Two Results on the Hamiltonicity of L_1 – Graphs

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Abstract

A graph G is called an L_1 - graph if, for each triple of vertices u, v, and w with $d(u,v) = 2$ and $w \in N(u) \cap N(v)$, $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| - 1$. Two results on the hamiltonicity of L_1 graphs are presented in this paper.

Keywords : claw – free graphs, L_1 – graphs, hamiltonicity

1. Introduction

We consider only finite undirected graphs without loops and multiple edges. Notation and terminology not defined here follow that in [2]. A graph G is locally connected if $G[N(u)]$ is connected for every vertex u in G. Let G and H be two graphs. A graph G is H – free if G contains no induced subgraph isomorphic to H. If H is $K_{1,3}$, then G is called claw – free. A graph G is 1 – tough if $\omega(G-S) \leq |S|$ for every subset S of $V(G)$ with $\omega(G-S) > 1$, where $\omega(G-S)$ denotes the number of components in the graph $G-S$. For an integer i, a graph G is called an L_i – graph if $d(u)+d(v) \geq |N(u) \cup N(v) \cup N(w)|$ $-i$ or equivalently $|N(u) \cap N(v)| \geq |N(w) - (N(u) \cup N(v))| - i$ for each triple of vertices u, v, and w with $d(u,v) = 2$ and $w \in N(u) \cap N(v)$. It can easily be verified that every claw – free graph is an L_1 – graph (see [1]).

The graph $Y = (V(Y), E(Y))$, where $V(Y) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $E(Y) = \{u_1u_2,\ u_1u_3,\ u_2u_3,\ u_2u_4,\ u_3u_4,\ u_4u_5,\ u_4u_6\}$ and the graph $Z_2 = (V(Z_2), E(Z_2))$, where $V(Z_2) = \{u_1, u_2, u_3, u_4, u_5\}$ and $E(Z_2) = \{u_1u_2, u_2u_3,$

$u_3u_4, u_3u_5, u_4u_5\}$ will be used in this paper. Obviously, every claw – free graph is Y – free. The family $\mathcal{K}$ of graphs is defined as $\{H : K_{p,p+1} \subseteq H \subseteq K_p + (p+1)K_1,\ p \geq 2\}$ and when a graph $H = (V(H), E(H)) \in \mathcal{K}$ we will always let $V(H)$ and $E(H)$ be $\{a_1, a_2, ..., a_p, b_1, b_2, ..., b_{p+1}\}$ and $\{a_ib_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq (p+1)\} \cup S$, where $S \subseteq \{a_ia_j : 1 \leq i < j \leq p\}$, respectively. The families $\mathcal{K}'$ and $\mathcal{K}''$of graphs are defined respectively as $\{H \in \mathcal{K} : H[a_1, a_2, ..., a_p]$ is connected and Y - free $\}$ and $\{H \in \mathcal{K} : H[a_1, a_2, ..., a_p]$ is Z_2 - free $\}$.

The following Theorem 1 was proved by Oberly and Sumner in [6] and Theorem 2 follows from Theorem 1 proved by Gould and Jacobson in [3].

Theorem 1. Every connected locally connected claw – free graph on at least three vertices is hamiltonian.

Theorem 2. If a 2 – connected graph G is claw – free and Z_2 – free, then G is hamiltonian.

Obviously, every connected locally connected graph is 2 – connected. Notice that if G is a noncomplete claw – free graph then G is 1 – tough if and only if G is 2 – connected (see [5]). Thus Theorem 1 and Theorem 2 are equivalent to the following Theorem A and Theorem B, respectively.

Theorem A. Every 1 – tough locally connected claw – free graph on at least three vertices is hamiltonian.

Theorem B. If a 1 – tough graph G is claw – free and Z_2 – free, then G is hamiltonian.

The objective of this paper is to prove the following Theorem 3 and Theorem 4 which generalize Theorem 1 and Theorem 2, respectively.

Theorem 3. Every connected locally connected Y – free L_1 – graph on at least three vertices is hamiltonian or in $\mathcal{K}'$.

Theorem 4. If a 2 – connected graph L_1 – graph G is Z_2 – free, then G is hamiltonian or in $\mathcal{K}''$.

Since every graph in $\mathcal{K}'$ or $\mathcal{K}''$ is not 1 – tough and every 1 – tough graph is 2 – connected, Theorem 3 and Theorem 4 have the following Corollary 1 and Corollary 2, respectively.

Corollary 1. Every 1 – tough locally connected Y – free L_1 – graph on at least three vertices is hamiltonian.

Corollary 2. If a 1 – tough L_1 – graph G is Z_2 – free, then G is hamiltonian.

Obviously, Corollary 1 and Corollary 2 are generalizations of Theorem A and Theorem B, respectively.

We need the following additional notations in the remainder of this paper. If C is a cycle of G, let $\overrightarrow{C}$ denote the cycle C with a given orientation. For u, $v \in C$, let $\overrightarrow{C}[u,v]$ denote the consecutive vertices on C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $\overleftarrow{C}[v,u]$. Both $\overrightarrow{C}[u,v]$ and $\overleftarrow{C}[v,u]$ are considered as paths and vertex sets. If u is on C, then the predecessor, successor, next predecessor and next successor of u along the orientation of C are denoted by u^-, u^+, u^{--} and u^{++} respectively. If $A \subseteq V(C)$, then A^- and A^+ are defined as $\{v^- : v \in A\}$ and $\{v^+ : v \in A\}$ respectively. If H is a connected component of a graph G and u and v are two vertices in H, let uHv denote a path between u and v in H.

2. Lemmas

Lemma 1. Let G be a 2 – connected nonhamiltonian L_1 – graph and $G \notin \mathcal{K}$. Suppose C is a longest cycle with a given orientation in G, H is any connected component of $G[V(G)-V(C)]$, $N(V(H)) \cap V(C) = \{a_1, a_2, ..., a_l\}$ such that $h_i a_i \in E$, where $h_i \in V(H)$ for each i, $1 \le i \le l$, and a_1, a_2, ..., a_l are labeled in the order of the orientation of C. Then

(1). $a_i^- a_i^+ \in E$ for each i, $1 \le i \le l$.

(2). $N(a_i^-) \cap \{ a_j^{--}, a_j^-, a_j \} = \emptyset$ if $1 \le i \ne j \le l$,
$N(a_i^+) \cap \{ a_j^{++}, a_j^+, a_j \} = \emptyset$ if $1 \le i \ne j \le l$.
$N(a_i) \cap \{ a_j^{--}, a_j^-, a_j^{++}, a_j^+ \} = \emptyset$ if $1 \le i \ne j \le l$.

(3). If $x_i \in N(a_i) - \{ a_i^-, h_i \}$, then $x_i a_i^- \in E$ or $x_i h_i \in E$ for each i, $1 \le i \le l$; If $x_i \in N(a_i) - \{ a_i^+, h_i \}$, then $x_i a_i^+ \in E$ or $x_i h_i \in E$ for each i, $1 \le i \le l$.

Proof of Lemma 1. The proof of (1) can be found in the early part of proof of Theorem 3 in [4]. In order to save the space, we will not repeat it here.

Now we prove the first claim in (2). If $a_j^{--} \in N(a_i^-)$, then G has a cycle

$$h_i \overrightarrow{C}[a_i, a_j^{--}] \overleftarrow{C}[a_i^-, a_j^+] a_j^- a_j h_j H h_i$$

which is longer than C, a contradiction. If $a_j^- \in N(a_i^-)$, then G has a cycle

$$h_i \overrightarrow{C}[a_i, a_j^-] \overleftarrow{C}[a_i^-, a_j] h_j H h_i$$

which is longer than C, a contradiction. If $a_j \in N(a_i^-)$, then G has a cycle

$$h_i \overrightarrow{C}[a_i, a_j^-] \overrightarrow{C}[a_j^+, a_i^-] a_j h_j H h_i$$

which is longer than C, a contradiction. Hence $N(a_i^-) \cap \{ a_j^{--}, a_j^-, a_j \} = \emptyset$ if $1 \leq i \neq j \leq l$,

Symmetrically, we can prove that the second claim in (2) is true.

Now we prove the third claim in (2). The proofs for $N(a_i) \cap \{ a_j^-, a_j^+ \} = \emptyset$ have been implicitly given in the proofs of the first and second claims in (2). If $a_j^{--} \in N(a_i)$, then G has a cycle

$$h_i a_i \overleftarrow{C}[a_j^{--}, a_i^+] \overleftarrow{C}[a_i^-, a_j^+] a_j^- a_j h_j H h_i$$

which is longer than C, a contradiction. If $a_j^{++} \in N(a_i)$, then G has a cycle

$$h_i a_i \overrightarrow{C}[a_j^{++}, a_i^-] \overrightarrow{C}[a_i^+, a_j^-] a_j^+ a_j h_j H h_i$$

which is longer than C, a contradiction. Thus $N(a_i) \cap \{a_j^{--}, a_j^-, a_j^{++}, a_j^+\} = \emptyset$ if $1 \leq i \neq j \leq l$.

Now we prove (3). Suppose, to the contrary, that for some i, $1 \leq i \leq h$ there exists a vertex $x_i \in N(a_i) - \{ a_i^-, h_i \}$ such that $x_i a_i^- \notin E$ and $x_i h_i \notin E$. Since G is an L_1 - graph, we have

$$|N(a_i^-) \cap N(h_i)| \geq |N(a_i) - (N(a_i^-) \cap N(h_i))| - 1 \geq |\{ x_i, a_i^-, h_i \}| - 1 = 2.$$

Thus there exists a vertex, say y, which is different from x_i, such that $y \in N(a_i^-) \cap N(h_i)$. If $y \in V(G) - V(C)$, we can easily find a cycle in G which is longer than C. If $y \in V(C)$, then we arrive at a contradiction with the first statement in (2). Hence we prove that the first statement in (3) is true. Similarly, we can prove that the second statement in (3) is also true. QED

Lemma 2. Let G be a connected graph. Then $G \in \mathcal{K}$ is locally connected and Y – free if and only if $G \in \mathcal{K}'$.

Proof of Lemma 2. If $G \in \mathcal{K}$ is locally connected and Y – free, then $G = (V(G), E(G))$, where $V(G) = \{ a_1, a_2, ..., a_p, b_1, b_2, ..., b_{p+1} \}$ and $E(G) = \{ a_i b_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq (p+1) \} \cup S$, where $S \subseteq \{ a_i a_j : 1 \leq i < j \leq p \}$. Since $G[N(b_1)]$ is connected, $G[a_1, a_2, ..., a_p]$ is connected. Obviously, $G[a_1, a_2, ..., a_p]$ is Y – free. Thus $G \in \mathcal{K}'$.

If $G \in \mathcal{K}'$, then $G \in \mathcal{K}$ and moreover $G = (V(G), E(G))$, where $V(G) = \{ a_1, a_2, ..., a_p, b_1, b_2, ..., b_{p+1} \}$, $E(G) = \{ a_i b_j : 1 \leq i \leq p$ and $1 \leq j \leq (p+1) \} \cup S$, where $S \subseteq \{ a_i a_j : 1 \leq i < j \leq p \}$, and $G[a_1, a_2, ..., a_p]$ is connected and Y – free. Since $G[a_1, a_2, ..., a_p]$ is connected, $G[N(b_i)]$, for each $1 \leq i \leq (p+1)$, is connected. Again since $G[a_1, a_2, ..., a_p]$ is connected, for each vertex a_i, where $1 \leq i \leq p$, there exists a vertex a_j, where $1 \leq j \leq p$ and $j \neq i$, such that $a_i a_j \in E$. Thus $G[N(a_i)]$, for each $1 \leq i \leq p$, is connected. Hence G is locally connected.

Assume that G is not Y – free. Then $Y = (V(Y), E(Y))$, where $V(Y) = \{ u_1, u_2, u_3, u_4, u_5, u_6 \}$ and $E(Y) = \{u_1u_2, u_1u_3, u_2u_3, u_2u_4, u_3u_4, u_4u_5, u_4u_6\}$, is an induced subgraph of G such that $\{ b_1, b_2, ..., b_{p+1} \} \cap \{ u_1, u_2, u_3, u_4, u_5, u_6 \} \neq \emptyset$.

When $u_1 \in \{ b_1, b_2, ..., b_{p+1} \}$, since $u_1u_4 \notin E(G)$ and $u_1u_5 \notin E(G)$, then both u_4 and u_5 are not in $\{ a_1, a_2, ..., a_p \}$, i.e., both u_4 and u_5 are in $\{ b_1, b_2, ..., b_{p+1} \}$, contradicting to $u_4u_5 \in E(G)$.

When $u_2 \in \{ b_1, b_2, ..., b_{p+1} \}$, since $u_2u_1 \in E(G)$ and $u_2u_5 \notin E(G)$, then $u_1 \in \{ a_1, a_2, ..., a_p \}$ and $u_5 \in \{ b_1, b_2, ..., b_{p+1} \}$, contradicting to $u_1u_5 \notin E(G)$. Similarly, we can arrive at a contradiction when $u_3 \in \{ b_1, b_2, ..., b_{p+1} \}$.

When $u_4 \in \{ b_1, b_2, ..., b_{p+1} \}$, since $u_4u_1 \notin E(G)$ and $u_4u_5 \in E(G)$, then $u_1 \in \{ b_1, b_2, ..., b_{p+1} \}$ and $u_5 \in \{ a_1, a_2, ..., a_p \}$, contradicting to $u_1u_5 \notin E(G)$.

When $u_5 \in \{ b_1, b_2, ..., b_{p+1} \}$, since $u_5u_1 \notin E(G)$ and $u_5u_2 \notin E(G)$, then both u_1 and u_2 are in $\{ b_1, b_2, ..., b_{p+1} \}$, contradicting to $u_1u_2 \in E(G)$. Similarly, we can arrive at a contradiction when $u_6 \in \{ b_1, b_2, ..., b_{p+1} \}$.

Thus the assumption that G is not Y – free is false. Hence $G \in \mathcal{K}$ is locally connected and Y – free. QED

Lemma 3. Let G be a graph. Then $G \in \mathcal{K}$ is Z_2 – free if and only if $G \in \mathcal{K}''$.

Proof of Lemma 3. Obviously, if $G \in \mathcal{K}$ is Z_2 – free, then $G \in \mathcal{K}''$. If $G \in \mathcal{K}''$, then $G \in \mathcal{K}$ and moreover $G = (V(G), E(G))$, where $V(G) = \{a_1, a_2, ..., a_p, b_1, b_2, ..., b_{p+1}\}$, $E(G) = \{a_ib_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq (p+1)\} \cup S$, where $S \subseteq \{a_ia_j : 1 \leq i < j \leq p\}$, and $G[a_1, a_2, ..., a_p]$ is Z_2 – free.

Assume that G is not Z_2 – free. Then $Z_2 = (V(Z_2), E(Z_2))$, where $V(Z_2) = \{u_1, u_2, u_3, u_4, u_5\}$ and $E(Y) = \{u_1u_2,\ u_2u_3,\ u_3u_4,\ u_3u_5,\ u_4u_5\}$, is an induced subgraph of G such that $\{b_1, b_2, ..., b_{p+1}\} \cap \{u_1, u_2, u_3, u_4, u_5\} \neq \emptyset$.

When $u_1 \in \{b_1, b_2, ..., b_{p+1}\}$, since $u_1u_4 \notin E(G)$ and $u_1u_5 \notin E(G)$, then both u_4 and u_5 are not in $\{a_1, a_2, ..., a_p\}$, i.e., both u_4 and u_5 are in $\{b_1, b_2, ..., b_{p+1}\}$, contradicting to $u_4u_5 \in E(G)$.

When $u_2 \in \{b_1, b_2, ..., b_{p+1}\}$, since $u_2u_4 \notin E(G)$ and $u_2u_5 \notin E(G)$, then both u_4 and u_5 are not in $\{a_1, a_2, ..., a_p\}$, i.e., both u_4 and u_5 are in $\{b_1, b_2, ..., b_{p+1}\}$, contradicting to $u_4u_5 \in E(G)$.

When $u_3 \in \{b_1, b_2, ..., b_{p+1}\}$, since $u_3u_1 \notin E(G)$ and $u_3u_4 \in E(G)$, then $u_1 \in \{b_1, b_2, ..., b_p, b_{p+1}\}$ and $u_4 \in \{a_1, a_2, ..., a_p\}$, contradicting to $u_1u_4 \notin E(G)$.

When $u_4 \in \{b_1, b_2, ..., b_{p+1}\}$, since $u_4u_1 \notin E(G)$ and $u_4u_3 \in E(G)$, then $u_1 \in \{b_1, b_2, ..., b_p, b_{p+1}\}$ and $u_3 \in \{a_1, a_2, ..., a_p\}$, contradicting to $u_1u_3 \notin E(G)$. Similarly, we can arrive at a contradiction when $u_5 \in \{b_1, b_2, ..., b_{p+1}\}$.

Thus the assumption that G is not Z_2 – free is false. Hence $G \in \mathcal{K}$ is Z_2 – free. QED

3. Proofs of Theorem 3 and Theorem 4

Proof of Theorem 3. Let G be a graph satisfying the conditions in Theorem 3. Then G is 2 – connected otherwise $G[N(v)]$ is disconnected, where v is a cut – vertex in G. Suppose that G is nonhamiltonian and $G \notin \mathcal{K}'$. Then from Lemma 2 and the facts that G is locally connected and Y – free we have that $G \notin \mathcal{K}$. Choose a longest cycle C in G and specify an orientation of C. Assume that H is any connected component of the graph $G[V(G) - V(C)]$, $N(V(H)) \cap V(C) = \{a_1, a_2, ..., a_h\}$ with $h_ia_i \in E$, where $h_i \in V(H)$ for each i, $1 \leq i \leq h$, and a_1, a_2, ..., a_h are labeled in the order of the orientation of C. Since G is 2 – connected, $h \geq 2$. From (1) in Lemma 1, we have $a_i^- a_i^+ \in E$ for each i, $1 \leq i \leq h$. Let $P_1 := h_1 p_1 p_2 ... p_r a_1^+$ be the shortest path between h_1 and a_1^+ and let $P_2 := h_1 q_1 q_2 ... q_s a_1^-$ be the shortest paths between h_1 and a_1^- in the connected graph $G[N(a_1)]$. Without loss of generality, we assume that $r \leq s$. This assumption implies that a_1^- is not on the path P_1.

Claim 1. $r = 2$.

Proof of Claim 1. Suppose, to the contrary, that $r \neq 2$. then $r = 1$ or $r \geq 3$. If $r = 1$, clearly, $p_1 \in V(C)$ otherwise we can easily find a cycle in G which is longer than C. Notice that $p_1 \in N(a_1)$. By the second statement in (2) of Lemma 1, we have $a_1^+ p_1 \notin E$, contradicting to $a_1^+ p_1 \in E$. If $r \geq 3$, then $p_2 \in N(a_1) - \{ a_1^+, h_1 \}$, $a_1^+ p_2 \notin E$, and $h_1 p_2 \notin E$ since P_1 is a shortest path between h_1 and a_1^+ in $G[N(a_1)]$. Thus we arrive at a contradiction with the second statement in (3) of Lemma 1.

Claim 2. $p_1 \in V(C)$ and $p_2 \in V(C)$.

Proof of Claim 2. If $p_2 \notin V(C)$, since P_1 is in $G[N(a_1)]$, we can easily find a cycle in G which is longer than C. If $p_1 \notin V(C)$ and $p_2 \in V(C)$, we arrive at a contradiction with the second statement in (2) of Lemma 1. Therefore $p_1 \in V(C)$ and $p_2 \in V(C)$. By (1) in Lemma 1, we have that $p_1^- p_1^+ \in E$.

Now we consider the case $p_1 \in \overrightarrow{C}[a_1^+, p_2]$.

Claim 3. $G[p_2, p_2^-, p_2^+, p_1, h_1, a_1]$ is isomorphic to Y if $p_1 \in \overrightarrow{C}[a_1^+, p_2]$.

Proof of Claim 3. First notice that $p_1 a_1 \in E$ and $p_2 a_1 \in E$ since P_1 is in $G[N(a_1)]$. If $p_2^- p_2^+ \in E$, then G has a cycle

$$h_1 p_1 p_2 \overrightarrow{C}[a_1^+, p_1^-] \overrightarrow{C}[p_1^+, p_2^-] \overrightarrow{C}[p_2^+, a_1] h_1$$

which is longer than C, a contradiction. If $p_2^- p_1 \in E$, then G has a cycle

$$h_1 \overleftarrow{C}[a_1, p_2] \overrightarrow{C}[a_1^+, p_1^-] \overrightarrow{C}[p_1^+, p_2^-] p_1 h_1$$

which is longer than C, a contradiction. If $p_2^- h_1 \in E$, then G has a cycle

$$h_1 \overleftarrow{C}[a_1, p_2] \overrightarrow{C}[a_1^+, p_2^-] h_1$$

which is longer than C, a contradiction. If $p_2^- a_1 \in E$, then G has a cycle

$$h_1 p_1 \overrightarrow{C}[p_2, a_1^-] \overrightarrow{C}[a_1^+, p_1^-] \overrightarrow{C}[p_1^+, p_2^-] a_1 h_1$$

which is longer than C, a contradiction. If $p_2^+ p_1 \in E$, then G has a cycle

$$h_1 p_1 \overrightarrow{C}[p_2^+, a_1^-] \overrightarrow{C}[a_1^+, p_1^-] \overrightarrow{C}[p_1^+, p_2] a_1 h_1$$

which is longer than C, a contradiction. If $p_2^+ h_1 \in E$, then G has a cycle

$$h_1 \overrightarrow{C}[p_2^+, p_1^-] \overrightarrow{C}[p_1^+, p_2] p_1 h_1$$

which is longer than C, a contradiction. If $p_2^+ a_1 \in E$, then G has a cycle

$$h_1 a_1 \overrightarrow{C}[p_2^+, a_1^-] \overrightarrow{C}[a_1^+, p_1^-] \overrightarrow{C}[p_1^+, p_2] p_1 h_1$$

which is longer than C, a contradiction. If $p_2 h_1 \in E$, then P_1 is not a shortest path between h_1 and a_1^+ in $G[N(a_1)]$, a contradiction.

Therefore $G[p_2, p_2^-, p_2^+, p_1, h_1, a_1]$ is isomorphic to Y if $p_1 \in \overrightarrow{C}[a_1^+, p_2]$.

Next we consider the case $p_2 \in \overrightarrow{C}[a_1^+, p_1]$.

Claim 4. $G[p_2, p_2^-, p_2^+, p_1, h_1, a_1]$ is isomorphic to Y if $p_2 \in \overrightarrow{C}[a_1^+, p_1]$.

Proof of Claim 4. First notice that $a_1^+ \neq p_2^-$ otherwise G has a cycle

$$h_1 p_1 \overrightarrow{C}[p_2, p_1^-] \overrightarrow{C}[p_1^+, a_1^-] a_1^+ a_1 h_1$$

which is longer than C. If $p_2^- p_2^+ \in E$, then G has a cycle

$$h_1 p_1 p_2 \overrightarrow{C}[a_1^+, p_2^-] \overrightarrow{C}[p_2^+, p_1^-] \overrightarrow{C}[p_1^+, a_1] h_1$$

which is longer than C, a contradiction. If $p_2^- p_1 \in E$, then G has a cycle

$$h_1 \overleftarrow{C}[a_1, p_1^+] \overleftarrow{C}[p_1^-, p_2] \overrightarrow{C}[a_1^+, p_2^-] p_1 h_1$$

which is longer than C, a contradiction. If $p_2^- h_1 \in E$, then G has a cycle

$$h_1 \overleftarrow{C}[a_1, p_2] \overrightarrow{C}[a_1^+, p_2^-] h_1$$

which is longer than C, a contradiction. If $p_2^- a_1 \in E$, then G has a cycle

$$h_1 p_1 \overrightarrow{C}[p_2, p_1^-] \overrightarrow{C}[p_1^+, a_1^-] \overrightarrow{C}[a_1^+, p_2^-] a_1 h_1$$

which is longer than C, a contradiction. If $p_2^+ p_1 \in E$, then G has a cycle

$$h_1 p_1 \overrightarrow{C}[p_2^+, p_1^-] \overrightarrow{C}[p_1^+, a_1^-] \overrightarrow{C}[a_1^+, p_2] a_1 h_1$$

which is longer than C, a contradiction. If $p_2^+h_1 \in E$, then G has a cycle

$$h_1\overrightarrow{C}[p_2^+, a_1^-]\overrightarrow{C}[a_1^+, p_2]a_1h_1$$

which is longer than C, a contradiction. If $p_2^+a_1 \in E$, then G has a cycle

$$h_1a_1\overrightarrow{C}[p_2^+, p_1^-]\overrightarrow{C}[p_1^+, a_1^-]\overrightarrow{C}[a_1^+, p_2]p_1h_1$$

which is longer than C, a contradiction. If $p_2h_1 \in E$, then P_1 is not a shortest path between h_1 and a_1^+ in $G[N(a_1)]$, a contradiction.

Therefore $G[p_2, p_2^-, p_2^+, p_1, h_1, a_1]$ is isomorphic to Y if $p_2 \in \overrightarrow{C}[a_1^+, p_1]$.

The combination of Claim 3 and Claim 4 gives contradictions. Therefore we complete the proof of Theorem 3. QED

Proof of Theorem 4. Let G be a graph satisfying the conditions in Theorem 4. Suppose that G is nonhamiltonian and $G \notin \mathcal{K}''$. Then from Lemma 3 and the fact that G is Z_2 – free we have that $G \notin \mathcal{K}$. Choose a longest cycle C in G and specify an orientation of C. Assume that H is any connected component of the graph $G[V(G)-V(C)]$, $N(V(H)) \cap V(C) = \{a_1, a_2, ..., a_h\}$ with $h_ia_i \in E$, where $h_i \in V(H)$ for each i, $1 \leq i \leq h$, and a_1, a_2, ..., a_h are labeled in the order of the orientation of C. Since G is 2 – connected, $h \geq 2$. From (1) in Lemma 1, we have $a_i^-a_i^+ \in E$ for each i, $1 \leq i \leq h$. Obviously, $a_1h_1Hh_2a_2$ is a path between a_1 and a_2 such that all the internal vertices of it are in H. Let $P := a_1p_1p_2...p_ra_2$, where $r \geq 1$, be the shortest path between a_1 and a_2 such that all the internal vertices of P are in H. Then we have the following possible cases.

Case 1. $r \geq 2$

By the choices of C and P, we have $G[a_1, a_1^-, a_1^+, p_1, p_2]$ is isomorphic to Z_2, a contradiction.

Case 2. $r = 1$

By the choices of C and P, (2) in Lemma 1, and $G[a_1, a_1^-, a_1^+, p_1, a_2]$ is not isomorphic to Z_2, we have that $a_1a_2 \in E$. From (3) in Lemma 1, we have that $a_2a_1^+ \in E$, a contradiction.

Therefore we complete the proof of Theorem 4. QED

4. Acknowledgments

The author would like to thank the referee for his or her helpful suggestions and comments.

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