On Mean Graphs

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Abstract:

We present mean and non mean graphs of order≤6, and give an upper bound for the number of edges of a graph with certain number of vertices to be a mean graph, and we show that the maximum vertex degree could be found in mean graphs depending on the number of edges. Also, we construct families of mean graphs depending on other mean and non mean graphs.

Introduction:

Somasundaram and Ponraj have introduced the notion of mean labelings of graphs, and they gave results concerning it in several papers [3,4,5,6,7]. A graph $G = (V(G), E(G))$ is said to be a mean graph, if there is an injective function f from $V(G)$ to $\{0, 1, 2, \ldots, |E(G)|\}$, such that when each edge uv is labeled with $\lceil (f(u) + f(v))/2 \rceil$, where $\lceil x \rceil$ denotes the smallest integer not less than x, then the resulting edge labels are distinct.
Gallian recorded these results in his well-known survey [1].

Here we extend the labeling of mean graphs given by Somasundaram and Ponraj to the labeling of mean graphs of order ≤6. We give an upper bound for the number of edges of a graph of certain number of vertices to be a mean graph, and we show that the maximum vertex degree in a mean graph could be found knowing the number of its edges. Also, we discuss unions of mean graphs and give some families of mean graphs.

All our graphs are finite and simple. The table of the graphs of order less than or equal to 6 is taken from [2]

Results:

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1. All mean graphs of order ≤ 6

Lemma 1.1 The following labeled graphs can't be subgraphs of any mean graph:

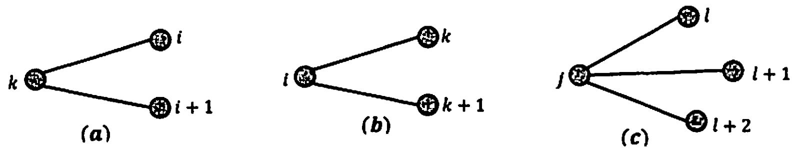


Figure 1.1

Where k is even, i is odd and j, l are arbitrary.

Proof: straight forward, since otherwise we have repeated edge labels.

The following lemmas are straight forward:

Lemma 1.2 In any connected mean graph there must be one of the following labeled paths to obtain the edge labels 1,2:

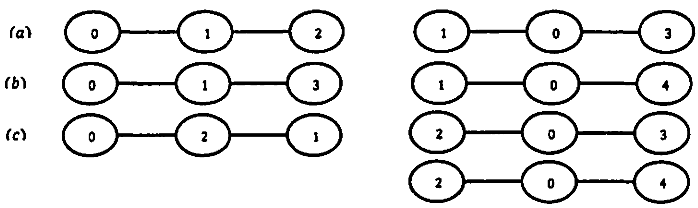


Figure 1.2

Lemma 1.3 In any connected mean graph $G(p, q)$ there must be one of the following labeled paths to obtain the edge labels $q, q-1$:

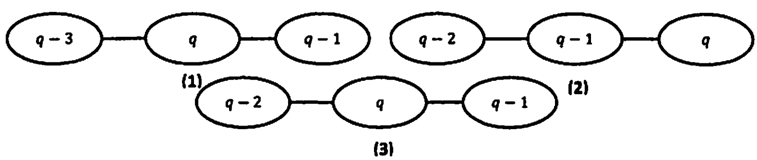


Figure 1.3

Lemma 1.4 If $G(p, q)$ is not a mean graph then $G(p, q) \cup \overline{K}_n$ is not a mean graph.

Lemma 1.5 If $G(p, q)$ is a mean graph then $G(p, q) \cup \overline{K}_{q+1-p}$ is a mean graph too.

Let H be the following graph:

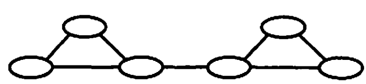


Figure 1.4

Theorem 1.1: Among all graphs $G(p < 7, q + 1 \geq p)$ the following graphs are not mean graphs:

1) $G(6, q \geq 12), G(5, q \geq 8), G(5, q \geq 8) \cup K_1$.
2) $G(6,11), G(6,10)$ that do not contain the graph H as a subgraph.
3) $K_{3,3}, K_{1,4}, K_{1,5}, K_4, K_{1,4} \cup K_1, K_4 \cup K_1, K_4 \cup \overline{K}_2$.
4) and the following graphs:

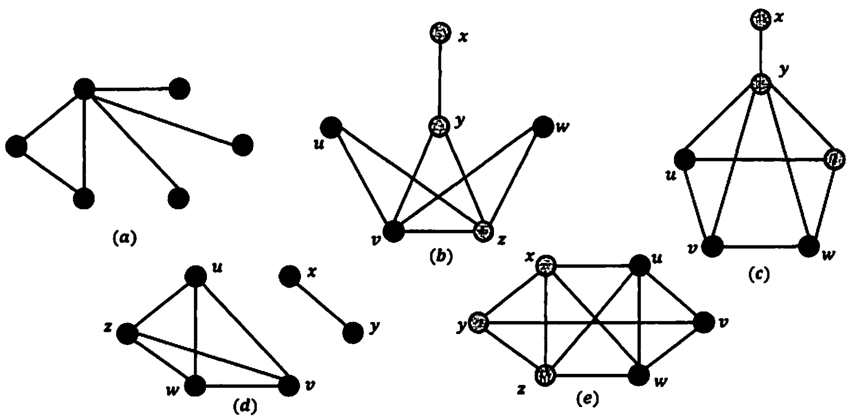
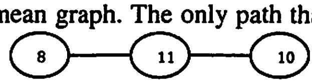
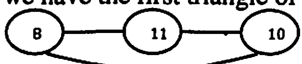


Figure 1.5

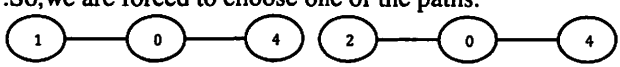
Proof:

1) The graphs $G(6, q \geq 12)$, $G(5, q \geq 8)$, are excluded by *Theorem2.1*, *Theorem2.2*, for the graphs $G(5, q \geq 8) \cup K_1$, consider *Lemma 1.4*.
2) Assume that one of the graphs $G(6,11)$ is a mean graph. The only path that works in those of *Lemma 1.3* is the first one 

, otherwise the minimum vertex label is 9, and that means the edge label 4 cannot exist using any path of those of *Lemma1.2* (*realize that we need at most the vertex label 8 to obtain the edge label 4*). Now if the ends of this path are adjacent, then we have the first triangle of the graph H:



If these ends were not adjacent, then there must exist a vertex label at least 6 to obtain the edge label 9, but no vertex of the paths of *Lemma 1.2* is labeled 6 i.e. we have certainly the mentioned triangle. To obtain the edge label 8 we need a vertex of label at least 4 which is adjacent to the vertex labeled 11.So,we are forced to choose one of the paths:

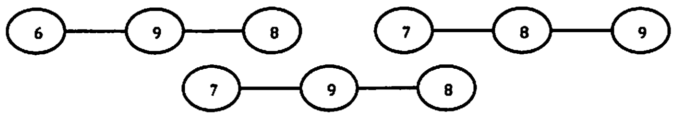


of *Lemma 1.2*. Again if the end vertices of one of these paths are adjacent, the result follows at once. If this were not the case, the edge label 3 cannot be achieved.

For the graphs $G(6,10)$ similar argument shows the assertion. It seems at first that there are much more cases to be investigated, but some of them are excluded at once such as the paths:

cannot be completed to be triangles so again we are in need to a vertex label at least 5, but no vertex of the paths in *Lemma 1.2* is labeled 5.

3) For $K_{3,3}$ we have by *Lemma 1.3* the possible paths:



To obtain the edge label 3, only the first one is possible, with a path of *Lemma 1.2 (realize that the edge label 3 need the vertex labeled 6 to be adjacent to the vertex labeled 0, since there is no triangles in* $K_{3,3}$*)*. The vertices labeled 0 and 6 must be in the two different partite sets of $K_{3,3}$. Now the vertex labeled 0 must be an end vertex of the chosen path of *Lemma 1.2*, since if it were not the case, the two vertices which are adjacent to 0 are in the same partite set, but the vertices labeled 6 and 8 are also in this partite set, so this partite set contains 4 elements: a contradiction. Now the vertex labeled 0 is an end vertex in the above sense so we have three possibilities for the two sets of $K_{3,3}$: $(a), (b) or (c)$ from *Lemma 1.2*:

a) The vertices labeled 9,0,2 are in one partite set, while the vertices labeled 6,8,1 are in the other one, we obtain a repeated edge label: 4: a contradiction.
b) The vertices labeled 9,0,3 are in one partite set, while the vertices labeled 6,8,1 are in the other one, we obtain a repeated edge label: 5: a contradiction.
c) The vertices labeled 9,0,1 are in one partite set, while the vertices labeled 6,8,2 are in the other one, we obtain a repeated edge label: 4: a contradiction.

The graphs $K_{1,4}, K_{1,5}, K_4$ are not mean graphs [4].

For the graphs $K_{1,4} \cup K_1, K_4 \cup K_1, K_4 \cup \overline{K}_2$ *Lemma 1.4* shows the assertion.

4) From *Theorem 2.3*, it follows directly that the graph (a) is not a mean graph.

For the graph (b) assume that it is a mean graph. There must be two labeled paths *(without any loss of generality)* as x, y, z and u, v, w given by *Lemma1.2* and *Lemma 1.3*, so we have the two following cases, as shown in *Figure 1.5*.

Case 1: the path u, v, w will be labeled 7,8,5; 7,8,6 or 8,7,6. To obtain the edge label 3 the vertex z should be labeled 0 or 1. To obtain the edge label 6 the vertex y should be labeled 3 or 4, but none of them could be found as a middle vertex label of the paths in *Lemma 1.2*: a contradiction. Case 2: the path x, y, z will be labeled 7,8,5; 7,8,6 or 8,7,6 (*the vertex labeled 5 or 6 should be given to the vertex z to obtain the edge label 3*). To obtain the edge label 6 the vertex v should be labeled 3 or 4, this is not possible as in case 1.

For the graph (c) assume that it is a mean graph. According to *Lemma 1.2* and *1.3*, without any loss of generality there must exist in the graph the two paths u, v, w labeled 8,9,6 and x, y, z (case 1); or the paths x, y, z labeled 8,9,6 and u, v, w (case 2); as shown in the *Figure 1.5* (*we take the path (1) from Lemma 1.3 which contains the vertex label 6, which is the only way to obtain the edge label 3, it is the same reason to apply the label 6 on the vertex z not x*).

In case 1, to obtain the edge label 7 the vertex y should be labeled 4, but this label does not exist in the middle of any path of *Lemma 1.2*: a contradiction.

In case 2, to get the edge label 7, v takes the label 4 which doesn't exist again in the middle of any path of *Lemma 1.2*; or u (*similarly* w) is labeled 4 and consequently v will be labeled 0 (*from lemma 1.2*) and the edge label 3 doesn't exist: a contradiction.

For graph (d): assume that it is a mean graph. The edge label 7 could be obtained by giving the labels 6 and 7 to the vertices x, y (case 1) or u, v (case 2) *Figure 1.5*.

In case 1, the edge label 6 cannot be obtained.

In case 2, according to *Lemma 1.1* the vertices w and z must take even labels, from the set $\{0,2,4\}$. To obtain the edge label 6 the label 4 should be given to the vertex z, without any loss of generality. The vertex w cannot be labeled 0, since the edge label 1 cannot exist. If the vertex w take the label 2, then we obtain the edge label 5 repeated twice: a contradiction.

For graph (e): assume that it is a mean graph. According to 2) the graph H is a subgraph of this graph, there is one possibility: the triangles of *Figure 1.4* are the triangles $x\,y\,z$ and $u\,v\,w$ in *Figure 1.5*. In the proof of 2) the vertex label 11 must be adjacent to the vertex labeled 4. Also the edge label 4 is obtained as a mean of the labels 0 and 8. Now without any loss of generality, we have the following cases:

Case(1) the vertices x, y, z are labeled 10,11,8 respectively and the vertices u, v, w are labeled 0,4,1 or 2. Here the edge label 5 will be repeated twice: a contradiction.

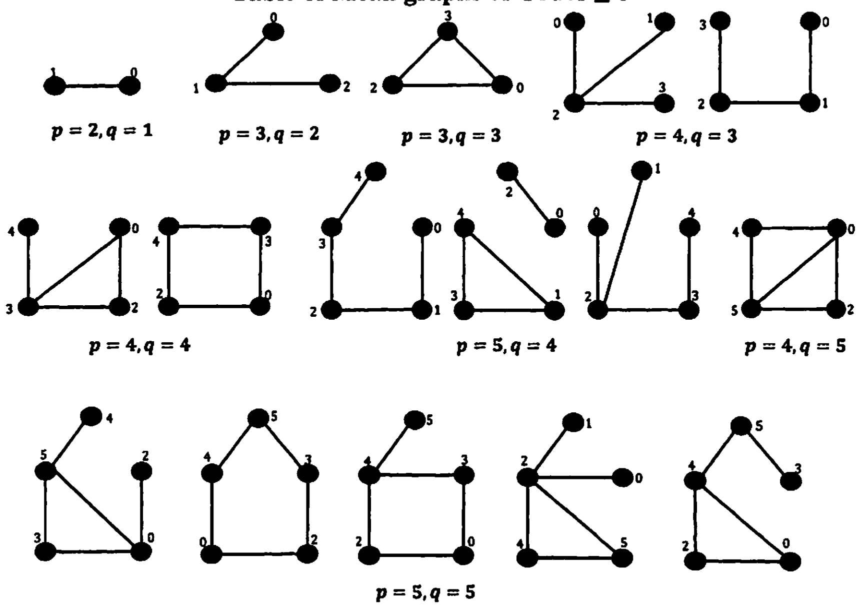
Case(2) x is labeled 11 u(*similarly* w) is labeled 4:

- y is labeled 8, v is labeled 0: the edge label 6 or 7 will be repeated: a contradiction.
- y is labeled 10, w is labeled 0: the edge label 6 will be repeated twice: a contradiction.

To complete the proof we present the graphs that are mean and a labeling for each one. Here *Lemma 1.4* is considered, (*for example when we present a mean labeling for the graph* C_4 *we do not present another one for* $C_4 \cup K_1$).

Table 1.1

Table of Mean graphs of Order ≤ 6



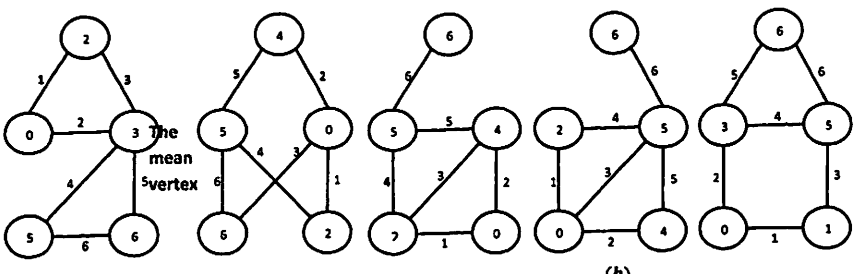
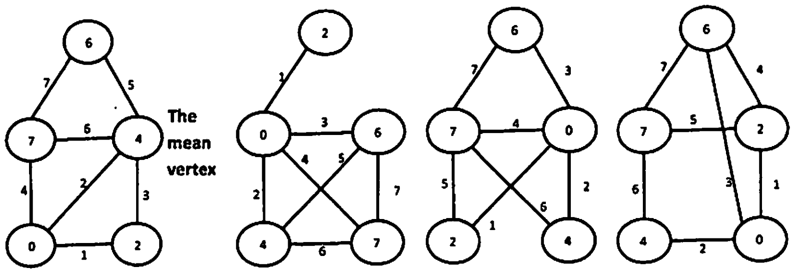
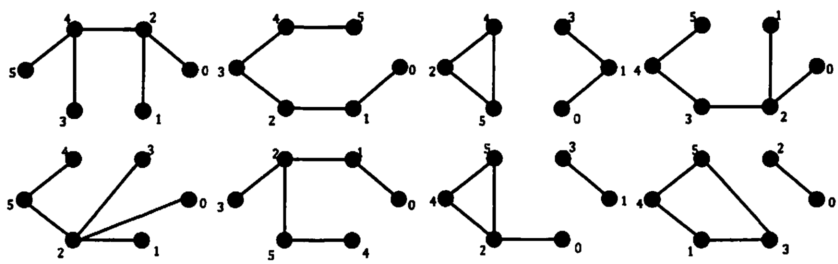


Figure 1.6 $p = 5, q = 6$

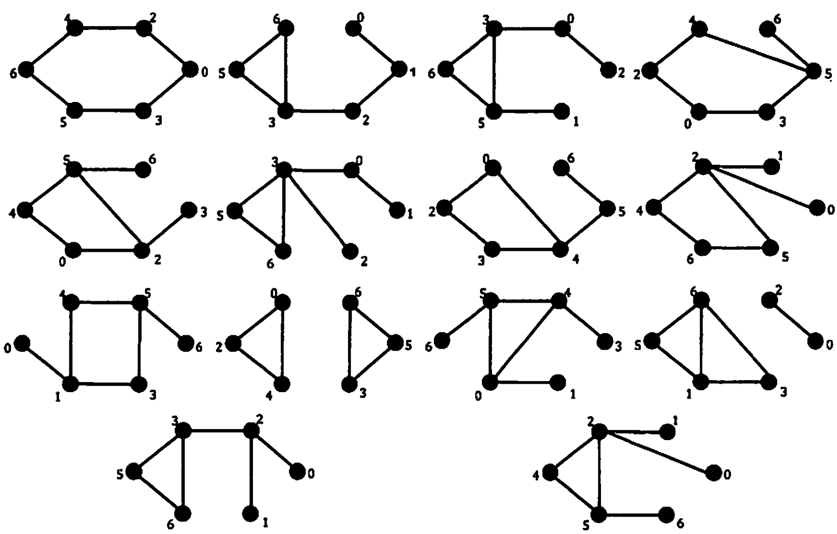


(*a*)

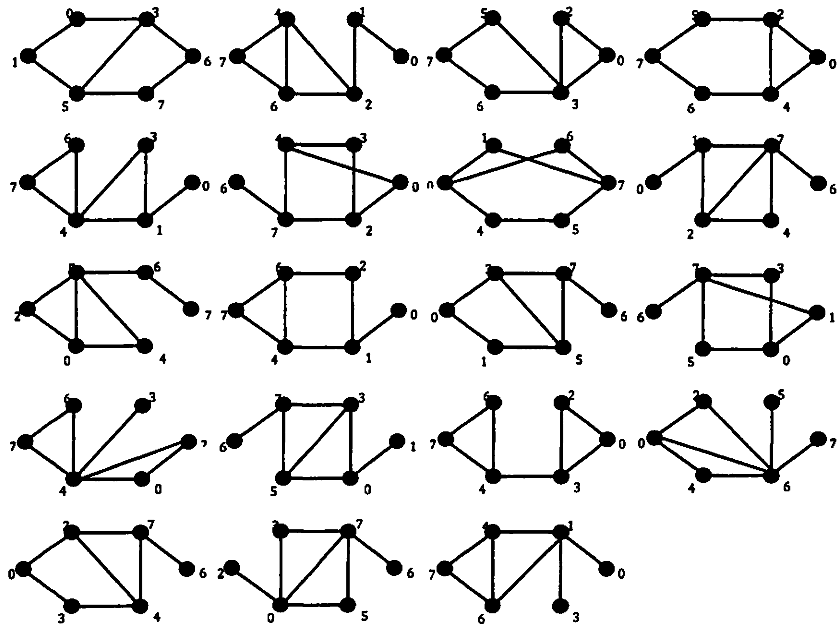
Figure 1.7 $p = 5, q = 7$



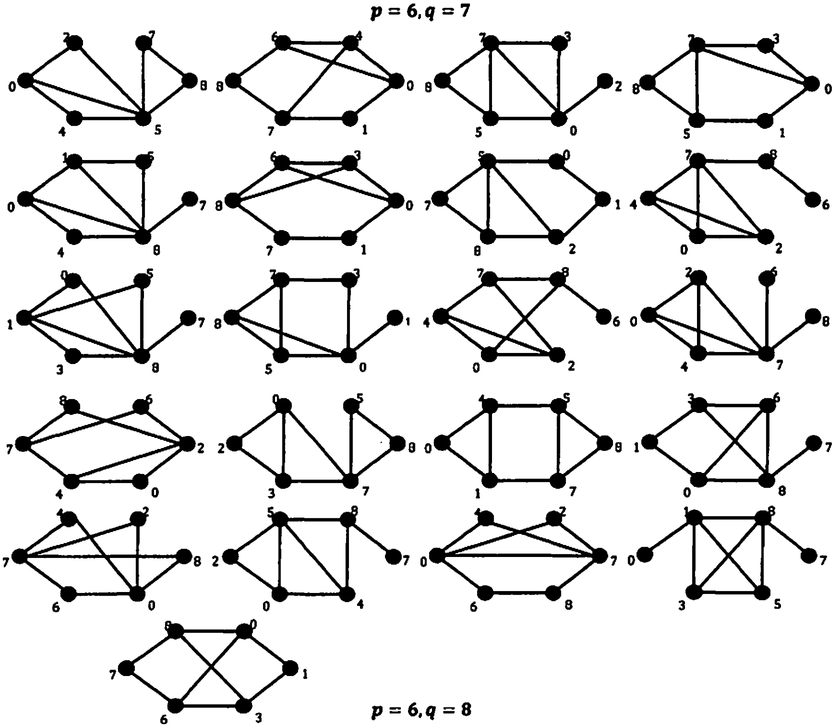
$p = 6, q = 5.$



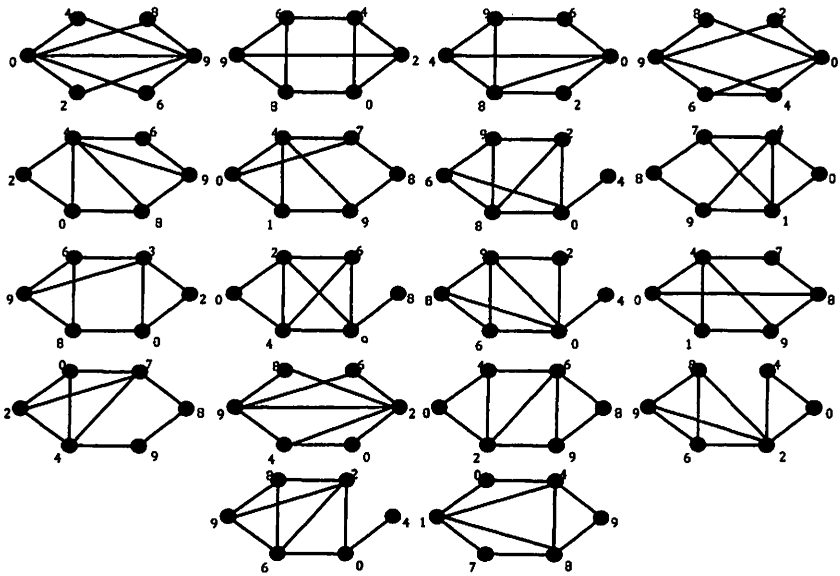
$p = 6, q = 6.$



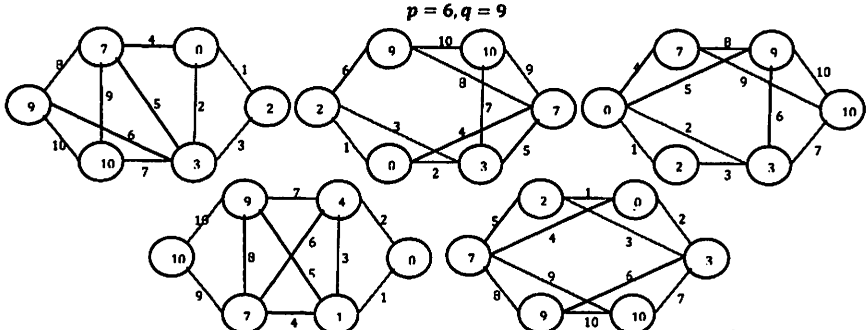
$p = 6, q = 7$

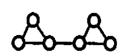


$p = 6, q = 8$

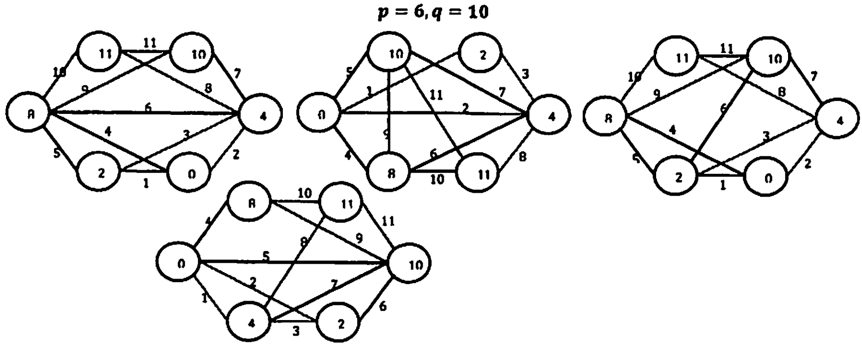


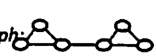
$p = 6, q = 9$



Realize the existence of the subgraph: 

$p = 6, q = 10$



Realize the existence of the subgraph: 

$p = 6, q = 11$

2. General Theorems:

Theorem 2.1: Let $G(p,q)$ be a graph of order $p = 2k, k \geq 2$, and let $u = 2(k^2 - 2k + 3)$.

Now if the number of the edges $q \geq u$, then the graph G is not a mean graph.

Proof: Let $q = u$, the edge labels: $1,2,3, \dots, \frac{(k-1)(k-2)}{2} + 1$ need at least k vertex labels the maximum of them can't exceed the number $(k-1)(k-2) + 2$.

On the other hand the edge labels: $u, u-1, u-2, \dots, u - (\frac{k(k-1)}{2} + 1) - 1 (= \frac{3k^2-7k+12}{2})$ need at least $k+1$ vertex labels with minimum vertex label, which could be found from the solution "y" of the equation: $\left\lceil \frac{y+u}{2} \right\rceil = u - \left(\frac{k(k-1)}{2} + 1\right) - 1$, is $y = (k-1)(k-2) + 3$. That means we are in need to $k + (k+1) = 2k + 1$ distinct vertex labels to get those edge labels i.e. the graph of $2k$ vertex can't be mean with such number of edges $q = u$.

In case $q > u$ then the solution "y" of the equation: $\left\lceil \frac{y+q}{2} \right\rceil = q - \left(\frac{k(k-1)}{2} + 1\right) - 1$ is greater than $(k-1)(k-2) + 3$, again we need at least $2k + 1$ vertex label and the result follows.

Theorem 2.2: Let $G(p,q)$ be a graph of order $p = 2k + 1, k \geq 2$, and let $u = 2(k^2 - k + 2)$.

Now if the number of the edges $q \geq u$, then the graph G is not a mean graph.

Proof: Let $q = u$, the edge labels $1,2,3, \dots, \frac{k(k-1)}{2} + 1$ need at least $k + 1$ vertex labels the maximum of them can't exceed the number $k(k-1) + 2$.

On the other hand the edge labels: $u, u-1, u-2, \dots, u - (\frac{k(k-1)}{2} + 1) - 1 (= \frac{3k^2-3k+8}{2})$ need at least $k+1$ vertex labels with minimum vertex label, which could be found from the solution "y" of the equation: $\left\lceil \frac{y+u}{2} \right\rceil = u - \left(\frac{k(k-1)}{2} + 1\right) - 1$, is $y = k(k-1) + 3$. That means we are in need to $(k+1) + (k+1) = 2k + 2$ distinct vertex labels to get those edge labels i.e. the graph of $2k + 1$ vertex can't be mean with such number of edges $q = u$.

In case $q > u$ then the solution "y" of the equation: $\left\lceil \frac{y+q}{2} \right\rceil = q - \left(\frac{k(k-1)}{2} + 1\right) - 1$ is greater than $k(k-1) + 3$, again we need at least $2k + 2$ vertex label and the result follows.

Note: For $p = 4,5,6,7,8,9$, there are mean graphs for such number of vertices when the number of edges $q = u - 1$ (*they are given for* $p = 4,5,6$ *in Table 1.1*).

Theorem 2.3: Let $G(p,q)$ be a mean graph and let $q = 2k + 4$; $k = 1,2,3, \dots$, then the largest vertex degree of G is $k + 3$; $k = 1,2,3, \dots$.

Proof: For $k = 1$, $q = 6$, (*there is a vertex of degree* $k + 3 = 4$), a mean graph in this case is shown in, *Table 1.1, Figure 1.6* - (b), i.e. such graph is exist.

Let there exists a vertex v of degree greater than $k + 3$.

We show by induction on k that this graph is not a mean graph. For deg(v)(= degree of v)=$k+4$, when $k=1$, then $q=6$, deg(v)=5, the adjacent vertices to v should be labeled by 5 elements of the set $\{0,1,2,3,4,5,6\}$, in any case the following two pairs should exist: $l_1=(i,i+1), l_2=(j,j+1)$, among those 5 elements, where i is odd and j is even. According to *Lemma 1.1*, the graph is not a mean graph.
Let the assertion be true for k (*i.e. there exist such pairs as l_1, l_2 between the $k+4$ labels of the adjacent vertices to v so the graph is not a mean graph*). We prove this assertion for $k+1$ as follows: $\deg(v)=k+1+4=k+5$ and $q=2(k+1)+4=2k+6$, the adjacent vertices to v should be labeled by $k+5$ elements of the set $\{0,1,2,3,\ldots,2k+3,2k+4,2k+5,2k+6\}$. If we do not use the labels $2k+5, 2k+6$, then the assertion follows immediately from the hypothesis. If we use only $2k+5$ *or* $2k+6$, then we are in the last situation, and the graph is not a mean graph. If we use the two mentioned labels, this means we use the labels in the pair l_1: if we do not exclude the element $2k+4$, then we have the labels $2k+4, 2k+5$ which represents the labels in the pair l_2, the graph is not a mean graph by *Lemma 1.1.* Now we exclude the label $2k+4$. We have to exclude $k+1$ elements from the set $\{0,1,2,3,\ldots,2k+3\}$, which contains $k+2$ pairs of the sort l_2 , hence it remains such a pair l_2. Hence the result. Now, it is easy to see that the graph is not a mean graph if $\deg(v) > k+4$.
Definition 2.1: Let $G(p,q)$ be a mean graph with a mean labeling. A vertex with label $\lceil q/2 \rceil$ is called a mean vertex.
Note: Every mean tree has a mean vertex.
Theorem 2.4: Let $G_1(p_1,q_1), G_2(p_2,q_2), G_3(p_3,q_3), \ldots, G_m(p_m,q_m)$ be mean graphs and let v_i, u_i be the vertices labeled $q_i, 0$ respectively of the graph G_i , $1 \le i \le m$.

1) If $q_1=q_2=\cdots=q_m=q$ then The graph which is made by attaching the vertex v_1 with the vertex u_2 and attaching u_2 with v_3 and v_3 with u_4 and so on until we reach last graph G_m, is again a mean graph.
2) If $q_1=q_2=\cdots=q_m=q$ and every graph G_i has a mean vertex w_i then the graph which is made by attaching the vertex w_1 with w_2 and w_2 with w_3 and so on until we reach the vertex w_m is again a mean graph.
3) The graph which is made by identifying v_1 with u_2 and v_2 with u_3 and v_3 with u_4 and so on until we identify v_{m-1} with u_m is also a mean graph.
4) If m is odd and $q_1=q_2=\cdots=q_m=q$ then the graph which is made by attaching the vertex u_1 with the vertex v_2 and attaching v_2 with u_3 and u_3 with v_4 and so on until we reach last graph G_m, we attach u_m to u_1 is again a mean graph.

Proof:

1) We add the number $\sum_{j=1}^{i-1} q_j + i - 1 = (i-1)(q+1)$ to all vertex labels of the mean graph G_i, $1 < i \le m$,(*the vertex labels remain distinct*) the edge

labels of the graph G_1 remain fixed, the edge labels of the graph G_i, $1 < i \leq m$ increase by the number$(i-1)(q+1)$ to be as follows:

$$1+(i-1)(q+1), 2+(i-1)(q+1), \dots, q+(i-1)(q+1)$$

Instead of: $1,2,\dots,q$. The bridge between the two graphs G_i and G_{i+1} gets the label:

(either $\lceil\{[(i-1)(q+1)] + [q+i(q+1)]\}/2\rceil$ *or*

$\lceil\{[q+(i-1)(q+1)] + [i(q+1)]\}/2\rceil)$

In both cases it is $iq+i, 1 \leq i \leq m-1$, and that means the edges labels in whole graph are distinct and they start with the label 1 and end with the label: $q+(m-1)(q+1) = mq+m-1$,

which is the total number of edges in the whole graph.

The vertex labels are taken from the set: $\{0,1,2,\dots,q\} \cup \{q+1, q+2, q+3, \dots, 2q+1\} \cup \{2q+2, 2q+3, 2q+4, \dots, 3q+2\}$

$\cup \dots \cup \{(m-1)(q-1), (m-1)(p-1)+1, \dots, (m-1)(q+1)+q\}$

, that means that the whole graph is a mean graph.

2) The same way of (1) is applied here, so the edge labels in the graph G_i will be the same as above, but the bridges between the graphs G_i and G_{i+1} will get the labels:

$$\lceil\{[\lceil q/2\rceil + (i-1)(q+1)] + [\lceil q/2\rceil + i(q+1)]\}/2\rceil$$

which is again $iq+i, 1 \leq i \leq m-1$, and that leads us to (1) again to end the proof.

3) First we give the label for the vertex which is common between the two graphs G_i and G_{i+1}:

$\sum_{j=1}^{i} q_j, 1 \leq i \leq m-1$, then we add the number $\sum_{j=1}^{i} q_j$ to all the remaining vertex labels of the graph G_{i+1}, $1 \leq i \leq m-1$, so the edge labels will be:

$1,2,\dots,q_1; q_1+1, q_1+2, \dots, q_1+q_2; q_1+q_2+1, q_1+q_2+2, \dots, q_1+q_2+q_3; \dots; \sum_{j=1}^{m-1} q_j + 1, \sum_{j=1}^{m-1} q_j + 2, \dots, \sum_{j=1}^{m} q_j$, the result follows.

4) We add the number $\sum_{j=1}^{i-1} q_j + i - 1 = (i-1)(q+1)$ to all vertex labels of the mean graph G_i, $1 < i \leq \frac{(m-1)}{2}$, and the number $\sum_{j=1}^{i-1} q_j + i = (i-1)(q+1)+1$ to all vertex labels of the mean graph G_i, $\frac{(m+1)}{2} \leq i \leq m$,*(the vertex labels remain distinct)*, the edge labels of the graph G_1 remain fixed, the edge labels of the graph G_i, $1 < i \leq \frac{(m-1)}{2}$ increase by the number $(i-1)(q+1)$ to be as follows:

$$1+(i-1)(q+1), 2+(i-1)(q+1), \dots, q+(i-1)(q+1)$$

instead of: $1,2,\dots,q$.

The edge between the two graphs G_i and G_{i+1}, $1 < i \leq \frac{(m-3)}{2}$ gets the label *(either* $\lceil\{[(i-1)(q+1)] + [q+i(q+1)]\}/2\rceil$ *or* $\lceil\{[q+(i-1)(q+1)] + [i(q+1)]\}/2\rceil)$ *In both cases it is* $iq+i$.

The edge labels of the graph G_i, $\frac{m+1}{2} \leq i \leq m$, increase by the number $(i-1)(q+1)+1$ to be as follows:

$$2+(i-1)(q+1), 3+(i-1)(q+1), \dots, q+1+(i-1)(q+1)$$

instead of: $1,2,\ldots,q$, and the edge between the two graphs G_i and $G_{i+1}, \frac{(m+1)}{2} \le i \le m-1$ gets the label

(either $\lceil\{[(i-1)(q+1)+1]+[q+i(q+1)+1]\}/2\rceil$ *or*

$\lceil\{[q+(i-1)(q+1)+1]+[i(q+1)+1]\}/2\rceil$).

In both cases it is $iq+i+1$.

The edge between the vertex u_1 , and the vertex u_m has the label $\frac{(m-1)}{2}q+\frac{(m+1)}{2}$ and the edge between the vertices, $v_{(\frac{m-1}{2})}$ and $u_{(\frac{m+1}{2})}$ where $m \equiv 1(mod4)$ or $u_{(\frac{m-1}{2})}$ and $v_{(\frac{m+1}{2})}$ where $m \equiv 3(mod4)$) has the label:

$$\left\lceil\frac{\left\{\left[\frac{(m-1)}{2}q+\frac{(m-3)}{2}\right]+\left[\frac{(m-1)}{2}q+\frac{(m+1)}{2}\right]\right\}}{2}\right\rceil = \frac{(m-1)}{2}q+\frac{(m-1)}{2} \quad \text{when}$$

$m \equiv 1(mod\ 4)$, or

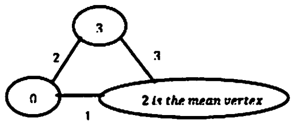
$$\left\lceil\frac{\left\{\left[\frac{(m-3)}{2}q+\frac{(m-3)}{2}\right]+\left[\frac{(m+1)}{2}q+\frac{(m+1)}{2}\right]\right\}}{2}\right\rceil = \frac{(m-1)}{2}q+\frac{(m-1)}{2} \quad \text{when}$$

$m \equiv 3(mod\ 4)$,

i.e. the edge labels in the whole graph start with the label 1 until the label $q+\frac{(m-3)}{2}(q+1)=\frac{(m-1)}{2}q+\frac{(m-3)}{2}$ in the graph $G_{(\frac{m-1}{2})}$, then we have the edge labeled $\frac{(m-1)}{2}q+\frac{(m-1)}{2}$ which is between $G_{(\frac{m-1}{2})}$ and $G_{(\frac{m+1}{2})}$ after that we have the edge labeled $\frac{(m-1)}{2}q+\frac{(m+1)}{2}$, which is between u_1 and u_m, then the edge labels continue from the edge label $q+1+\frac{(m-1)}{2}(q+1)=\frac{(m+1)}{2}q+\frac{(m+1)}{2}$ until we reach the edge label $q+1+(m-1)(q+1)=mq+m$, which is total number of edges in the whole graph.

Definition, results and examples: The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 was defined by Frucht and Harary [2] as the graph H obtained by taking one copy of G_1 (which has p vertices) and p copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

In ,*Figure 2.1,* there is an example of $P_4 \odot K_2$ labeled as a mean graph twice by taking the graph C_3 as G_i in the *Theorem 2.4* -(1) and (2) respectively.



A mean labeling of C_3

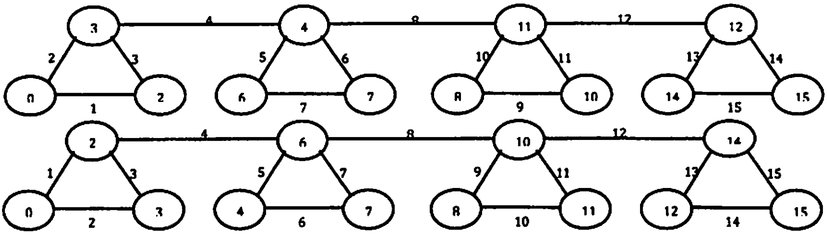


Figure2.1 $P_4 \odot K_2$

Similarly the following graphs are mean by *Theorem 2.4*: $P_n \odot K_2$ (the previous example for n cycles C_3), $P_n \odot K_{1,m}$ by applying *Theorem 2.4*-(1) on n copies of $K_{1,1,m}$, (*Figure 2.2*). Similarly $C_n \odot K_2$ and $C_n \odot K_{1,m}$ are mean graphs when n is odd by *Theorem 2.4*-(4). $P_n \odot \overline{K}_2$ and $P_n \odot \overline{K}_3$ are mean graphs by applying *Theorem 2.4*-(2) on n copies of the stars $K_{1,2}, K_{1,3}$ respectively *(see Table 1.1)*. $P_n \odot P_4$ is a mean graph by applying *Theorem 2.4*-(2) on n copies of the graph (a) in *Table 1.1, Figure 1.7* $P_n \odot 2K_2$, (*where* $2K_2 = K_2 \cup K_2$) and $P_n \odot 2K_{1,m}$ are mean graphs by applying *Theorem 2.4*-(3) on two copies of the graphs C_3 and $K_{1,1,m}$, *Figure 2.2*, respectively, then applying 2) again on n copies of each of the resulting graphs.

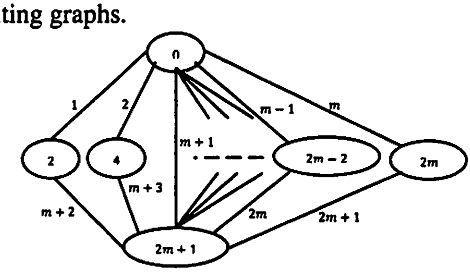


Figure 2.2 a mean labeling for $K_{1,1,m}$ $(q = 2m + 1)$

Theorem 2.5: Let $G_1(p_1, q_1), G_2(p_2, q_2)$ be mean graphs, the graph, which is made by attaching the vertex labeled q_1 from G_1 with the vertex labeled 0 from G_2, is a mean graph.

Proof: We add the number $q_1 + 1$ to all vertex labels of the graph G_2 (the vertex labels stay distinct), all its edge labels increase by the number $q_1 + 1$ i.e. the edge labels start from $q_1 + 2$ and end by $q_1 + q_2 + 1$. Also the bridge between G_1 and G_2 has the label $q_1 + 1$ i.e. the edge labels of total graph start from 1 and end with $q_1 + q_2 + 1$, and the vertex labels are distinct, that completes the proof.

Note: In *Theorem 2.5* if G_1 has a mean vertex w_1 and G_2 has a vertex labeled with the same label of w_1, then by joining them we get a mean graph again.

2. Unions of Mean Graphs.

Definitions 3.1:

1- In a mean labeling the labels of vertices come out from the set $\{0,1,2,\dots,q\}$. We call the subset of all unused elements in this labeling a *Free set*, and if we concentrate on one unused number t we call this labeling $t-$ *free labeling*

2- Let $G(p,q)$ be a mean graph and let v be the vertex with the label q, and let one of the mean labelings of G be satisfying the following: If q is *odd* (*even*) and all the labels of the vertices which are adjacent to v are *even* (*odd*), then we call this mean labeling *extra mean labeling*, and the graph G *extra mean graph.*

Examples:

1- Every cycle C_n has a mean labeling which is 1-free:

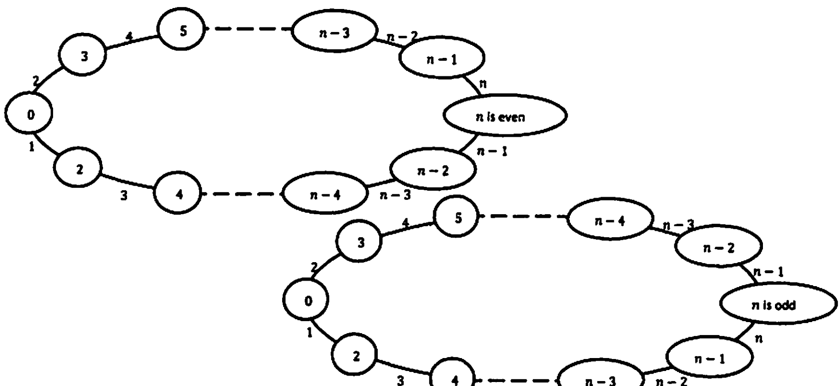


Figure 3.1 a 1-free labeling for odd and even cycle C_n

2- Every path P_n and every Odd cycle C_n have extra mean labelings :

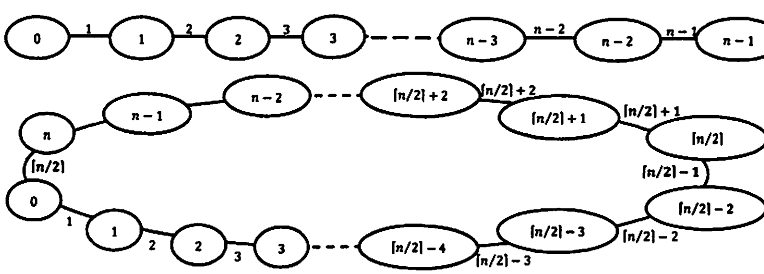


Figure 3.2 an extra mean labeling for the path P_n, an extra mean labeling for a cycle C_n of odd length

Lemma 3.1: For every integer $t \geq 1$ there are at least two t-free mean graphs.
Proof: For each t we will present two t- free mean labelings of C_{2t+1} and C_{2t+2}.

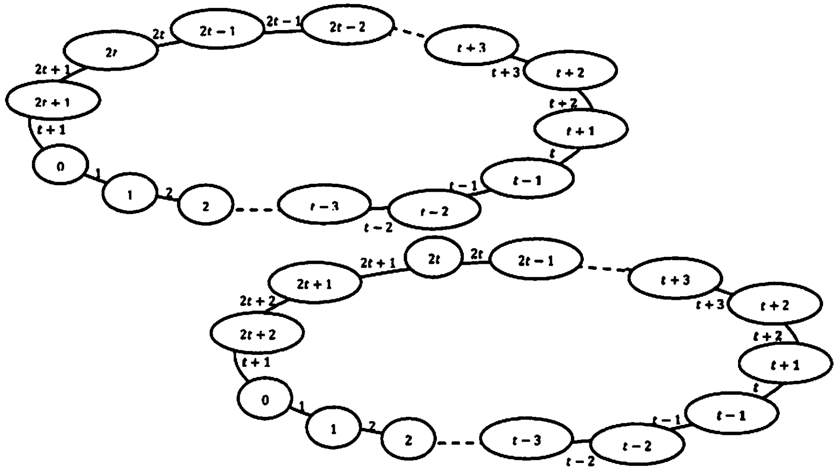


Figure 3.3 C_{2t+1}, C_{2t+2} **are two** t**-free mean graphs**

Theorem 3.1: Let $G_1(p_1, q_1)$, $G_2(p_2, q_2)$, $G_3(p_3, q_3)$, … , $G_m(p_m, q_m)$ be extra mean graphs with extra mean labelings that are 1-free. Then the following graph: $H = \bigcup_{i=1}^{m} G_i$ is a mean graph.

Proof: Let v_i be the vertex labeled q_i in the related extra 1- free mean labeling of the graph G_i , $1 \le i \le m$. First we add 1 to each label q_i of the vertex v_i , $1 \le i < m$, since the graphs are extra mean, the edge labels do not change. The label of v_m has not changed yet. Let us add the number $\sum_{j=1}^{i-1} q_j$ to all vertices labels of the graph G_i , $1 < i \le m$,

Therefore the edge labels of the graph G_i are changed to:

$\{ 1 + \sum_{j=1}^{i-1} q_j , 2 + \sum_{j=1}^{i-1} q_j , 3 + \sum_{j=1}^{i-1} q_j , \ldots , q_i + \sum_{j=1}^{i-1} q_j = \sum_{j=1}^{i} q_j\}$ instead of: $\{1,2,3, \ldots , q_i\}$ $1 < i \le m$. Which mean that all edge labels in H are distinct.

Realize that: the graph G_1 takes its vertex labels from the set: $\{0,2, \ldots , q_1 - 1, q_1 + 1\}$,

the graph G_2 takes its vertex labels from the set: $\{q_1, q_1 + 2, q_1 + 3, \ldots , q_1 + q_2 - 1, q_1 + q_2 + 1 \}$

The graph G_i , $1 < i < m$ takes its vertex labels from the set:

$\{\sum_{j=1}^{i-1} q_j , \sum_{j=1}^{i-1} q_j + 2, \sum_{j=1}^{i-1} q_j + 3, \ldots , \sum_{j=1}^{i-1} q_j + q_i - 1, \sum_{j=1}^{i-1} q_j + q_i + 1 = \sum_{j=1}^{i} q_j + 1\}$

The graph G_m takes its vertex labels from the set:

$\{\sum_{j=1}^{m-1} q_j , \sum_{j=1}^{m-1} q_j + 2, \sum_{j=1}^{m-1} q_j + 3, \ldots , \sum_{j=1}^{m-1} q_j + q_m - 1, \sum_{j=1}^{m-1} q_j + q_m = \sum_{j=1}^{m} q_j\}$

that means that all vertex labels are distinct and they are taken from the total set: $\{0,2, \ldots , \sum_{j=1}^{m} q_j\}$

hence the result.

Remarks and example:

- There are many graphs which are extra mean graphs and have 1-free labeling in *Table 1.1* to apply *Theorem 3.1* on them.
- We can take the first of those graphs in *Theorem 3.1*, without having a 1-free mean labeling, and the result remains without changes.
- In the following example C_3 will act as G_i in *Theorem 3.1* so $\boldsymbol{mC_3}$ is a mean graph:

  (C_3 is an extra mean graph and has a 1- free mean labeling, *Figure 3.4*):

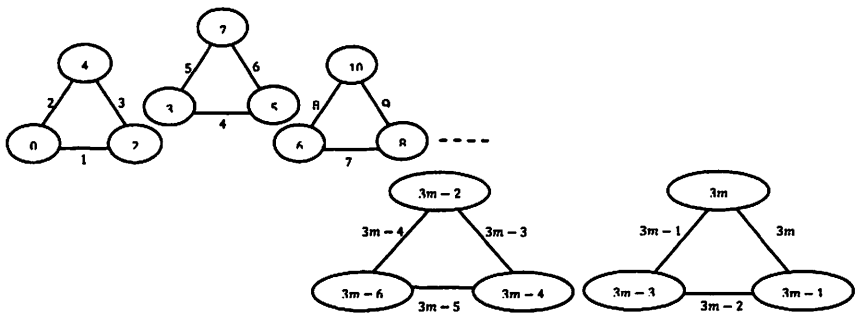


Figure 3.4 $\boldsymbol{mC_3}$ **is a mean graph**

Similarly $PG(= Petersen\ Graph), K_{1,1,m}$ are extra mean graphs and they have 1-free labeling, *Figure 3.5, Figure 3.6,* so $\boldsymbol{rPG}$, $\boldsymbol{rK_{1,1,m}}$ are mean graphs for arbitrary r by *Theorem 3.1*.

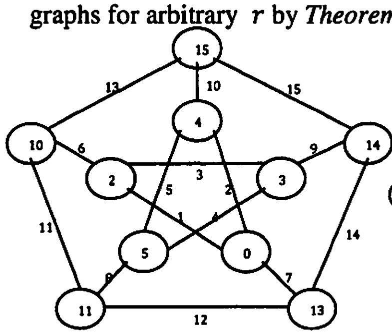


Figure 3.5 a 1-free mean labeling for Petersen graph PG

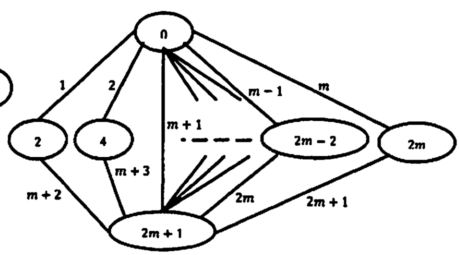


Figure 3.6 a 1-free mean labeling for $K_{1,1,m}$ $(q = 2m + 1)$

Theorem 3.2: Let $G_1(p_1, q_1)$ be an extra mean graph with an extra mean labeling, and let $G_2(p_2, q_2)$ be a mean graph with a 1-free labeling, then $G_1 \cup G_2$ is a mean graph.

Proof: Let $v_1 \in V(G_1)$ be the vertex labeled q_1. We add 1 to the label q_1, then we add the number q_1 to all vertex labels of the graph G_2. Since G_1 is an extra mean graph, its edge labels will not change, but in the graph G_2 they will be: $\{q_1 + 1, q_1 + 2, q_1 + 3, \dots, q_1 + q_2 - 1, q_1 + q_2\}$ instead of: $\{1, +2, +3, \dots, q_2 - 1, q_2\}$, i.e. the edge labels of $G_1 \cup G_2$ are distinct. The graph G_1 takes its vertex labels from the set: $\{0,1,2, \dots, q_1 - 1, q_1 + 1\}$, the graph G_2 takes its vertex labels from the set: $\{q_1, q_1+2, q_1+3, \dots, q_1+q_2-1, q_1+q_2\}$, i.e. the vertex labels of the graph $G_1 \cup G_2$ are distinct too. So $G_1 \cup G_2$ is a mean graph.

Example: The cycle C_n with an odd length has an extra mean labeling as given in *Figure 3.2,* also in general C_m has a 1-free mean labeling as shown in *Figure 3.1,* so by *Theorem 3.2* $(C_n \cup C_m)$ is a mean graph for any m, and an odd n as shown in *Figure 3.7*

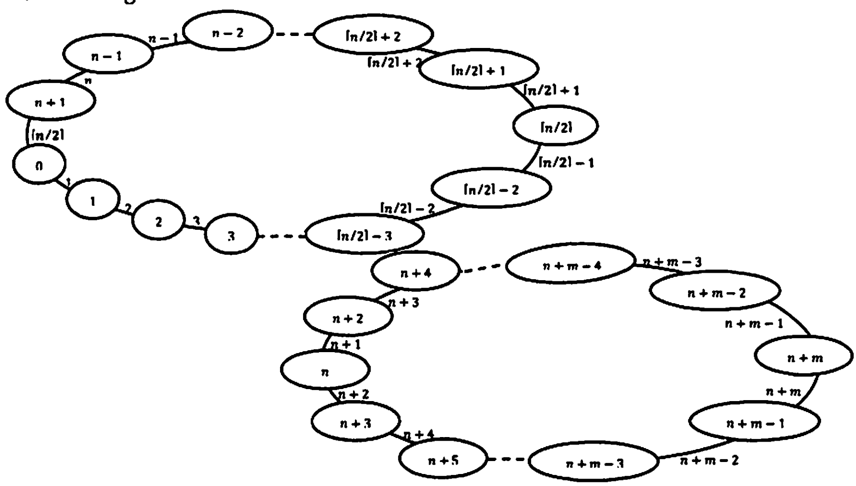


Figure 3.7 $C_n \cup C_m$ **is a mean graph** (n *is odd*)

Definition 3.2: Let $G(p,q)$ be a graph and let l be a positive integer, we define an *l-semi mean labeling* to be the injective function:

$$f: V(G) \longrightarrow \{0,1,2,\dots,q-1,q+l\},$$

where every edge uv has the label $\lceil (f(u)+f(v))/2 \rceil$, such that all edges have different labels from 1 to q exactly.

Note: The difference between this labeling and the mean labeling is just replacing q from the set $\{0,1,2,\dots,q-1,q\}$ by $q+l$.

Examples: K_4 is a 1-semi mean graph, K_5 is a 3-semi mean graph and $K_{3,3}$ is 2 and 3-semi mean graph, *Figure 3.8.* (*Note that K_4, K_5 and $K_{3,3}$ are not mean graphs from Theorem1.1*).

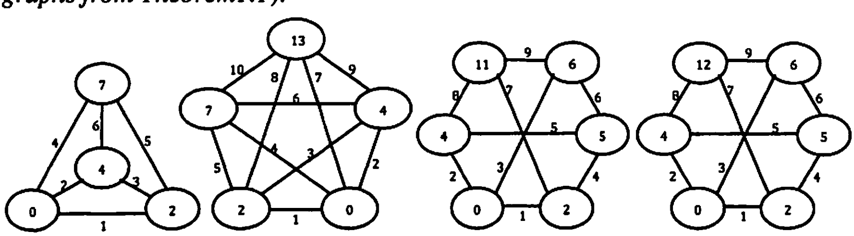


Figure 3.8 K_4 **is a 1-semi mean graph,** K_5 **is a 3- semi mean graph and** $K_{3,3}$ **is 2 and 3-semi mean graph**

Corollary 3.1: Every extra mean graph is a 1- semi mean graph, but the converse is not necessarily true.

Proof: If we change the largest vertex label q to $q+1$ then the edge labels will not change. K_4 is a 1-semi mean graph (*Figure 3.8*), but it is not extra mean graph. (*It is not mean graph*).

Lemma 3.2: Every cycle is a 1-semi mean graph.

Proof: shown in *Figure 3.9*:

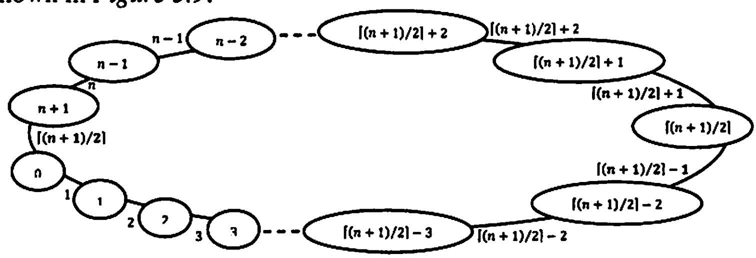


Figure 3.9 C_n **is a 1-semi mean graph**

Lemma 3.3: Let $G(p,q)$ be an l-semi mean graph and let it has a labeling which is l- free, then mG is again an l-semi mean graph for any positive integer m.

Proof: We add the number $((i-1)q)$ to each vertex label of the i^{th} copy of G, then the edge labels in this copy are: $\{1+(i-1)q, 2+(i-1)q, 3+(i-1)q, \dots, q+(i-1)q = iq\}$, instead of $\{1,2,3,\dots,q\}$, the vertex labels for the i^{th} copy of G are taken from the set:

$\{0+(i-1)q, 1+(i-1)q, 2+(i-1)q, \dots, l-1+(i-1)q, l+1+(i-1)q, \dots, (i-1)q+q+l\}$,

i.e. mG is an l-semi mean graph.

Example: K_4 is a 1-semi mean graph and also it has a 1-free labeling so mK_4 is a 1-semi mean graph as shown in *Figure 3.10*:

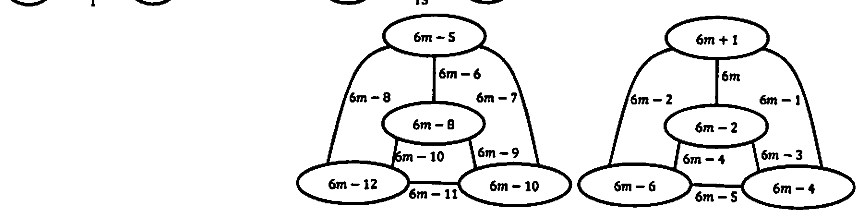


Figure 3.10 mK_4 **is a 1-semi mean graph**

Theorem 3.3: Let $G(p,q)$ be an l-semi mean graph, $H(p',q')$ be a mean graph, with an l-free mean labeling, then the graph $G \cup H$ is a mean graph.
Proof: Let $f: V(G) \longrightarrow \{0,1,2,\dots,q-1,q+l\}$, and $f': V(H) \longrightarrow \{0,1,2,\dots,l-1,l+1,\dots,q'\}$ be the two injective functions for the vertices of the graphs G,H respectively. Now let us define a new injective function for the graph $G \cup H$ as follows:
$g: V(G \cup H) \longrightarrow \{0,1,2,\dots,q-1,q+l\} \cup \{0+q,1+q,\dots,l-1+q,l+1+q,\dots,q'+q\}$
$g(u) = f(u),\ \forall\, u \in V(G);\quad g(v) = f'(v) + q,\ \forall\, v \in V(H)$. Now if every edge xy of the graph $G \cup H$ takes the label $\lceil (f(x)+f(y))/2 \rceil$, then the labels of the edges will be: $1,2,3,\dots,q,q+1,q+2,\dots,q+q'$, which means that $G \cup H$ is a mean graph.
Corollary3.2: If G mentioned above has an l-free labeling, then $mG \cup H$ is a mean graph for any positive integer m. The proof is straight forward by *Lemma 3.3.*
Example: We know that K_4 is a 1-semi mean graph and has a 1-free labeling, so by *Lemma 3.3* mK_4 is a 1-semi mean graph. Also the cycle C_n is a mean graph which has always a 1-free mean labeling, (*see Figure 3.1*), so $\boldsymbol{mK_4} \cup \boldsymbol{C_n}$ is a mean graph for any m,n (by *Corollary 3.2*).
Corollary3.3: $\boldsymbol{C_n} \cup \boldsymbol{C_m}$ is a mean graph for any n,m.
Proof: *Lemma 3.2* states that the cycle C_n is a 1-semi mean graph, so by *Theorem 3.3* $C_n \cup H$ is a mean graph, for every mean graph H, with a 1-free mean labeling, but *Figure 3.1* shows that C_m has a 1-free mean labeling for any m, so by taking $H = C_m$, the result follows.
Note : C_3 is a 1-semi mean graph and it has a 1-free semi mean labeling, so mC_3 is a 1-semi mean graph by *Theorem 3.3*, and by taking $H = C_n$, we get that $\boldsymbol{mC_3} \cup \boldsymbol{C_n}$ is a mean graph. As a special case $mC_3 \cup C_3 = \boldsymbol{(m+1)C_3} = \boldsymbol{rC_3}$ is a mean graph. When $n > 3$: C_n is a 1-semi mean graph by (*Lemma 3.2*), but it has not a 1-free semi mean labeling, so mC_n is not known to be a 1-semi mean graph i.e. mC_n is not known to be a mean graph, where $n > 3$, $m > 2$. Here we have a conjecture states that: $C_{n_1} \cup C_{n_2} \cup C_{n_3} \cup \dots \cup C_{n_m}, n_i > 4, 1 \le i \le m, m \ge 4$ is not a mean graph.

- We present some graphs, that are mean graphs by means of *Theorem 3.3* and *Lemma 3.3:*
  $\boldsymbol{mC_3} \cup \boldsymbol{C_n}, \boldsymbol{mPG} \cup \boldsymbol{C_n}, \boldsymbol{rK_{1,1,m}} \cup \boldsymbol{C_n}, \boldsymbol{mK_4} \cup \boldsymbol{C_n}, \boldsymbol{C_n} \cup \boldsymbol{C_m}$.
  $\boldsymbol{mK_5} \cup \boldsymbol{H_1}$, $\boldsymbol{mK_{3,3}} \cup \boldsymbol{H_2}$, *where H_1 is any 3-free mean graph (there are at least C_7, C_8), H_2 is any 2-free or 3- free mean graph (there are at least C_5, C_6, C_7, C_8).*

3. Some Families of Mean Graphs.

In the following figures we present a mean labeling for each of the graphs $\boldsymbol{C_n \odot \overline{K}_2}$, $\boldsymbol{C_n \odot K_2}$

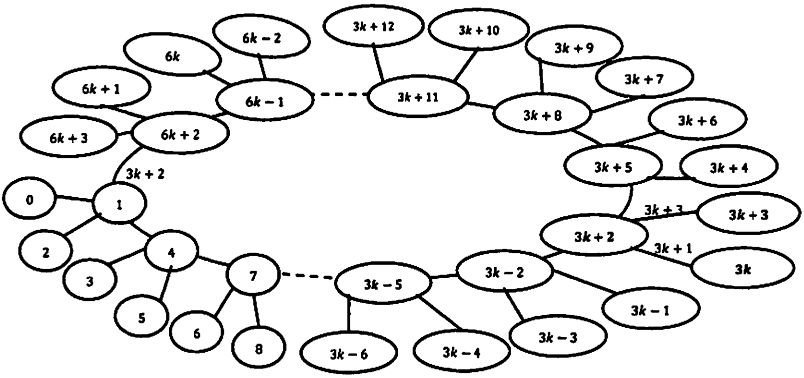


Figure 4.1 a mean labeling for $C_n \odot \overline{K}_2, n = 2k + 1$

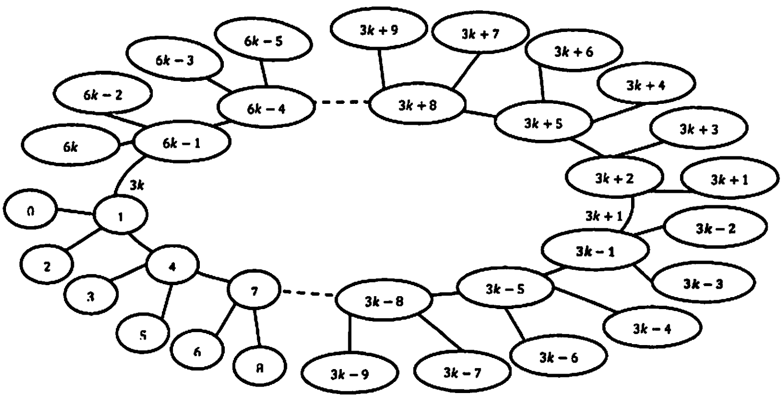


Figure 4.2 a mean labeling for $C_n \odot \overline{K}_2, n = 2k$

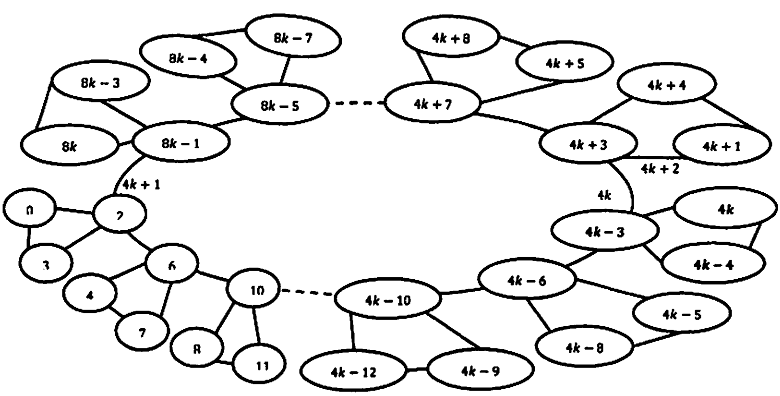


Figure 4.3 a mean labeling for $C_n \odot K_2, n = 2k$

For $C_n \odot K_2$, when n is odd, see *Theorem 2.4* - (4).
In *Figure 4.4,* we present a mean labeling for the graph P_n^2.

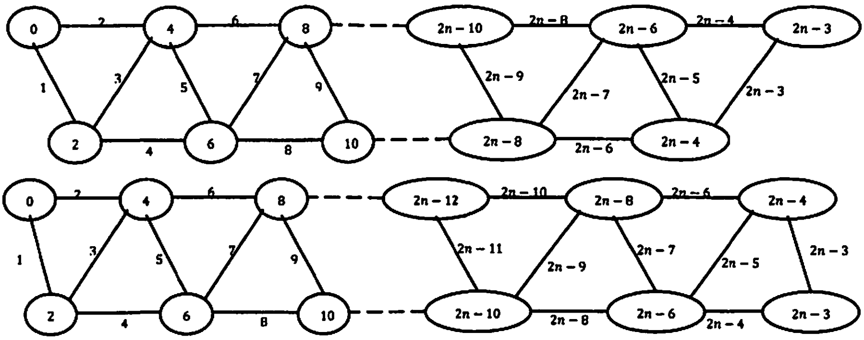


Figure 4.4 a mean labeling for the graph P_n^2, for odd and even n

Remark: Note that $C_n \odot K_2$ and P_n^2 are extra mean graphs and they have 1-free mean labelings, so by *Theorem 3.1* the unions $m(C_n \odot K_2)$ and mP_n^2 are mean graphs.

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