The minimal Kirchhoff index of graphs with k edge-disjoint cycles*

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Abstract

Let $S_n(k;|C_1|,\cdots,|C_k|)$ $(k \geq 3)$ denote the n-vertex connected graph obtained from k cycles $C_1,\cdots,C_k$ with unique common vertex by attaching $n-\sum_i |C_i|+k-1$ pendent edges to it. In this paper, we show that among all n-vertex graphs with k edge-disjoint cycles, the following graphs have minimal Kirchhoff indices: (i) $n \leq 12$. $S_7(3;3,3,3)$; $S_8(3;3,3,4)$; $S_9(3;3,4,4)$; $S_n(3;4,4,4)$ $(n=10,11)$; $S_{12}(3;3,3,3)$, $S_{12}(3;3,3,4)$, $S_{12}(3;3,4,4)$ and $S_{12}(3;4,4,4)$; $S_9(4;3,3,3,3)$; $S_{10}(4;3,3,3,4)$; $S_{11}(4;3,3,4,4)$; $S_{12}(4;3,3,3,3)$; $S_{12}(4;3,3,3,4)$; $S_{12}(4;3,3,4,4)$ and $S_{12}(4;3,4,4,4)$; $S_{11}(5;3,3,3,3,3)$; $S_{12}(5;3,3,3,3,3)$ and $S_{12}(5;3,3,3,3,4)$; (ii) $n>12$. $S_n(k;3,\cdots,3)$. In addition, we obtain the lower bounds of Kirchhoff index of n-vertex graphs with k edge-disjoint cycles

1 Introduction

In 1993, Klein and Randić [1] defined a new distance function named resistance distance on the basis of electrical network theory. Let G be a connected graph with vertices labelled as $v_1, v_2, \cdots, v_n$. They view G as an electrical network N by replacing each edge of G with a unit resistor. The resistance distance between v_i and v_j, denoted by $r(v_i, v_j)$ (if more than one graphs are considered, we write $r_G(v_i, v_j)$ in order to avoid confusion), is defined to be the effective resistance between them in N. Recall that the conventional distance between vertices v_i and v_j, denoted by $d(v_i, v_j)$, is the length of a shortest path between them and the famous Wiener index

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$W(G)$ [2] is the sum of distances between all pairs of vertices; that is,

$$W(G) = \sum_{i<j} d(v_i, v_j).$$

Analogue to Wiener index, the Kirchhoff index $Kf(G)$ [2] is defined as:

$$Kf(G) = \sum_{i<j} r(v_i, v_j).$$

As an intrinsic graph metric and a relevant tool to characterize wave- or fluid-like communication between two vertices [3], resistance distance has been studied extensively in mathematics and chemistry, such as [4, 5, 6, 7]. It is computed in a variety of ways, such as [1, 8, 9, 10, 11]. As a new structure-descriptor [7], Kirchhoff index has attracted more and more attention. On one hand, simple closed-form formulae or numerical values of Kirchhoff index of some classes of graphs have been obtained, such as complete graphs [12], cycles [12, 13], platonic solids [12, 14], some fullerenes including buckminsterfullerene [15, 16, 17], distance transitive graphs [18], circulant graphs [19], linear hexagonal chains [20], and so on [3, 17, 18, 21]. On the other hand, although the formulae of Kirchhoff index of some classes of graphs are difficult to get, their sharp bounds for Kirchhoff index can be obtained and graphs with extremal Kirchhoff index can be characterized as well. Such as general connected graph [13], circulant graph [19].

Since Kirchhoff index and Wiener index of trees coincide [1] and Wiener indices of graphs are extensively studied, so it is natural to consider Kirchhoff index of graphs with cycles. In Ref. [22] and [23], unicyclic graphs and bicyclic graphs with extremal Kirchhoff index are characterized and sharp bounds for Kirchhoff index of such graphs are obtained as well, respectively. Now we generalize one result, that is, consider the lower bounds of Kirchhoff index of graphs with k edge-disjoint cycles ($k \geq 3$) and characterize the extremal graphs. A graph with k edge-disjoint cycles contains k cycles and any two cycles have at most one common vertices.

For convenience, we employ some notations. Let $\mathcal{G}_n^k$ be the set of n-vertex connected graph with k edge-disjoint cycles $C_1, C_2, \cdots, C_k$. We call cycles C_i and C_j adjacent if there is only one path P_{ij} connecting the two cycles (that is, the internal vertices of the path are not on any cycle), and denote $C_i \sim C_j$. Let $\mathcal{S} = \bigcup_{i=1}^{k} V(C_i) \bigcup (\bigcup_{C_i \sim C_j} V(P_{ij}))$. Trees T_j ($1 \leq j \leq |\mathcal{S}|$) is rooted $v_j \in \mathcal{S}$. We say tree T trivial if $|V(T)| = 1$, i.e., T is a singleton vertex. Fig. 1 illustrates an example of graphs in $\mathcal{G}_n^k$. Let $S_n(k; |C_1|, \cdots, |C_k|)$ denote the n-vertex connected graph obtained from cycles $C_1, \cdots, C_k$ with unique common vertex by attaching $n - \sum_i |C_i| + k - 1$ pendent edges to it. See Fig. 2.

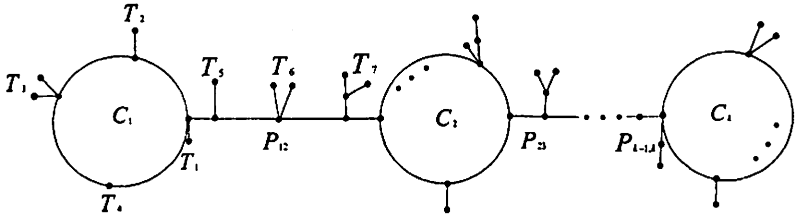

Fig. 1. Illustration the graphs in $\mathcal{G}_n^k$.

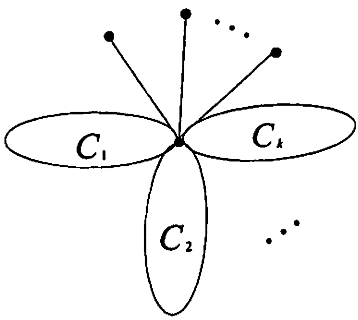

Fig. 2. $S_n(k; |C_1|, \cdots, |C_k|)$.

In the second section, we characterize the extremal graphs with minimal Kirchhoff index among all n-vertex graphs with k edge-disjoint cycles. By Theorem 2.6, we obtain that the following graphs have the minimal Kirchhoff indices: (i) $n \leq 12$. For $k = 3$, $S_7(3; 3, 3, 3)$; $S_8(3; 3, 3, 4)$; $S_9(3; 3, 4, 4)$; $S_n(3; 4, 4, 4)$ $(n = 10, 11)$; $S_{12}(3; 3, 3, 3), S_{12}(3; 3, 3, 4), S_{12}(3; 3, 4, 4)$ and $S_{12}(3; 4, 4, 4)$; for $k = 4$, $S_9(4; 3, 3, 3, 3)$; $S_{10}(4; 3, 3, 3, 4)$; $S_{11}(4; 3, 3, 4, 4)$; $S_{12}(4; 3, 3, 3, 3)$, $S_{12}(4; 3, 3, 3, 4)$, $S_{12}(4; 3, 3, 4, 4)$ and $S_{12}(4; 3, 4, 4, 4)$; for $k = 5$, $S_{11}(5; 3, 3, 3, 3, 3)$; $S_{12}(5; 3, 3, 3, 3, 3)$ and $S_{12}(5; 3, 3, 3, 3, 4)$. (ii) $n > 12$. $S_n(k; 3, \cdots, 3)$. Then the lower bound of Kirchhoff index of n-vertex graphs with k edge-disjoint cycles is derived as well.

2 The extremal graphs with minimal Kirchhoff index in $\mathcal{G}_n^k$

In this section, we first find out the graphs with minimal Kirchhoff index when the length of the k cycles is fixed. Furthermore, we characterize the extremal graphs of Kirchhoff index in $\mathcal{G}_n^k$.

Lemma 2.1. *[24] Let T be a n-vertex tree different from P_n and S_n. Then*

$$W(S_n) < W(T) < W(P_n).$$

It is obtained in [24] that

$$W(S_n) = (n-1)^2.$$

$$Kf(C_n) = \frac{n^3 - n}{12}.$$

Lemma 2.2. *[1] Let x be a cut-vertex of a connected graph, and let a and b be vertices occurring in different components which arise upon deletion of x. Then*

$$r(a,b) = r(a,x) + r(x,b).$$

For $v_i \in V(G)$, define $Kf_{v_i}(G)$ as the sum of resistance distances between v_i and other vertices of G; that is,

$$Kf_{v_i}(G) = \sum_{v_j \in V(G)} r_G(v_i, v_j).$$

$$Kf_v(C_n) = \frac{n^2 - 1}{6},$$

where v is a vertex of C_n.

By Lemma 2.2, we obtain the following theorem in [23].

Theorem 2.3. *[23] Let x be a cut-vertex of a connected graph G such that $G - x$ has exactly two components G_1 and G_2. Let G'_i be the subgraph of G induced by $V(G_i) \bigcup \{x\}$ $(i = 1, 2)$. Then*

$$Kf(G) = Kf(G'_1) + Kf(G'_2) + (|V(G'_1)| - 1)Kf_x(G'_2) + (|V(G'_2)| - 1)Kf_x(G'_1).$$

Lemma 2.4. *Let G be a connected graph. Assume that S_1 and S_2 are two stars of G with centers v_1 and v_2, and u_1 and u_2 are leaves of S_1 and S_2, respectively. If $Kf_{u_1}(G) \leq Kf_{u_2}(G)$, let $G' = G - v_2u_2 + v_1u_2$. Then*

$$Kf(G') < Kf(G).$$

Proof. For any two vertices $v_k, v_l \in V(G) \backslash \{u_2\}, r_G(v_k, v_l) = r_{G'}(v_k, v_l)$. While

$$Kf_{u_2}(G') = Kf_{u_1}(G') = Kf_{u_1}(G) - r_G(u_1, u_2) + 2 < Kf_{u_1}(G) \leq Kf_{u_2}(G).$$

Therefore,

$$Kf(G') = Kf(G) - Kf_{u_2}(G) + Kf_{u_2}(G') < Kf(G).$$

□

The following Theorem show that among $\mathcal{G}_n^k$, when the length of cycles $C_1, C_2, \cdots, C_k$ is fixed, $S_n(k; |C_1|, \cdots, |C_k|)$ has minimal Kirchhoff index.

Theorem 2.5. *Let $G \in \mathcal{G}_n^k$ with cycles $C_1, C_2, \cdots, C_k$. If $G \neq S_n(k; |C_1|, \cdots, |C_k|)$, then $Kf(G) > Kf(S_n(k; |C_1|, \cdots, |C_k|))$.*

Proof. Suppose that graph $G_0 \in \mathcal{G}_n^k$ with cycles $C_1, C_2, \cdots, C_k$ has minimal Kirchhoff index. For G_0, we prove the following Claims.

Claim 1. For any C_i and C_j, if $C_i \sim C_j$, then P_{ij} is of 0 length.

Suppose to the contrary that there is one path P_{ij} joining C_i and C_j with length l ($l \geqslant 1$) (whose endpoints lie in C_i and C_j, respectively). Let $e = uv$ be an edge of P_{ij}. Let G_1 be the graph obtained from G_0 by first contracting e, then attaching a pendent edge ua to u. Assume that G_{01} and G_{02} are two components of $G_0 - e$ and G_{11} and G_{12} are copies of G_{01} and G_{02} in G_1, respectively. See Fig. 3.

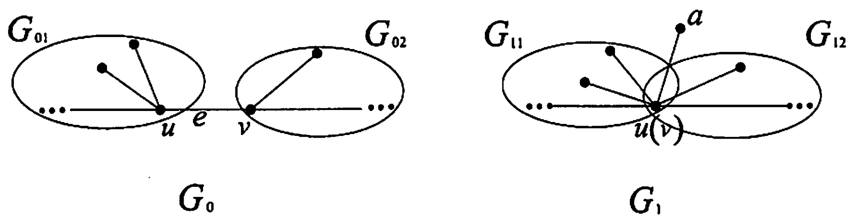

Fig. 3. Graphs G_0 and G_1.

In the following, we prove $Kf(G_1) < Kf(G_0)$.

For $x, y \in V(G_{01})\backslash\{u\}$ or $V(G_{02})\backslash\{v\}$, $r_{G_1}(x, y) = r_{G_0}(x, y)$; and for $x \in V(G_{01})\backslash\{u\}, y \in V(G_{02})\backslash\{v\}$, $r_{G_1}(x, y) = r_{G_0}(x, y) - 1 < r_{G_0}(x, y)$. On the other hand,

$$\sum_{x \in V(G_{01})} r_{G_0}(x, v) = \sum_{x \in V(G_{11})} r_{G_1}(x, a),$$

$$\sum_{x \in V(G_{02})} r_{G_0}(x, u) = \sum_{x \in V(G_{12})} r_{G_1}(x, a).$$

So

$$\begin{aligned}
&Kf_u(G_0)+Kf_v(G_0)\\
=&Kf_u(G_{01})+\sum_{x\in V(G_{02})} r_{G_0}(x,u)+Kf_v(G_{02})+\sum_{x\in V(G_{01})} r_{G_0}(x,v)\\
=&Kf_u(G_{11})+\sum_{x\in V(G_{12})} r_{G_1}(x,a)+Kf_v(G_{12})+\sum_{x\in V(G_{11})} r_{G_1}(x,a)\\
=&(Kf_u(G_{11})+Kf_u(G_{12})+r_{G_1}(a,u))\\
&+(\sum_{x\in V(G_{12})} r_{G_1}(x,a)+\sum_{x\in V(G_{11})} r_{G_1}(x,a)-r_{G_1}(a,u))\\
=&Kf_u(G_1)+Kf_a(G_1).
\end{aligned}$$

By the definition of Kirchhoff index, we obtain $Kf(G_1) < Kf(G_0)$. This contradicts the hypothesis. Hence Claim 1 holds.

Claim 2. T_i is star with $v_i \in \mathcal{S}$ as center $(1 \le i \le |\mathcal{S}|)$.

By Claim 1, we know that $\mathcal{S} = \bigcup_{i=1}^{k} V(C_i)$. Without loss of generality, suppose that tree T_j rooted $v_j \in \mathcal{S}$ is not star. Let G_2 be constructed from G_0 by first deleting all edges of T_j, then connecting all isolated vertices to v_j; that is, T_j is a star in G_2 with center v_j, denoted by S_j. For G_0, v_j is a cut vertex, T_j and $G_0-(V(T_j)-v_j)$ are two induced subgraphs of G_0. By Lemma 2.1, $Kf(S_j) < Kf(T_j)$. On the other hand, $Kf_{v_j}(S_j) < Kf_{v_j}(T_j)$, by Theorem 2.3, $Kf(G_2) < Kf(G_0)$, which contradicts the choice of G_0. Hence Claim 2 holds.

By Claim 1, we know $\mathcal{S} = \bigcup_{i=1}^{k} V(C_i)$ and let $A \subset \mathcal{S}$ collect all the vertices shared by at least two cycles.

Claim 3. Let $B = \mathcal{S}\setminus A = \{v_1, \cdots, v_r\}$. If $\sum_{i=1}^{n} |C_i| < n+k-1$, then T_i $(1 \le i \le r)$ is trivial.

Suppose that T_j rooted $v_j \in B$ is not trivial, then there is a vertex $u \in A$ such that u and v_j are on the same cycle, and T_j and T_u are both stars by Claim 2. Let G_{01}^* be the component of G_0-u containing T_j and G_{01} be the subgraph induced by $V(G_{01}^*)\bigcup V(T_u)$. And let G_{02} be the subgraph of G_0 induced by $V(G_0)\backslash V(G_{01})\bigcup\{u\}$.

If $Kf_{v_j}(G_{01}) < Kf_u(G_{01})$, construct G_0' by identifying v_j of G_{01} with u of G_{02}. Then by Theorem 2.3, $Kf(G_0') < Kf(G_0)$, a contradiction. If $Kf_{v_j}(G_{01}) \ge Kf_u(G_{01})$, assume that a and b are leaves of T_u and T_j, respectively. Then

$$Kf_a(G_{01}) = Kf_u(G_{01}) - 1 + |V(G_{01})| - 1,$$

$$Kf_b(G_{01}) = Kf_{v_j}(G_{01}) - 1 + |V(G_{01})| - 1,$$

so $Kf_b(G_{01}) \geq Kf_a(G_{01})$. Let $G'_0 = G_0 - v_j b + ub$, and let G'_{02} be the copy of G_{02} in G'_0 and G'_{01} be the subgraph of G'_0 induced by $V(G'_0)\backslash V(G'_{02}) \bigcup \{u\}$. By Lemma 2.4, $Kf(G'_{01}) \leq Kf(G_{01})$. Also $Kf_u(G'_{01}) < Kf_u(G_{01})$, since $Kf_u(G'_{01}) = Kf_u(G_{01}) - r_{G_{01}}(b,u) + 1$. By Theorem 2.3, we obtain $Kf(G'_0) < Kf(G_0)$, a contradiction. Hence Claim 3 holds.

Claim 4. All cycles have exactly one common vertex.

Conveniently, we denote the common vertex u_1. If not, without losing generality, there is cycle C_i which does not contain u_1 and has a common vertex u_i with some cycle C_j. Let G_{01} be the subgraph of G_0 induced by $V(C_i) \bigcup V(T_{u_i})$ and G_{02} be the subgraph of G_0 induced by $V(G_0)\backslash V(G_{01}) \bigcup \{u_i\}$. Then $Kf_u(G_{02}) < Kf_{u_i}(G_{02})$. Since

$$\begin{aligned} & Kf_{u_i}(G_{02}) \\ &= \sum_{x \in V(C_j)} r(x,u_i) + \sum_{x \in (V(G_{02})\backslash V(C_j))} r(x,u_i) \\ &= \sum_{x \in V(C_j)} r(x,u) + \sum_{x \in (V(G_{02})\backslash V(C_j))} r(x,u) + |V(G_{02})\backslash V(C_j)| r(u,u_i) \\ &= Kf_u(G_{02}) + |V(G_{02})\backslash V(C_j)| r(u,u_i). \end{aligned}$$

Construct G'_0 by identifying u_i of G_{01} with u of G_{02}. By Theorem 2.3,

$$\begin{aligned} & Kf(G'_0) = Kf(G_{01}) + Kf(G_{02}) + (|V(G_{01})| - 1) Kf_u(G_{02}) + (|V(G_{02})| - 1) \\ & Kf_u(G_{01}) < Kf(G_{01}) + Kf(G_{02}) + (|V(G_{01})| - 1) Kf_{u_i}(G_{02}) + (|V(G_{02})| \\ & - 1) Kf_{u_i}(G_{01}) = Kf(G_0). \end{aligned}$$

A contradiction. Hence Claim 4 holds.

Summing up, we prove that G_0 is $S_n(k; |C_1|, \cdots, |C_k|)$. □

By Theorem 2.5, we know the graph with minimal Kirchhoff index in $\mathcal{G}_n^k$ must be in $\{S_n(k; |C_1|, \cdots, |C_k|) \mid 3 \leq |C_i| \leq n - 2k - 2 \text{ for } 1 \leq i \leq k\}$. In the following, we will investigate how long the length of the cycle is, the Kirchhoff index is minimal.

Theorem 2.6. *In $\mathcal{G}_n^k$, the following graphs have minimal Kirchhoff indices:*
(i) $n \leqslant 12$.
(a) $k = 3$. $S(7; 3,3,3)$;$S_9(3; 3,4,4)$;$S_n(3; 4,4,4)$ $(n = 10, 11)$; $S_{12}(3; 3,3,3)$, $S_{12}(3; 3,3,4)$, $S_{12}(3; 3,4,4)$ *and* $S_{12}(3; 4,4,4)$.
(b) $k = 4$. $S_9(4; 3,3,3,3)$; $S_{10}(4; 3,3,3,4)$; $S_{11}(4; 3,3,4,4)$; $S_{12}(4; 3,3,3,3)$, $S_{12}(4; 3,3,3,$
$4)$, $S_{12}(4; 3,3,4,4)$ *and* $S_{12}(4; 3,4,4,4)$.
(c) $k = 5$. $S_{11}(5; 3,3,3,3,3)$; $S_{12}(5; 3,3,3,3,3)$ *and* $S_{12}(5; 3,3,3,3,4)$.
(ii) $n > 12$. $S_n(k; 3,3,\ldots,3)$.

Proof. When $n = 7$, $S_7(3;3,3,3)$ is the unique graph, and has minimal Kirchhoff index. When $n = 8$, there are only two graphs: $S_8(3;3,3,3)$ and $S_8(3;3,3,4)$. Since $Kf(S_8(3;3,3,3) = 33$ and $Kf(S_8(3;3,3,4) = \frac{97}{3}$, then $S_8(3;3,3,4)$ has minimal Kirchhoff index.

For $n \geq 9$, in $S_n(k;|C_1|,\cdots,|C_k|)$, all cycles have unique common vertex u and there is a unique tree rooted u. Since all cycles are symmetric, without loss of generality, we take cycle C_1 with length l as an example and analysis how long it is, the Kirchhoff index is minimal. Let G_{01} be the subgraph of $S_n(k;|C_1|,\cdots,|C_k|)$ induced by $V(S_n(k;|C_1|,\cdots,|C_k|) - C_1)\bigcup\{u\}$ and G_{02} be cycle C_1. Let G'_{01} be the copy of G_{01} in $S_n(k;|C_1|-1,\cdots,|C_k|)$ and G'_{02} be the subgraph of $S_n(k;|C_1|-1,\cdots,|C_k|)$ induced by $V(S_n(k;|C_1|-1,\cdots,|C_k|) - G'_{01})\bigcup\{u\}$. By Theorem 2.3, we compute

$$\begin{aligned}
&Kf(S_n(k;|C_1|,\cdots,|C_k|)) - Kf(S_n(k;|C_1|-1,\cdots,|C_k|))\\
=&Kf(G_{01}) + Kf(G_{02}) + (|G_{01}|-1)kf_{u_1}(G_{02}) + (|G_{02}|-1)kf_{u_1}(G_{01})\\
&-[Kf(G'_{01}) + Kf(G'_{02}) + (|G'_{01}|-1)kf_{u_1}(G'_{02}) + (|G'_{02}|-1)kf_{u_1}(G'_{01})]\\
=&\frac{-3l^2 + (4n+3)l + (12-14n)}{12}.
\end{aligned}$$

Let

$$g(l) := -3l^2 + (4n+3)l + (12-14n) \quad (3 \leq l \leq n-4),$$

and

$$h(n) := \Delta_{g(l)} = 16n^2 - 144n + 153 \quad (n \geq 7).$$

When $n \geq 9$, $h(n) > 0$. The two roots of $g(l)$ are

$$l_1 = \frac{3 + 4n - \sqrt{(4n-18)^2 - 171}}{6},$$

and

$$l_2 = \frac{3 + 4n + \sqrt{(4n-18)^2 - 171}}{6}.$$

Simple calculations show that $l_2 > n-4$, hence $l \leq n-4 < l_2$.

Case 1. $9 \leq n \leq 11$. Then $4 < l_1 < 5$ and $l \leq 7$. This indicates that for $l \geq 5$, $g(l) > 0$, namely, $Kf(S_n(k;|C_1|,\ldots,|C_k|)) > Kf(S_n(k;|C_1|-1,\ldots,|C_k|))$. On the other hand, $g(4) < 0$, $Kf(S_n(k;4,\ldots,|C_k|) < Kf(S_n(k;3,\ldots,|C_k|)$. So when the length of the cycle is 4, the Kirchhoff index is minimal. By the symmetric of cycles in $S_n(k;|C_1|,\cdots,|C_k|)$, we obtain that $S_9(3;3,4,4)$, $S_9(4;3,3,3,3)$, $S_n(3;4,4,4)(n = 10,11)$, $S_{10}(4;3,3,3,4)$, $S_{11}(4;3,3,4,4)$ and $S_{11}(5;3,3,3,3,3)$ have minimal Kirchhoff index.

Case 2. $n = 12$. $l_1 = 4$ and $l \leqslant 8$, namely, $g(4) = 0$. Hence $Kf(S_{12}(k;4,\ldots,|C_k|) = Kf(S_{12}(k;3,\ldots,|C_k|)$. On the other hand, for $l \geq 5$, $g(l) > 0$. That is, $Kf(S_{12}(k;|C_1|,\cdots,$

$|C_k|)) > Kf(S_{12}(k;|C_1|-1,\ldots,|C_k|))$. So when the length of cycle is 4 or 3, the Kirchhoff index is minimal. By the symmetric of cycles in $\{S_n(k;|C_1|,\cdots,|C_k|)$, we have that $S_{12}(3;3,3,3)$, $S_{12}(3;3,3,4)$, $S_{12}(3;3,4,4)$ and $S_{12}(3;4,4,4)$; $S_{12}(4;3,3,3,3)$, $S_{12}(4;3,3,3,4)$, $S_{12}(4;3,3,4,4)$ and $S_{12}(4;3,4,4,4)$; $S_{12}(5;3,3,3,3,3)$ and $S_{12}(5;3,3,3,3,4)$ have the minimal Kirchhoff index.

Case 3. $n > 12$. Then $l_1 < 4$. For $l \geq 4$, $g(l) > 0$, namely, $Kf(S_n(k;|C_1|,\ldots,|C_k|)) > Kf(S_n(k;|C_1|-1,\ldots,|C_k|))$. It means when the length of cycle is 3, the Kirchhoff index is minimal. Hence $S_n(k;3,\ldots,3)$ has the minimal Kirchhoff index .

Summing up, Theorem 2.6 is proved. □

3 The lower bounds of Kirchhoff index of graphs in $\mathcal{G}_n^k$

In this section, we compute the Kirchhoff indices of extremal graphs with minimal Kirchhoff index and the lower bounds of Kirchhoff index of graphs in $\mathcal{G}_n^k$ are derived, too.

For $S_n(k;3,3,\ldots,3)$, assume that a is common vertex of k 3-cycles, let G_1 be the subgraph spanned by the k 3-cycles and G_2 be the star induced by $V(S_n(k;3,3,\ldots,3))\backslash V(G_1)$
$\bigcup\{a\}$. Then we have

$$\begin{aligned}
&Kf(G_1) = 2k + C_k^2 \times \frac{4}{3} \times 4 = \frac{8}{3}k^2 - \frac{2}{3}k;\\
&Kf(G_2) = (n-2k-1)^2;\\
&Kf_a(G_1) = \frac{4k}{3};\\
&Kf_a(G_2) = n-2k-1.
\end{aligned}$$

By Theorem 2.3,

$$\begin{aligned}
&Kf(S_n(k;3,3,\ldots,3))\\
&= Kf(G_1) + Kf(G_2) + (|G_1|-1)Kf_a(G_2) + (|G_2|-1)Kf_a(G_1)\\
&= n^2 + (-\frac{2}{3}k - 2)n + 1.
\end{aligned}$$

Put n and k into the equation, we obtain

$$Kf(S_7(3;3,3,3)) = 22; Kf(S_{12}(3;3,3,3)) = Kf(S_{12}(3;3,3,4)) = Kf(S_{12}(3;3,4,4)) = Kf(S_{12}(3;4,4,4)) = 97; Kf(S_9(4;3,3,3,3)) = 40; Kf(S_{12}(4;3,3,3,3)) = Kf(S_{12}(4;3,3,3,4)) = Kf(S_{12}(4;3,3,4,4)) = Kf(S_{12}(4;3,4,4,4)) = 89; Kf(S_{11}(5;3,3,3,3,3)) = \frac{190}{3}; Kf(S_{12}(5;3,3,3,3,3)) = Kf(S_{12}(5;3,3,3,3,4)) = 81.$$

By simple computation, we have

$$Kf(S_8(3;3,3,4) = \frac{94}{3}; Kf(S_9(3;3,4,4)) = 45; Kf(S_{10}(3;4,4,4)) = 60; Kf(S_{11}(3;4,4,4)) = 77.5; Kf(S_{10}(4;3,3,3,4)) = 54; Kf(S_{11}(4;3,3,4,4)) = \frac{211}{3};$$

Combining Theorem 2.6, we can obtain the lower bounds of Kirchhoff index of graphs in $\mathcal{G}_n^k$.

Theorem 3.1. *For any graph* $G \in \mathcal{G}_n^k$,

(i) $n = 7.\ k = 3,\ Kf(G) = 22;$

(ii) $n = 8.\ k = 3,\ Kf(G) \geqslant \frac{94}{3};$

(iii) $n = 9.\ k = 3,\ Kf(G) \geqslant 45;\ k = 4,\ Kf(G) = 40;$

(iv) $n = 10.\ k = 3,\ Kf(G) \geqslant 60;\ k = 4,\ Kf(G) \geqslant 54;$

(v) $n = 11.\ k = 3,\ Kf(G) \geqslant 77.5;\ k = 4,\ Kf(G) \geqslant \frac{211}{3};\ k = 5,\ Kf(G) = \frac{190}{3};$

(vi) $n \geqslant 12.\ Kf(G) \geqslant n^2 + (-\frac{2}{3}k - 2)n + 1.$

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