}_{m-1} & 0 \end{pmatrix} & I_k\otimes A(H) \end{pmatrix}$$

The result refines the arguments used to prove Proposition 1. □

2.2 The Laplacian spectrum of graphs with pockets

In this section, we give the Laplacian characteristic polynomial of $G[r,l,H_v]$ and all other Laplacian eigenvalues of $G[r,l,H_v]$ except $2l$ eigenvalues using the Laplacian spectrum of $S_{r,l}$ and H_v, respectively.

Proposition 4 Let $G=G[r,l,H_v]$ and $T_{L(H)}(\lambda)$ be the L-coronal of H. Then the characteristic polynomial of $L(G)$ is

$$f_{L(G)}(\lambda) = (f_{L(H)}(\lambda-1))^{rl}det(\lambda I_{rl+1}-((m-1)+T_{L(H)}(\lambda-1)) \begin{pmatrix} 0 & 0^T \\ 0 & I_{rl} \end{pmatrix} - L(S_{r,l})).$$

Proof Since v is of degree $m-1$, H_v can be written as $H_v=\{v\}\vee H$, where H is the graph obtained from H_v, after deleting the vertex v and the edges incident to it, the Laplacian matrix of G can be written as

$$L(G)=\begin{pmatrix} L(S_{r,l})+\begin{pmatrix} 0 & 0^T \\ 0 & (m-1)I_{rl} \end{pmatrix} & -\mathbf{1}_{m-1}^T\otimes\begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix} \\ -\mathbf{1}_{m-1}\otimes\begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}^T & (L(H)+I_{m-1})\otimes I_{rl} \end{pmatrix},$$

let $C=\begin{pmatrix} 0 & 0^T \\ 0 & (m-1)I_{rl} \end{pmatrix}$, $D=\begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}$, and $F=(L(H)+I_{m-1})\otimes I_{rl}$. Thus the Laplacian characteristic polynomial of G is given by

$$\begin{aligned} f_{L(G)}(\lambda) &= det\begin{pmatrix} \lambda I_{rl+1}-L(S_{r,l})-C & \mathbf{1}_{m-1}^T\otimes D \\ \mathbf{1}_{m-1}\otimes D^T & \lambda I_{rl(m-1)}-F \end{pmatrix} \\ &= det\begin{pmatrix} \lambda I_{rl+1}-L(S_{r,l})-C & \mathbf{1}_{m-1}^T\otimes D \\ \mathbf{1}_{m-1}\otimes D^T & ((\lambda-1)I_{m-1}-L(H))\otimes I_{rl} \end{pmatrix} \\ &= det(((\lambda-1)I_{m-1}-L(H))\otimes I_{rl})\times detB, \end{aligned}$$

where

$$B = \lambda I_{rl+1}-L(S_{r,l})-\begin{pmatrix} 0 & 0^T \\ 0 & (m-1)I_{rl} \end{pmatrix}(\mathbf{1}_{m-1}^T\otimes\begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}) (((\lambda-1)I_{m-1}-L(H))\otimes I_{rl})^{-1}(\mathbf{1}_{m-1}\otimes\begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix})^T$$

is the Schur complement of $((\lambda-1)I_{m-1}-L(H))\otimes I_{rl})$. The result follows from

$$det((\lambda-1)I_{m-1}-L(H))\otimes I_{rl})=(det(\lambda-1)I_{m-1}-L(H)))^{rl}$$

and

$$\begin{aligned} detB &= det((\lambda I_{rl+1}-L(S_{r,l})-\begin{pmatrix} 0 & 0^T \\ 0 & (m-1)I_{rl} \end{pmatrix}-(\mathbf{1}_{m-1}^T\otimes \\ &\begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix})(\lambda I_{m-1}-(L(H)+I_{m-1})\otimes I_{rl})^{-1} \\ &(\mathbf{1}_{m-1}\otimes\begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}^T) \\ &= det((\lambda I_{rl+1}-L(S_{r,l})-\begin{pmatrix} 0 & 0^T \\ 0 & (m-1)I_{rl} \end{pmatrix}-(\mathbf{1}_{m-1}^T\otimes \\ &\begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix})((\lambda-1)I_{m-1}-L(H))\otimes I_{rl})^{-1}(\mathbf{1}_{m-1}\otimes\begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}^T) \\ &= det(\lambda I_{rl+1}-L(S_{r,l})-(m-1)\begin{pmatrix} 0 & 0^T \\ 0 & I_{rl} \end{pmatrix}-T_{L(H)}(\lambda-1) \\ &\begin{pmatrix} 0 & 0 \\ 0 & I_{rl} \end{pmatrix}) \\ &= det(\lambda I_{rl+1}-((m-1)+T_{L(H)}(\lambda-1))\begin{pmatrix} 0 & 0 \\ 0 & I_{rl} \end{pmatrix})-L(S_{r,l})). \end{aligned}$$

Hence, the Laplacian characteristic polynomial of G is

$$\begin{aligned} f_{L(G)}(\lambda) &= (f_{L(H)}(\lambda-1))^{rl}det(\lambda I_{rl+1}-((m-1)+T_{L(H)}(\lambda-1)) \\ &\begin{pmatrix} 0 & 0^T \\ 0 & I_{rl} \end{pmatrix}-L(S_{r,l})). \end{aligned}$$

The proof is completed. □

The following Theorem 5, first addressed in [6], is an immediate consequence of Proposition 4. We remark that here our method is straightforward and different from that of Theorem 4 in [6]

Theorem 5 Let $G=G[r,l,H_v]$. Suppose that $I(H_v)=(0=\gamma_1,\gamma_2,...,\gamma_m)$ and $I(S_{r,l})=(0=\delta_1,\delta_2,...,\delta_{rl+1})$. Then

(i) $0\in I(G)$ with multiplicity 1,

(ii) $\gamma_j\in I(G)$ with multiplicity rl for every eigenvalue $j=2,...,m-1$ and

(iii) The eigenvalues

$$\frac{\delta_i+m\pm\sqrt{(\delta_i+m)^2-4\delta_i}}{2}$$

with multiplicity $(r-1)$ for each eigenvalue $\delta_i(i=2,r+2,...,(l-1)r+2)$ of $L(S_{r,l})$.

Proof Since $H_v = H \vee \{v\}$, using Lemma 1, we have the eigenvalues of H are $-1, \gamma_2 - 1, ..., \gamma_m - 1$. By Proposition 4, $\gamma_j (j = 2, ..., m-1)$ are the eigenvalues of G repeated rl times, 0 is an eigenvalue of G with multiplicity 1. Since the sum of all entries on every row of Laplacian matrix is zero, (2) implies that

$$T_{L(H)}(\lambda - 1) = \frac{m-1}{\lambda - 1}$$

The only pole of $T_{L(H)}(\lambda - 1)$ is $\lambda = 1$, so the remaining eigenvalues of G are obtained by solving $\lambda - (m-1) - \frac{m-1}{\lambda-1} = \delta_i$ for each $i = 2, r+2, ..., (l-1)r+2$ and this yields the eigenvalues in (iii).

Note we obtain $mrl + 1 - 2l$ eigenvalues of G. The other $(mrl+1) - (mrl + 1 - 2l) = 2l$ eigenvalues of G must come from the only pole $\lambda = 1$ of $T_{L(H)}(\lambda - 1)$. This completes the proof of theorem. □

Next, we consider a more general case where $G = G[F, u_1, u_2, ..., u_k, H_v]$. Then we describe the Laplacian characteristic polynomial of G using the Laplacian characteristic polynomial of F and H_v, when v has degree $m-1$.

Theorem 6 Let $G = G[F, u_1, u_2, ..., u_k, H_v]$. Suppose that $I(H_v) = (0 = \gamma_1, \gamma_2, ..., \gamma_m)$ and $I(F) = (0 = \alpha_1, \alpha_2, ..., \alpha_{n+k})$, then

$$f_{L(G)}(\lambda) = (f_{L(H)}(\lambda - 1))^k det(\lambda I_n - ((m-1) + T_{L(H)}(\lambda - 1)) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix} - L(F)).$$

Proof With a proper labeling of vertices, the Laplacian matrix of G can be written as

$$L(G) = \begin{pmatrix} L(F) + \begin{pmatrix} (m-1)I_k & 0^T \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} I_k \otimes \text{-}\mathbf{1}^T \\ 0 \end{pmatrix} \\ \begin{pmatrix} I_k \otimes \text{-}\mathbf{1} & 0 \end{pmatrix} & I_k \otimes (L(H) + I_{m-1}) \end{pmatrix}$$

The result refines the arguments used to prove Theorem 5. The proof is completed. □

2.3 The signless Laplacian spectrum of graphs with pockets

In this section, we give the signless Laplacian characteristic polynomial of $G[r, l, H_v]$ and all other signless Laplacian eigenvalues of $G[r, l, H_v]$ except $2l + 1$ eigenvalues using the signless Laplacian spectrum of $S_{r,l}$ and H_v, respectively.

Proposition 7 Let $G = G[r, l, H_v]$ and $T_{Q(H)}(\lambda)$ be the Q-coronal of H. Then the characteristic polynomial of $Q(G)$ is

$$f_{Q(G)}(\lambda) = (f_{Q(H)}(\lambda-1))^{rl} det(\lambda I_{rl+1} - ((m-1) + T_{Q(H)}(\lambda-1)) \begin{pmatrix} 0 & 0^T \\ 0 & I_{rl} \end{pmatrix} - Q(S_{r,l})).$$

Proof Since v is of degree $m-1$, H_v can be written as $H_v = \{v\} \vee H$, where H is the graph obtained from H_v, after deleting the vertex v and the edges incident to it. Thus the signless laplacian matrix of G can be written as

$$Q(G) = \begin{pmatrix} Q(S_{r,l}) + \begin{pmatrix} 0 & 0^T \\ 0 & (m-1)I_{rl} \end{pmatrix} & \mathbf{1}_{m-1}^T \otimes \begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix} \\ \mathbf{1}_{m-1} \otimes \begin{pmatrix} 0^T \\ I_{rl} \end{pmatrix}^T & (Q(H) + I_{m-1}) \otimes I_{rl} \end{pmatrix},$$

The rest of the proof is similar to that of Proposition 4 and hence we omit details. □

Theorem 8 Let $G = G[r, l, K_m]$. Suppose that $S(S_{r,l}) = (\eta_1, \eta_2, ..., \eta_{rl+1})$. Then

(i) $2(m-2)$ with multiplicity rl and
(ii) The eigenvalues

$$\frac{(3m + \eta_i - 4) \pm \sqrt{(m - \eta_i)^2 + 4\eta_i}}{2}$$

with multiplicity $(r-1)$ for each eigenvalue $\eta_i (j = 2, r+2, ..., (l-1)r+2)$ of $Q(S_{r,l})$.

Proof Since K_{m-1} is $(m-2)$-regular with $m-1$ vertices, by (1) we have

$$T_{Q(H)}(\lambda - 1) = \frac{m-1}{\lambda - 1 - 2(m-2)}$$

The only pole of $T_{Q(H)}(\lambda)$ is $\lambda = 1+2(m-2)$. By Proposition 7, $2(m-2)$ is an eigenvalue of G with multiplicity of rl. The remaining eigenvalues are obtained by solving $\lambda - (m-1) - \frac{m-1}{\lambda-1-2(m-2)} = \eta_i$ for each $i = 2, r+2, ..., (l-1)r+2$, and this yields the eigenvalue in (ii).

Note we obtain $mrl - 2l$ eigenvalues of G. The other $(mrl+1) - (mrl - 2l) = 2l+1$ eigenvalues of G must come from the only pole $\lambda = 1+2(m-2)$ of $T_{Q(H)(\lambda)}$. This completes the proof of theorem. □

Next, we describe the signless Laplacian characteristic polynomial of G using the signless Laplacian characteristic polynomial of F and H_v, when v has degree $m-1$.

Theorem 9 Let $G = G[F, u_1, u_2, ..., u_k, H_v]$. Then

$$f_{Q(G)}(\lambda) = (f_{Q(H)}(\lambda-1))^k det(\lambda I_n - ((m-1) + T_{Q(H)}(\lambda-1)) \begin{pmatrix} I_k & 0^T \\ 0 & 0 \end{pmatrix} - Q(F)).$$

Proof With a proper labeling of vertices, the signless Laplacian matrix of G can be written as

$$L(G) = \begin{pmatrix} Q(F) + \begin{pmatrix} (m-1)I_k & 0^T \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} I_k \otimes \mathbf{1}^T \\ 0 \end{pmatrix} \\ \begin{pmatrix} I_k \otimes \mathbf{1} & 0 \end{pmatrix} & I_k \otimes (Q(H) + I_{m-1}) \end{pmatrix}$$

The result refines the arguments used to prove the Proposition 7. □

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