Association schemes on the sets of lines of regular near hexagons

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Abstract

We examine under what conditions there exists an association scheme on the set of lines of a regular near hexagon with quads of order (s, t_2) through every two points at distance 2. All regular near hexagons with such an association scheme are determined in the case $s \geq t_2$. Unfortunately, the case $t_2 > s$ is still open.

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1 Definitions and overview

A *near polygon* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a partial linear space with the property that every line L contains a unique point q nearest to any given point p. Here distances $\mathrm{d}(\cdot,\cdot)$ are measured in the collinearity graph. This is the graph whose vertices are the points of $\mathcal{S}$, with two different vertices adjacent whenever they are collinear in $\mathcal{S}$. If d is the maximal distance between two points, then the near polygon is called a near $2d$-gon. A near 0-gon consists of one point, a near 2-gon is a line, and the class of the near quadrangles coincides with the class of the generalized quadrangles (GQ's, [7]) which were introduced by Tits in [9]. Near polygons themselves were introduced by Shult and Yanushka in [8] because of their relationship with certain line systems in Euclidean spaces. Generalized $2d$-gons ([10]) and dual polar spaces ([8]) form two important classes of near polygons.
A near polygon $\mathcal{S}$ is said to have *order* (s, t) if every line is incident with exactly $s + 1$ points and if every point is incident with exactly $t + 1$ lines. We will assume that $\mathcal{S}$ is a *regular* near hexagon with parameters (s, t, t_2),

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i.e. $\mathcal{S}$ has order (s,t) and every two points at distance 2 have exactly t_2+1 common neighbours. We will also assume that every two points at distance 2 are contained in a unique quad, i.e. a set Q of points satisfying

- if $x, y \in Q$ and $d(x,y) = 1$, then every point on the line through x and y belongs to Q;
- if $x, y \in Q$ and $d(x,y) = 2$, then every common neighbour of x and y belongs to Q;
- the points and lines of $\mathcal{S}$ which are completely contained in Q define a GQ of order (s, t_2), $t_2 \geq 1$.

By Proposition 2.5 of [8], quads certainly exist if $s \geq 2$ and $t_2 \geq 1$. In the sequel, we will use the same notation for the quad and its corresponding generalized quadrangle. The unique quad through two points x and y at distance 2 will be denoted by $Q(x,y)$. Similarly, the unique quad through two intersecting lines K and L will be denoted by $Q(K,L)$. For a point x and a line K of $\mathcal{S}$, let $d(x,K)$ denote the minimum distance between x and a point of K. For every point x (respectively every line K) and every $i \in \mathbb{N}$, let $\Gamma_i(x)$ (respectively $\Gamma_i(K)$) denote the set of all points at distance i from x (respectively K). If $x \in \Gamma_1(K)$, then the unique quad through x and K will be denoted by $Q(x,K)$. For two lines K and L of $\mathcal{S}$, let $d(K,L)$ denote the minimum of $d(k,l)$ over $(k,l) \in K \times L$. By Lemma 1 of [5], there are two possibilities. Either there exist unique points $k \in K$ and $l \in L$ such that $d(K,L) = d(k,l)$, or, for every $k \in K$, there exists a unique $l \in L$ such that $d(K,L) = d(k,l)$. In the latter case K and L are called *parallel* ($\|$). Taking into account the possible values of $d(K,L)$, one can even distinguish into five possibilities:

- $K = L$: we say that $(K,L) \in \mathrm{R}_0$;
- $K \cap L$ is a point: we say that $(K,L) \in \mathrm{R}_1$;
- $K \| L$ and $d(K,L) = 1$: we say that $(K,L) \in \mathrm{R}_2$;
- $K \nparallel L$ and $d(K,L) = 1$: we say that $(K,L) \in \mathrm{R}_3$;
- $K \| L$ and $d(K,L) = 2$: we say that $(K,L) \in \mathrm{R}_4$.

Here R_i, $i \in \{0,\ldots,4\}$, is a relation on the line set $\mathcal{L}$ such that $\mathcal{L} \times \mathcal{L}$ is the disjoint union of R_0, R_1, R_2, R_3 and R_4. If $(K,L) \in R_2$, then the unique quad through K and L will be denoted by $Q(K,L)$. For a fixed line K of $\mathcal{S}$, we define $n_i(K) = |\{L \in \mathcal{L} | (K,L) \in \mathrm{R}_i\}|$. In the following section we will show that $n_i(K)$ does not depend on K. For every pair $(K,L) \in \mathrm{R}_i$, the *intersection number* $p^i_{jk}(K,L)$ is defined as the number of lines M satisfying $(K,M) \in \mathrm{R}_j$ and $(L,M) \in \mathrm{R}_k$. If $p^i_{jk}(K,L)$ is independent from the pair $(K, L) \in R_i$ or if no confusion is possible about the choice of the pair (K, L) we will write p^i_{jk} instead of $p^i_{jk}(K, L)$. In the following section we will prove that all intersection numbers p^i_{jk} are constant if i is not equal to 4. An example shows that the intersection numbers p^4_{jk} are not necessarily constant. Nevertheless we will prove that all intersection numbers p^4_{jk} are constant as soon as p^4_{22} is constant. If all intersection numbers are constant, then the structure $(\mathcal{L}, \{R_0, R_1, R_2, R_3, R_4\})$ is a so-called *symmetric association scheme*, see [1]. In the last section all regular near hexagons with $s \geq t_2$ and constant intersection numbers are determined. Unfortunately, the case $s < t_2$ is still open.

2 The intersection numbers

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a regular near hexagon with parameters (s, t, t_2) and suppose that every two points at distance 2 are contained in a unique quad. The total number v of points in $\mathcal{S}$ is then equal to $1 + s(t+1) + \frac{s^2t(t+1)}{t_2+1} + \frac{s^3t(t-t_2)}{t_2+1}$. For every point-line pair (x, K), we have one of the following possibilities.

(A) If $d(x, K) = 2$, then $t_2 + 1$ lines through x have relation R_3 with K (the $t_2 + 1$ lines through x which are contained in the quad $Q(x, y)$ with y the unique point of K nearest to x). The remaining $t - t_2$ lines through x have relation R_4 with K.

(B) If $d(x, K) = 1$, then one line through x has relation R_1 with K and the other t_2 lines of $Q(x, K)$ through the point x have relation R_2 with K. The remaining $t - t_2$ lines through x have relation R_3 with K.

(C) If $d(x, K) = 0$, then one line through x has relation R_0 with K, while the other t lines have relation R_1 with K.

For every line K of $\mathcal{S}$, we have $|\Gamma_1(K)| = st(s+1)$ and $\Gamma_2(K) = v - (s+1) - |\Gamma_1(K)| = \frac{s^2(s+1)t(t-t_2)}{t_2+1}$. By (A), (B) and (C), we then immediately have that $n_0(K) = 1$, $n_1(K) = (s+1)t$, $n_2(K) = \frac{|\Gamma_1(K)| \cdot t_2}{s+1} = st_2t$, $n_3(K) = |\Gamma_1(K)| \cdot (t - t_2) = st(s+1)(t-t_2)$ and $n_4(K) = \frac{|\Gamma_2(K)| \cdot (t-t_2)}{s+1} = \frac{s^2t(t-t_2)^2}{t_2+1}$. As a consequence the intersection numbers p^0_{ij} are constant for all $i, j \in \{0, \ldots, 4\}$. Clearly, also p^i_{0j} is constant for all $i, j \in \{0, \ldots, 4\}$. If $(K, L) \in R_i$, then $|K \cap \Gamma_j(L)|$ does only depend on i and j and not on the particular choice of (K, L) in R_i. Because of (A), (B) and (C), it then immediately follows that also the intersection numbers p^i_{1j} are constant for all $i, j \in \{0, \ldots, 4\}$.

Lemma 1 *Let $i \in \{0,\ldots,4\}$ be fixed. Then p^i_{jk} is constant for all $j,k \in \{0,\ldots,4\}$ if and only p^i_{22}, p^i_{23} and p^i_{33} are constant.*

Proof. This lemma follows from the following facts:

(1) p^i_{jk} is constant if $j \in \{0,1\}$;

(2) If p^i_{jk} is constant, then also p^i_{kj} is constant and equal to p^i_{jk};

(3) $\sum_{0\le k\le 4} p^i_{jk}(K,L) = n_j(K)$ for every $(K,L) \in \mathrm{R}_i$. □

The intersection numbers p^1_{22}, p^1_{23} and p^1_{33}

Let $(K,L) \in \mathrm{R}_1$, put $K \cap L = \{x\}$ and let $Q := Q(K,L)$. If $(K,M) \in \mathrm{R}_2$ and $(L,M) \in \mathrm{R}_2$, then K and L are lines of $Q(x,M)$; hence $Q(x,M) = Q(K,L)$. In the generalized quadrangle $Q(K,L)$ there are exactly $st_2(t_2-1)$ lines disjoint from K and L; hence $p^1_{22} = st_2(t_2-1)$. Since $(K,M) \in \mathrm{R}_2$ implies that $\mathrm{d}(M,L) \le 1$, we necessarily have that $p^1_{24} = 0$; hence also p^1_{23} is constant. Now, define $X = \{M \in \mathcal{L} \,|\, (K,M) \in \mathrm{R}_3 \text{ and } (L,M) \in \mathrm{R}_3\}$. Suppose now that $M \in X$. Let u and v denote the unique points of M at distance 1 from K and L respectively. If $u \ne v$, then $\mathrm{d}(x,u) = \mathrm{d}(x,v) = 2$, such that $\mathrm{d}(x,w) = 1$ for a certain point w on uv. This is impossible since u is the unique point of M at distance 1 from K. Hence $u = v$. If $u \not\sim x$, then u is collinear with two points of Q and hence is contained in Q. There are now s^2t_2 points in Q not collinear with x and every such point is contained in $t-t_2$ lines not contained in Q. In this way, we obtain $s^2t_2(t-t_2)$ lines which have relation R_3 with both K and L. There are also $s(t_2-1)(t-t_2)$ lines intersecting $Q \setminus (K \cup L)$ in a point collinear with x and each of these lines belongs to X. Through every point y of $\Gamma_1(x) \setminus Q$, there are exactly $(t-2t_2)$ lines not contained in $Q(y,K) \cup Q(y,L)$ and each of these lines belongs to X. Adding all contributions, we find that $p^1_{33} = s^2t_2(t-t_2)+s(t_2-1)(t-t_2)+s(t-t_2)(t-2t_2) = s(t-t_2)(st_2+t-t_2-1)$.

The intersection numbers p^2_{22}, p^2_{23} and p^2_{33}

Let $(K,L) \in \mathrm{R}_2$ and put $Q := Q(K,L)$. Suppose that $(K,M) \in \mathrm{R}_2$, then either $Q(K,M) = Q$ or $Q(K,M) \cap Q = K$. If $Q(K,M) \cap Q = K$, then every point of M has distance 2 to every point of L. Now Q contains $st_2^2 - st_2 - t_2 + s$ lines disjoint with K and L and each of these lines has relation R_2 with both K and L. Hence $p^2_{22} = st_2^2 - st_2 - t_2 + s$ and $p^2_{23} = 0$. We now prove that also p^2_{33} is constant. Put $X := \{M \in \mathcal{L} \,|\, (K,M) \in \mathrm{R}_3 \text{ and } (L,M) \in \mathrm{R}_3\}$. For every $M \in X$, let u_M, respectively v_M, denote the point on M which is collinear with a point u'_M of K, respectively a point v'_M of L. If $u_M = v_M$ then this point belongs to Q since Q is geodetically closed.

If $u_M \neq v_M$, then M is disjoint from Q and $\mathrm{d}(u'_M, v'_M) = 1$. In the quad Q, there are $(s+1)(st_2-1)$ points disjoint from $K \cup L$. Through each of these points, there are $t-t_2$ lines not contained in Q. All $(s+1)(t-t_2)(st_2-1)$ lines obtained this way have relation R_3 with K and L. There are now $s+1$ lines intersecting K and L. Through each such line, there are $\frac{t}{t_2}-1$ quads different from Q and each such a quad contains st_2^2 lines disjoint from Q. Each of these lines has relation R_3 with both K and L. Hence $p_{33}^2 = (s+1)(t-t_2)(st_2-1) + (s+1)\frac{t-t_2}{t_2}st_2^2 = (s+1)(t-t_2)(2st_2-1)$.

The intersection numbers p_{22}^3, p_{23}^3 and p_{33}^3

Since $p_{23}^2 = 0$, also $p_{22}^3 = 0$. We now prove that also p_{23}^3 and p_{33}^3 are constant. Let $(K, L) \in \mathrm{R}_3$ and let p and q denote those points of K and L respectively such that $\mathrm{d}(p,q) = 1$. Put $X := \{M \in \mathcal{L} \mid (K,M) \in \mathrm{R}_2$ and $(L,M) \in \mathrm{R}_3\}$ and $Y := \{M \in \mathcal{L} \mid (K,M) \in \mathrm{R}_3$ and $(L,M) \in \mathrm{R}_3\}$. For every line $M \in X$, let r_M denote the unique point of M collinear with a point u_M of L, and let v_M denote the unique point of K collinear with r_M. Since $\mathrm{d}(u_M, v_M) \leq 2$, one of the following possibilities occurs:

(a) $v_M = p$ and $u_M = q$;

(b) $v_M = p$ and $u_M \neq q$;

(c) $v_M \neq p$ and $u_M = q$.

Let N_i, $i \in \{a,b,c\}$, denote the total number of lines of type (i). If M is a line of type (a), then the point r_M belongs to the line pq. There are now $s-1$ points x in $pq \setminus \{p,q\}$. In the quad $Q(x,K)$, there are then t_2 lines through x different from pq. Since $p_{22}^3 = 0$ each of these lines belongs to X. Hence $N_a = (s-1)t_2$. Counting triples (x,y,M) with $x \in L \setminus \{q\}$, $y \in \Gamma_1(p) \cap \Gamma_1(x) \cap M$, $q \neq y$ and $(K,M) \in \mathrm{R}_2$ gives $N_b = st_2^2$. Counting triples (x,y,M) with $x \in K \setminus \{p\}$, $y \in \Gamma_1(q) \cap \Gamma_1(x) \cap M$, $p \neq y$, $(K,M) \in \mathrm{R}_2$ and $M \cap L = \emptyset$ gives $N_c = st_2(t_2-1)$. It is now clear that $p_{23}^3 = N_a + N_b + N_c$ is constant. If $M \in Y$, let u_M and v_M denote those points of M which are collinear with $u'_M \in K$ and $v'_M \in L$. We distinguish the following cases:

(1) $u_M = v_M$, $u'_M = p$ and $v'_M = q$;

(2) $u_M = v_M$, $u'_M = p$ and $v'_M \neq q$;

(3) $u_M = v_M$, $u'_M \neq p$ and $v'_M = q$;

(4) $u_M = v_M$, $u'_M \neq p$ and $v'_M \neq q$;

(5) $u_M \neq v_M$, $u'_M = p$ and $v'_M = q$;

(6) $u_M \neq v_M$, $u'_M = p$ and $v'_M \neq q$;

(7) $u_M \neq v_M$, $u'_M \neq p$ and $v'_M = q$;

(8) $u_M \neq v_M$, $u'_M \neq p$ and $v'_M \neq q$.

Let N_i, $i \in \{1, \ldots, 8\}$, denote the number of lines of type (i). If M is a line of type (1), then u_M lies on the line pq. From the $t+1$ lines through a point x of $pq \setminus \{p, q\}$, one coincides with pq, $2t_2$ have relation R_2 with either K or L, and the other $t - 2t_2$ have relation R_3 with both K and L. Hence $N_1 = (s-1)(t-2t_2)$. Counting triples (x, y, M) with $x \in L \setminus \{q\}$, $y \in \Gamma_1(p) \cap \Gamma_1(x) \cap M$, $y \neq q$ and $M \in Y$, gives $N_2 = st_2(t - 2t_2)$. For reasons of symmetry, we have that $N_3 = N_2$. If M is a line of type (4), then (u'_M, u_M, v'_M) is a path of length 2. But $d(u'_M, v'_M) \leq 2$ implies that either $u'_M = p$ or $v'_M = q$. So $N_4 = 0$. If M is a line of type (5), then $(pq, M) \in R_2$ and hence the quad $Q(pq, M)$ can be defined. There are now $\frac{t}{t_2} - 2$ quads through pq different from $Q(pq, K)$ and $Q(pq, L)$. In each such a quad there are st_2^2 lines disjoint from pq and every such line belongs to Y. Hence $N_5 = st_2(t - 2t_2)$. Suppose now that M is a line of type (6). Since $d(q, u_M) = d(q, v_M) = 2$, there exists a point w on M collinear with q, a contradiction. Hence $N_6 = 0$. Similarly $N_7 = 0$. If M is a line of type (8), then (u'_M, u_M, v_M, v'_M) is a path of length 3. Moreover $u_M \notin Q(pq, K)$ and $v_M \notin Q(pq, L)$. Fix now a point $x \in K \setminus \{p\}$ and a point $y \in L \setminus \{q\}$ and consider all $(t+1)(t_2+1)$ paths of the form (x, z_1, z_2, y).

- Exactly one path satisfies $z_1 = p$ and $z_2 = q$.
- Exactly t_2 paths satisfy $z_1 = p$ and $z_2 \neq q$.
- Exactly t_2 paths satisfy $z_1 \neq p$ and $z_2 = p$.
- Exactly t_2^2 paths satisfy $z_1 \in Q(pq, K)$ and $z_2 \notin Q(pq, L)$.
- Exactly t_2^2 paths satisfy $z_1 \notin Q(pq, K)$ and $z_2 \in Q(pq, L)$.
- The remaining $tt_2 + t - t_2 - 2t_2^2$ paths satisfy $z_1 z_2 \in Y$.

Hence $N_8 = s^2(tt_2 + t - t_2 - 2t_2^2)$. It is now clear that also $p^3_{33} = N_1 + \cdots + N_8$ is constant.

The intersection numbers p^4_{22}, p^4_{23} and p^4_{33}

We have seen that the numbers $n_2(K)$ and $n_4(K)$ are independent from the chosen line K. We respectively have $n_2 = st_2t$ and $n_4 = \frac{s^2t(t-t_2)^2}{t_2+1}$. The intersection number p^2_{24} is also constant and equal to $n_2 - p^2_{20} - p^2_{21} - p^2_{22} - p^2_{23} = st_2t - 1 - (s+1)(t_2-1) - (st_2^2 - st_2 - t_2 + s) - 0 = st_2(t - t_2)$.

If p^4_{22} is constant, then $n_2p^2_{24} = n_4p^4_{22}$ and hence $\frac{n_2p^2_{24}}{n_4} = \frac{t_2^2(t_2+1)}{t-t_2} \in \mathbb{N}$. The unique near hexagon with parameters $(s,t,t_2) = (2,11,1)$, see [3] and [8], proves that this condition is not always satisfied. Nevertheless we are able to prove the following result.

Theorem 1 *If p^4_{22} is constant, then all intersection numbers are constant.*

Proof. Suppose that p^4_{22} is constant. Let $(K,L) \in \mathrm{R}_4$. Counting triples (x,y,M) with $x \in K$, $y \in \Gamma_1(x) \cap \Gamma_1(L) \cap M$ and $M \| K$ gives $p^4_{23} + (s+1)p^4_{22} = (s+1)(t_2+1)t_2$, proving that p^4_{23} is constant. It remains to show that also p^4_{33} is constant. Put $X := \{M \in \mathcal{L} \,|\, (K,M),(L,M) \in \mathrm{R}_3\}$. For every $M \in X$, let u_M, respectively v_M, denote the unique points of M at distance 1 from a point u'_M of K, respectively a point v'_M of L. We consider the following cases:

(1) $u_M \neq v_M$ and $\mathrm{d}(u'_M, v'_M) = 2$;

(2) $u_M = v_M$;

(3) $u_M \neq v_M$ and $\mathrm{d}(u'_M, v'_M) = 3$.

Let N_i, $i \in \{1,2,3\}$, denote the number of lines of type (i). Suppose now that the line M is of type (1). Let $Q := Q(u'_M, v'_M)$. Since u'_M has distance 2 to v_M and v'_M, there exists a point w on $v_Mv'_M$ collinear with u_M. Since Q is geodetically closed, $w \in Q$, $v_M \in Q$ and $u_M \in Q$. Hence the line M is completely contained in Q. There are now $s+1$ quads R intersecting K and L and all $(1+t_2)(1+st_2) - 2(1+t_2) - (1+t_2)(t_2-1)$ lines in such a quad not containing $R \cap K$, $R \cap L$ or any common neighbour of these two points have relation R_3 with both K and L. This proves that $N_1 = (s+1)[(1+t_2)(1+st_2) - 2(1+t_2) - (1+t_2)(t_2-1)]$. Counting all triples (x,y,M) with $x \in K$, $y \in \Gamma_1(x) \cap \Gamma_1(L) \cap M$ and $M \cap (K \cup L) = \emptyset$ gives that $(s+1)(t_2+1)(t-1) = N_2 + p^4_{23} + p^4_{32} + (s+1)p^4_{22}$, proving that N_2 is constant. Counting all 4-tuples (x,y,z,M) with $x \in K$, $y \in L \cap \Gamma_3(x)$, $z \in \Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(K) \cap \Gamma_2(L) \cap M$ and $\mathrm{d}(y,M) = 1$ gives $(s+1)s(t-t_2-1)(t_2+1) = N_3 + sp^4_{23}$, proving that N_3 is constant. Hence $p^4_{33} = N_1 + N_2 + N_3$ is also constant. □

Examples

In [8] it was proved that the following incidence structure $\mathcal{T}$ is a regular near hexagon with parameters $(s,t,t_2) = (2,14,2)$. The points, respectively lines, of $\mathcal{T}$ are the blocks, respectively triples of two by two disjoint blocks, of the unique Steiner system $S(5,8,24)$ ([2]). By [4], $\mathcal{T}$ is the unique regular near hexagon with parameters $(s,t,t_2) = (2,14,2)$. Also by [4], page 59, p^4_{22} is constant and equal to 1. Hence all intersection numbers are constant. Every other known regular near hexagon with $s \geq 2$, $t_2 \geq 1$ and constant intersection numbers is *classical*, i.e. satisfies $\mathrm{d}(x, Q) \leq 1$ for every point x and every quad Q. By Cameron [6], a near hexagon with this property necessarily is a dual polar space, i.e. the points, respectively lines, of the near hexagon are the maximal, respectively next-to-maximal, singular subspaces of a polar space of rank 3, and incidence is reverse containment. The cube with $(s+1)^3$ vertices and parameters $(s, t, t_2) = (s, 2, 1)$ is an example of a classical near hexagon. The other classical near hexagons are listed in the following table. The classical near hexagon related to $Q(6, q)$ is isomorphic to the one related to $W(5, q)$ if and only if q is even.

POLAR SPACE	TYPE	(s, t, t_2)
$Q(6, q)$	quadratic	(q, q^2+q, q)
$W(5, q)$	symplectic	(q, q^2+q, q)
$Q^-(7, q)$	quadratic	(q^2, q^2+q, q)
$H(5, q^2)$	hermitian	(q, q^4+q^2, q^2)
$H(6, q^2)$	hermitian	(q^3, q^4+q^2, q^2)

Proposition 1 *Let $\mathcal{S}$ be a regular near hexagon which is also classical, then p^4_{22} is constant.*

Proof. Let K and L be two lines such that $\mathrm{d}(K, L) = 2$ and let k and l be two points on K and L such that $\mathrm{d}(k, l) = 2$. If M is a line having relation R_2 with K and M, then there is a point on M collinear with k and l. The quad through that point and the line K necessarily contains M. Now, let m be one of the $t_2 + 1$ common neighbours of k and l, and put $Q := Q(m, K)$. Let q denote an arbitrary point on L different from l. Since $\mathcal{S}$ is classical, q is collinear with a (necessarily unique point) q' of Q. Since Q is geodetically closed $\mathrm{d}(q, x) = \mathrm{d}(q, q') + \mathrm{d}(q', x)$ for every point $x \in Q$. Hence m and q' are collinear. The line mq' clearly is the unique line through m having relation R_2 with K and L. This proves that $p^4_{22} = t_2 + 1$. □

3 The case $s \geq t_2$

Theorem 2 *Every nonclassical regular near hexagon with parameters (s, t, t_2) satisfying:*

(i) every two points at distance 2 are contained in a unique quad,

(ii) $s \geq t_2$,

(iii) p^4_{22} is constant,

is either isomorphic to $\mathcal{T}$ or to the unique near hexagon whose collinearity graph is the incidence graph of the unique biplane of order 2.

Proof. Suppose that $\mathcal{S}$ satisfies the conditions of the theorem. If $s = 1$ then $t_2 = 1$, and the condition $p^4_{22} = \frac{(1+t_2)t_2^2}{t-t_2} \in \mathbb{N}$ implies that $t \in \{2, 3\}$. If $t = 2$, then $\mathcal{S}$ necessarily is a cube, but this contradicts the fact that $\mathcal{S}$ is nonclassical. If $t = 3$, then the collinearity graph Γ of $\mathcal{S}$ is the incidence graph of a block design $\mathcal{D}$. From the parameters of $\mathcal{S}$, we easily derive that $\mathcal{D}$ has 7 points and 7 blocks, every point (block) is incident with 4 blocks (points) and every two points (blocks) are incident with 2 blocks (points). By [2] $\mathcal{D}$ is the unique biplane of order 2. A description of Γ easily follows. Take an arbitrary vertix and label it ∞. Label the vertices adjacent with ∞ with the elements of $X := \{1, 2, 3, 4\}$. Every vertex at distance two from ∞ has two common neighbours adjacent with ∞ and hence corresponds to a subset of size two of X. This correspondence clearly is bijective. Take now a point u at distance 3 from ∞, and let $\{a, b\}$ and $\{c, d\}$ denote the two points at distance 2 from ∞ which are not adjacent to u. We necessarily have that $\{a, b\} \cap \{c, d\} = \emptyset$. [If for instance $a = c = 1$, $b = 2$ and $d = 3$, then $\{1, 4\}$, $\{2, 4\}$ and $\{3, 4\}$ would be three common neighbours of u and the point with label 4.] As a consequence the three points at distance 3 from ∞ correspond to the three partitions of X in two subsets of size 2. If K and L are two lines satisfying $\mathrm{d}(K, L) = 2$, then we may suppose that $K = \{\infty, a\}$ and $L = \{\{b, c\}, \{\{a, b\}, \{c, d\}\}\}$ with $\{a, b, c, d\} = \{1, 2, 3, 4\}$. The line $M = \{c, \{a, c\}\}$ clearly is the unique line having relation R_2 with both K and L; hence $p^4_{22} = 1$. If $s \geq 2$ and if $\mathcal{S}$ is not a dual polar space then, by Lemma 25 of [5], $1 + t \geq (1 + s)(1 + st_2)$. Since $s \geq t_2$, $1 + t \geq (1 + t_2)(1 + t_2^2)$, or equivalently $p^4_{22} = \frac{t_2^2(t_2+1)}{t-t_2} \leq 1$. Since $p^4_{22} \geq 1$, $1 + t = (1 + st_2)(1 + s)$ and $s = t_2$. By Theorem 5 of [5] these properties are sufficient to conclude that $\mathcal{S}$ is isomorphic to $\mathcal{T}$, the near hexagon related to $S(5, 8, 24)$. □

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