An Improved Algorithm for Cyclic Edge Connectivity of Regular Graphs

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Abstract

In this paper, we develop an $O(k^9V^6)$ time algorithm to determine the cyclic edge connectivity of k-regular graphs of order V for $k \geq 3$ which is an improvement of a known algorithm by Lou and Wang.

1. Introduction and terminology

Let G be a finite, undirected, connected and simple graph, except the graphs otherwise stated.

Let G = (V, E), where V is the vertex set of G and E is the edge set of G. We denote |V| by $\nu(G)$. Sometimes, we use V to represent $\nu(G)$. Let H be a subgraph of G. Then we denote by V(H) the vertex set of H and by E(H) the edge set of H. Let F be another subgraph of G. The $H \cap F$ is the subgraph with vertex set $V(H) \cap V(F)$ and edge set $E(H) \cap E(F)$. Let $S \subseteq V(G)$. The induced subgraph of G on S is denoted by G[S]. Let $v \in V(G)$, and $N(v) = \{u \mid u \in V(G) \text{ and } uv \in E(G)\}$. Let u, $v \in V(G)$. Then d(u, v) is the distance between u and v which is the length of a shortest path from u to v in G. Let C be a cycle in G. A chord of C is an edge xy in G such that x, $y \in V(C)$ but $xy \notin E(C)$. A minimal cycle C of G is a cycle without chords. An odd (even) cycle is a cycle of odd (even) length. Let C be a cycle without chord, then the co-cycle of C is the set of all edges with exactly one end on C. We denote the length of a shortest cycle in G by g(G), which is called the girth of G. Let T be a tree with a root $u \in V(T)$. Then $Q_r(u) = \{v \mid v \in V(T) \text{ and } d(u, v) = r-1\}$ is called the r-th layer of T. The maximum r such that $Q_r(u) \neq \emptyset$ is the number of layers of T. For a positive real number y, we denote by $\lfloor y \rfloor$ the maximum integer not larger than y.

A cyclic edge cutset S of G is an edge cutset whose deletion disconnects G such that at least two of the components of G−S contain a cycle. The cyclic edge connectivity, denoted by $c\lambda(G)$, is the cardinality of a minimum cyclic edge cutset of G. If no cyclic edge cutset exists in G, we say that $c\lambda(G) = \infty$.

For the terminology and notation not defined in this paper, the reader is referred to [1].

The concept of cyclic edge connectivity was introduced by Tait [14] in the proof of the Four Colour Theorem. Plummer [13] shows the cyclic edge connectivity of 5-connected planar graphs is at most 13, but that of 4-connected planar graphs can be any natural number at least 4. In [3] and [5], Holton, Lou and Plummer show the relation between cyclic edge connectivity and n-extendable graphs. In a paper of Peroche [12], several sorts of connectivity, including cyclic edge connectivity, and their relation are studied. Nedela and Skoviera [10] introduce the concept of atom. If B is a cyclic edge cutset of size $c\lambda(G)$, then a subgraph P of G such that each edge in B has exactly one incident vertex in P is called a cyclic part. An atom is a cyclic part that is minimal under inclusion. They also show that, for a connected cubic graph G, if $c\lambda(G) = k$ and P is not a cycle, then either (1) P has at least $2k-3$ vertices, or (2) $k = 6$ and P consists of two vertices joined by three openly disjoint paths of length 3. Moreover, if P is an atom, then either (1) P has at least 2k vertices, or (2) $k = 3$ and P consists of two vertices joined by three openly disjoint paths of length 2. For a summary of research in connectivity and edge connectivity, the reader is referred to [11]. In [6] and [7], Lou and Wang obtained the first efficient algorithm to determine the cyclic edge connectivity of k-regular graphs for $k \geq 3$ and an efficient algorithm to determine whether a general graph has infinite cyclic edge connectivity. However, the time complexity of the first algorithm is $O(k^{11}V^8)$, which is a little too large for practical use. In this paper, we improve the algorithm to obtain an $O(k^9V^6)$ time algorithm to determine the cyclic edge connectivity of k-regular graphs for $k \geq 3$.

2. An improved algorithm for cyclic edge connectivity

In this section, we give an improved algorithm for the cyclic edge connectivity of k-regular graphs which has time complexity $O(k^9V^6)$.

Algorithm 1:
1. Use a breadth first search strategy to find a shortest cycle containing v for each vertex v in G, then we can find the girth g of G; $//O(k|V|^2)$

2. If $\nu(G) < 2g$, then $c\lambda(G) = \infty$ and is returned; // O(1)

3. For each edge $e \in E(G)$, use a breadth first search strategy to find all minimal odd cycles C containing e such that $|V(C)| \leq 2\lfloor \log_{k-1}\nu(G)\rfloor+5$ and all minimal even cycles C containing e such that $|V(C)| \leq 2\lfloor \log_{k-1}\nu(G)\rfloor+6$. Let C_e be the set of all such cycles containing e and let $F = \bigcup_{e\in E(G)} C_e$;

// $O(k^4|V|^2)$

4. $s := (k-2)g$; // O(1)

5. For any two different cycles C_1 and C_2 in F do // $O(k^8|V|^4)$

BEGIN

6. If $V(C_1) \cap V(C_2) = \emptyset$, then we construct a new graph G′ such that $V(G') = V(G) \cup \{x, y\}$, where $x, y \notin V(G)$, and E(G′) contains all the edges in E(G), for each vertex u on C_1, we put (k−2) multiple edges between x and u, and for each vertex v on C_2, we put (k−2) multiple edges between y and v; // O(|V|)

7. Use the algorithm of [9] to find a minimum edge cutset S_{xy} which separates x and y; // $O(k|V|^2)$

8. $s := \min\{s, |S_{xy}|\}$; // O(1)

END;

9. Then $c\lambda(G) = s$ and is returned; // O(1).

3. Correctness and time complexity of the algorithm

In this section, we prove the correctness of Algorithm 1, and analyse time complexity of the algorithm. In the proof of correctness, we need the following two lemmas.

Lemma 1([7]): Let G be a connected k-regular graph. Then $c\lambda \neq \infty$ if and only if $\nu(G) \geq 2g$, where g is the girth of G.

Proof. For convenience of the reader to read Lemma 2 and its proof, we include the proof of this lemma again.

Suppose $\nu(G) \geq 2g$. We shall prove that the co-cycle of a minimum cycle C of G is a cyclic edge cutset, then $c\lambda(G) \leq (k-2)g$, hence $c\lambda(G) \neq \infty$.

Now we prove that $G-V(C)$ has a cycle. Suppose not. Then $G-V(C)$ is a forest. Then $k(\nu-g)-2(\nu-g-1) \leq (k-2)g$. We have $\nu \leq 2g-2/(k-2)$. But ν is an integer. So $\nu \leq 2g-1$, which contradicts the assumption that $\nu(G) \geq 2g$.

Suppose $\nu(G) < 2g$ and $c\lambda \neq \infty$. Then G has a cyclic edge cutset S such that G−S has two components C_1 and C_2 that both C_1 and C_2 have a cycle. So $|V(C_1)| \geq g$ and $|V(C_2)| \geq g$, which contradicts the assumption that $\nu(G) < 2g$. □

Lemma 2: Let G be a connected k-regular graph. If $c\lambda(G) < (k-2)g$, then, deleting a minimum cyclic edge cutset S, G−S has two components each of which contains a cycle C such that $|V(C)| \leq 2\lfloor \log_{k-1}\nu(G)\rfloor+5$ if C is an odd cycle or $|V(C)| \leq 2\lfloor \log_{k-1}\nu(G)\rfloor+6$ if C is an even cycle.

Proof. For a connected k-regular graph G, let C_1 be a shortest cycle of G, by the proof of Lemma 1, the co-cycle of C_1 is a cyclic edge cutset of size $(k-2)g$. So $c\lambda(G) \leq (k-2)g$.

Suppose $c\lambda(G) < (k-2)g$. Then $c\lambda(G) \leq (k-2)g-1$. Let S be a minimum cyclic edge cutset of G of size $c\lambda(G)$, D be a component of G−S containing a cycle and let C be a shortest cycle in D, where $|V(C)| \geq g$. Let $C = a_0a_1\cdots a_{c-1}a_0$, where $c = |V(C)|$. Since C is a shortest cycle in D, C does not have any chord. Let $N_0(a_i) = \{a_i\}$ and let $a_{i1}, a_{i2}, \cdots, a_{i,k-2}$ be the vertices in $N(a_i)\backslash V(C)$. Let $N_1(a_{ij})\backslash C = \{a_{ij}\}$, $N_r(a_{ij})\backslash C = \{u \mid u \in V(G)\backslash V(C), d(a_{ij}, u) = r-1, \text{ and } \exists x \in N_{r-1}(a_{ij})\backslash C, xu \in E(G)\}$ $(r \geq 2)$. Let $M_r(a_{ij}) = N_0(a_i) \bigcup_{k=1}^{r} N_k(a_{ij})\backslash C$.

Suppose $(N_1(a_{ij})\backslash C) \cap (N_1(a_{pq})\backslash C) \cap D \neq \emptyset$ for some $i \neq p$ or $j \neq q$. Then either we have a cycle C′ in D of length less than c, contradicting the assumption that C is a shortest cycle in D, or $c \leq 3 \leq 2\lfloor \log_{k-1}\nu(G)\rfloor+5$ if C is an odd cycle or $c \leq 4 \leq 2\lfloor \log_{k-1}\nu(G)\rfloor+6$ if C is an even cycle, as we required in Lemma 2.

Now suppose $(N_1(a_{ij})\backslash C) \cap (N_1(a_{pq})\backslash C) \cap D = \emptyset$ for all $i \neq p$ or $j \neq q$. If $c \leq 4$, it also satisfies the requirement of Lemma 2. So we suppose that $c > 4$.

The main idea of our proof is that first a lower bound for $\nu(D)$ will be found which will then enable us to find the upper bound of c.

With respect to the congruence of the length of the shortest cycle C modulo 4, the proof is divided into four cases.

Case 1: $c = 4m$ and m is a positive integer.

Let $G_{ij} = G[M_{c/4}(a_{ij})]$ $(i = 0, 1, \cdots, c-1; j = 1, 2, \cdots, k-2)$. Since every cycle in D has length at least c, $G_{ij} \cap D$ is a tree, and $E(G_{ij} \cap D) \cap E(G_{pq} \cap D) = \emptyset$ ($i \neq p$ or $j \neq q$). Since $|S| \leq (k-2)g-1$ and $c \geq g$, the edges of S lie in at most $(k-2)g-1 \leq (k-2)c-1$ of G_{ij} $(i = 0, 1, 2, \cdots, c-1; j = 1, 2, \cdots, k-2)$. So at least one G_{ij} does not contain any edge of S. Without loss of generality, assume $G_{0,1}$ does not contain any edge in S. Since every cycle in D has length at least c, $G_{0,1} \cap D$ and $G_{c/2,j} \cap D$ may share at most one vertex in $N_{c/4}(a_{0,1})\backslash C \cap N_{c/4}(a_{c/2,j})\backslash C$ for one j such that $1 \leq j \leq k-2$. Otherwise, if we have more than one vertex in $(G_{0,1} \cap D) \cap (\bigcup_{j=1}^{k-2}(G_{c/2,j} \cap D))$, then we can find a cycle shorter than C in $(G_{0,1} \cap D) \cup (\bigcup_{j=1}^{k-2}(G_{c/2,j} \cap D)) \cup C$, which is a contradiction.

Let $T_{0,1} = G[M_{c/2-1}(a_{0,1})]$. Then $T_{0,1} \cap D$ is a tree as every cycle in D has length at least c.

Now we use G_{ij} and $T_{0,1}$ ($i = 1, 2, \cdots, c-1$; $j = 1, 2, \cdots, k-2$; and, $i = 0$ and $j \neq 1$) to represent the trees in the case that G_{ij} and $T_{0,1}$ are contained in D, and we use $G_{ij} \cap D$ and $T_{0,1} \cap D$ to represent the trees in the real case. Notice that $T_{0,1}$ may share vertices with $G_{c/4+1,j}$, $G_{c/4+2,j}$, $\cdots$, $G_{3c/4-1,j}$ for some $1 \leq j \leq k-2$, whereas does not intersect with $G_{1,j}$, $G_{2,j}$, $\cdots$, $G_{c/4,j}$, $G_{3c/4,j}$, $G_{3c/4+1,j}$, $\cdots$, $G_{c-1,j}$ for any $1 \leq j \leq k-2$ and $G_{0,j}$ for $j \neq 1$ because every cycle in D has length at least c. Otherwise, if $T_{0,1}$ intersects with a G_{ij} ($1 \leq i \leq c/4$ or $3c/4 \leq i \leq c-1$; $1 \leq j \leq k-2$; and, $i = 0$ and $j \neq 1$), then $G_{ij} \cup T_{0,1} \cup C$ contains a cycle shorter than C, a contradiction.

Let $b \in N_{c/4+1}(a_{0,1})\backslash C$, $N_1(b)\backslash G_{0,1} = \{b\}$, $N_r(b)\backslash G_{0,1} = \{u \mid u \in V(G)\backslash V(G_{0,1}),\ d(u, b) = r-1, \text{ and } \exists x \in N_{r-1}(b)\backslash G_{0,1},\ xu \in E(G)\}$ ($r \geq 2$). Let $a \in N_{c/4}(a_{0,1})\backslash C$ such that $ab \in E(T_{0,1})$. Let $M_{c/4,r}(b) = \{a\} \bigcup_{k=1}^{r} N_k(b)\backslash G_{0,1}$ and $T_r(a,b) = G[M_{c/4,r}(b)]$. Then $T_r(a,b) \cap D$ is a subtree of $T_{0,1} \cap D$ for $1 \leq r \leq c/4-1$. We also use $T_r(a, b)$ to represent the subtree of $T_{0,1}$ in the case that $T_{0,1}$ is contained in D, and $T_r(a, b) \cap D$ to represent the subtree of $T_{0,1} \cap D$ in the real case. Notice that $T_{c/4-1}(a, b)$ is a branch of $T_{0,1}$ above $G_{0,1}$ and $G_{0,1}$ is contained in $T_{0,1}$.

The main idea of the following proof is that, in the first step, we find an edge cutset S' such that $|S'| = |S|$ and after cutting some vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$ by S', the remaining subgraph of $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$ is contained in D with no more vertices than D; in the second step, we do a modification of S' to calculate the lower bound of the order of the above subgraph of $\bigcup_{i,j} G_{i,j} \cup T_{0,1} \cup C$, which is also a lower bound of $\nu(D)$. Here and in the following, $\bigcup_{i,j} G_{ij}$ is the union of all G_{ij}'s ($i = 1, 2, \cdots, c-1$; $j = 1, 2, \cdots, k-2$; and, $i = 0$ and $j \neq 1$).

From S, we construct S' as follows. First, let $S' = \emptyset$. For each edge e of S which does not lie in all G_{ij}'s and $T_{0,1}$, then e does not cut any vertex from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$, and we do following Operation 1.

Operation 1: We find a G_{ij} ($1 \leq i \leq c/4$ or $3c/4 \leq i \leq c-1$; $1 \leq j \leq k-2$;

or, i = 0 and j $\neq$ 1) which does not contain any edge of $S \cup S'$, and put $a_i a_{ij}$ in S'; if no such G_{ij} exists, then we find a $T_{c/4-1}(a, b)$ for some $a \in N_{c/4}(a_{0,1})\backslash C$ and $b \in N_{c/4+1}(a_{0,1})\backslash C$ with $ab \in E(T_{0,1})$ such that $T_{c/4-1}(a, b)$ does not contain any edge of $S \cup S'$, and put ab in S'.

For each G_{ij} ($1 \leq i \leq c-1$, $1 \leq j \leq k-2$; or, i = 0 and j $\neq$ 1) containing an edge of S, if $a_i a_{ij} \in S$, then put $a_i a_{ij}$ in S'; if $a_i a_{ij} \notin S$, we find an edge $e_1 \in S \cap E(G_{ij})$, and put $a_i a_{ij}$ in S', then for each edge $e \in S \cap E(G_{ij})$ other than e_1, we do Operation 1.

For each $T_{c/4-1}(a, b)$ containing an edge of S for $a \in N_{c/4}(a_{0,1})\backslash C$ and $b \in N_{c/4+1}(a_{0,1})\backslash C$ with $ab \in E(T_{0,1})$, if $ab \in S$, then put ab in S'; if ab $\notin$ S, we find an edge $e_1 \in S \cap E(T_{c/4-1}(a, b))$, and put ab in S', then for each edge $e \in S \cap E(T_{c/4-1}(a,b))$ other than e_1, we do Operation 1.

Notice that for each G_{ij} (and $T_{c/4-1}(a, b)$) containing an edge of S, all vertices of G_{ij} except a_i (all vertices of $T_{c/4-1}(a, b)$ except a) are deleted by S' from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$. There may be other G_{ij}'s ($1 \leq i \leq c/4$ or $3c/4 \leq i \leq c-1$, $1 \leq j \leq k-2$; or, i = 0 and j $\neq$ 1) (and ($T_{c/4-1}(a, b)$'s for some $a \in N_{c/4}(a_{0,1})\backslash C$ and $b \in N_{c/4+1}(a_{0,1})\backslash C$ with $ab \in E(T_{0,1})$)) not containing any edge of S which are also deleted except a_i (except a) by S'. Hence the remaining subgraph of $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$ is contained in D. Since, for each edge in S, S' has a corresponding edge, so $|S'| = |S|$.

Now, for each G_{ij} containing an edge e of S', $e = a_i a_{ij}$; for each $T_{c/4-1}(a, b)$ containing an edge e of S', e = ab. To cut an edge $a_i a_{ij} \in S'$ in G_{ij}, we can delete at most all vertices of G_{ij} except a_i from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$, that is, to delete $(k-1)^0 + (k-1)^1 + \cdots + (k-1)^{c/4-1} = [(k-1)^{c/4} - 1]/(k-2)$ vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$. To cut an edge $ab \in S$ in $T_{c/4-1}(a, b)$ for one $a \in N_{c/4}(a_{0,1})\backslash C$ and one $b \in N_{c/4+1}(a_{0,1})\backslash C$ such that $ab \in E(T_{0,1})$, we can delete at most all vertices of $T_{c/4-1}(a, b)$ except a from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$, that is, to delete $(k-1)^0 + (k-1)^1 + \cdots + (k-1)^{c/4-2} = [(k-1)^{c/4-1} - 1]/(k-2)$ vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$.

Suppose there are m G_{ij}'s ($1 \leq i \leq c/4$ or $3c/4 \leq i \leq c-1$, $1 \leq j \leq k-2$; or, i = 0 and j $\neq$ 1) which do not contain any edge of S'; and there are n G_{ij}'s ($c/4+1 \leq i \leq 3c/4-1$, $1 \leq j \leq k-2$) which do not contain any edge of S'. Then the number of the subtrees $T_{c/4-1}(a, b)$ containing an edge ab of S' is not larger than n + m since $|S'| = |S| \leq (k-2)c-1$, where $(k-2)c-1$ is the number of all G_{ij}'s except $G_{0,1}$.

Now we have $V(T_{0,1}) \cap V(G_{ij}) = \emptyset$, to delete an edge $a_i a_{ij}$ in G_{ij} instead of an edge ab in $T_{c/4-1}(a, b)$ will delete more vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$ ($1 \leq i \leq c/4$ or $3c/4 \leq i \leq c-1$; $1 \leq j \leq k-2$; or, i = 0 and j

$\neq 1$).To calculate a lower bound of the order of the remaining subgraph of $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$ after deleting S′, we do the modification of S′ to assume that all G_{ij}'s have an cut edge $a_i a_{ij}$ in S′ ($1 \leq i \leq c/4$ and $3c/4 \leq i \leq c-1$, $1 \leq j \leq k-2$; and, $i = 0$ and $j \neq 1$) and there are n G_{ij}'s ($c/4+1 \leq i \leq 3c/4-1$, $1 \leq j \leq k-2$) not containing any cut edge in S′ which may share vertices with $T_{0,1}$ and there are at most n $T_{c/4-1}(a, b)$ containing cut edge ab in S′ for some $a \in N_{c/4}(a_{0,1})\backslash C$ and $b \in N_{c/4+1}(a_{0,1})\backslash C$ with $ab \in E(T_{0,1})$. Then the order of remaining subgraph of $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$ after cutted by S′ is a lower bound of $\nu(D)$.

Let a_0 be the root of $T_{0,1}$. Notice that $T_{0,1}$ originally has c/2 layers, G_{ij} ($c/4+1 \leq i \leq 3c/4-1$, $1 \leq j \leq k-2$) can provide at most $n \leq (k-2)(c/2-1)$ cut edges to $T_{c/4-1}(a, b)$'s, and $N_{c/4}(a_{0,1})\backslash C$ has $(k-1)^{c/4-1}$ vertices. By k-regularity of G, if $(k-2)(c/2-1) \leq (k-2)(k-1)^{c/4-1}$, then we cut at most $(k-2)(k-1)^{c/4-1}$ subtrees $T_{c/4-1}(a, b)-a$ from $T_{0,1}$, the resulting tree of $T_{0,1}$ equivalently has at least the number of vertices of $c/2-1$ layers plus a layer of $(k-1)^{c/4-1}$ vertices, which is a lower bound of $\nu(D)$. So

$$\begin{aligned}\nu(D) &\geq (k-1)^0+ (k-1)^1+ \cdots + (k-1)^{c/2-3}+ c + (k-1)^{c/4-1}\\ &= [(k-1)^{c/2-2}- 1]/(k-2) + c + (k-1)^{c/4-1}\\ &\geq [(k-1)^{c/2-2}- 1]/(k-2).\end{aligned}$$

However, $(k-2)(c/2-1) \leq (k-2)(k-1)^{c/4-1}$ is equivalent to $c/4-1 = (c/4 - 1/2) - 1/2 \geq \log_{k-1}(c/2 - 1) = \log_{k-1}2 + \log_{k-1}(c/4-1/2)$. Since $x-\log_{k-1}x \geq \log_{k-1}2 + 1/2$ is satisfied when $x \geq 7/2$ and $k \geq 3$, when $c/4 - 1/2 \geq 7/2$, i.e. $c \geq 16$, the above inequality holds.

Now we discuss the special cases of c = 4, 8 and 12.

Case (1.1): c = 4.

Then c satisfies the requirement of Lemma 2.

Case (1.2): c = 8.

Notice that $[(k-1)^{c/2-2}-1]/(k-2) = [(k-1)^2-1]/(k-2) = (k-2)k/(k-2) = k$. $\nu(D) \geq \nu(G_{0,1}) \geq k+1 \geq k$.

Case (1.3): c = 12.

Let a_0 be the root of $G_{0,1}$. Now $G_{0,1}$ has 4 layers $N_0(a_0)$, $N_1(a_{0,1})\backslash C$, $N_2(a_{0,1})\backslash C$ and $N_3(a_{0,1})\backslash C$. Let $T'_{0,1} = G[M_{c/4+1}(a_{0,1})] = G[N_0(a_0) \cup N_1(a_{0,1})\backslash C \cup N_2(a_{0,1})\backslash C \cup N_3(a_{0,1})\backslash C \cup N_4(a_{0,1})\backslash C]$. We shall prove that D has at least as many vertices as $T'_{0,1}$.

Let $T_{c/4-2}(a, b) = G[M_{c/4,c/4-2}(b)]$ for some $a \in N_{c/4}(a_{0,1})\backslash C$ and $b \in N_{c/4+1}(a_{0,1})\backslash C$ such that $ab \in E(T_{0,1})$ and let a be the root of $T_{c/4-2}(a, b)$. Since c = 12, $T_{c/4-2}(a, b)$ has 2 layers and $c/4-2 = 1$. Let $G'_{ij} = G[M_{c/4-1}(a_{ij})]$ ($i = c/4+1, c/4+2, \cdots, 3c/4-1$; $1 \leq j \leq k-2$).

Similar to the previous proof, $T'_{0,1}$ does not intersect with G_{ij} ($i = 1, 2, \cdots, c/4, 3c/4, 3c/4+1, \cdots, c-1$; $j = 1, 2, \cdots, k-2$; and, $i = 0$ and $j \neq 1$), and $T'_{0,1}$ may share at most one vertex with $G'_{6,j}$ for one j such that 1

$\leq$ j $\leq$ k−2, but does not intersect with the other G'_{ij}'s (4 = c/4+1 $\leq$ i $\leq$ 3c/4−1 = 8; 1 $\leq$ j $\leq$ k−2) as every cycle in D has length at least c. Then G_{ij} (i = 1, 2, 3, 9, 10, 11; 1 $\leq$ j $\leq$ k−2; and, i = 0 and j $\neq$ 1) contains an edge $a_i a_{ij} \in S'$ as previous proof, and there are n G'_{ij}'s (4 $\leq$ i $\leq$ 8, 1 $\leq$ j $\leq$ k−2) not containing cut edges of S′ whereas there are at most n $T_{c/4-2}$(a, b)'s containing cut edges ab in S′. Notice that each cut edge ab can delete at most one vertex from $T_{c/4-2}$(a, b) and hence from $T'_{0,1}$ and the n G'_{ij}'s not containing any edge in S′ have n $[(k-1)^0 + (k-1)^1]$ vertices. Also considering the common vertex of $T'_{0,1}$ and $G'_{6,j}$, we have

$$\nu(D) \geq \nu(T'_{0,1}) - n + n\nu(G'_{ij}) - 1 + c$$
$$= [(k-1)^0 + (k-1)^1 + (k-1)^2 + (k-1)^3 - n] + n[(k-1)^0 + (k-1)^1] -1 + c$$
$$\geq [(k-1)^4 - 1]/(k-2) + n[((k-1)^2-1)/(k-2) - 1]$$
$$\geq [(k-1)^4 - 1]/(k-2)$$
$$= [(k-1)^{c/2-2} - 1]/(k-2).$$

So, in all subcases of Case 1, $\nu(D) \geq [(k-1)^{c/2-2}-1]/(k-2)$.

Case 2: c = 4m+1 and m is a positive integer.

The main idea of proof is similar to that of Case 1.

Let $G_{ij} = G[M_{(c-1)/4}(a_{ij})]$ (i = 1, 2, ⋯, c−1; j = 1, 2, ⋯, k−2). Then $G_{ij} \cap D$ is a tree, and $E(G_{ij} \cap D) \cap E(G_{pq} \cap D) = \emptyset$, in fact, $V(G_{ij} \cap D) \cap V(G_{pq} \cap D) = \emptyset$ (i $\neq$ p or j $\neq$ q) as every cycle in D has length at least c. Since |S| $\leq$ (k−2)g−1 and c $\geq$ g, without loss of generality, we can assume that $G_{0,1}$ does not contain any edge in S.

Let $T_{0,1} = G[M_{(c-1)/2}(a_{0,1})]$. Then $T_{0,1} \cap D$ is a tree as every cycle in D has length at least c. We use G_{ij} and $T_{0,1}$ (i = 1, 2, ⋯, c−1; j = 1, 2, ⋯, k−2; and, i = 0 and j $\neq$ 1) to represent the trees in the case that G_{ij} and $T_{0,1}$ are contained in D. We use $G_{ij} \cap D$ and $T_{0,1} \cap D$ (i = 1, 2, ⋯, c−1; j = 1, 2, ⋯, k−2; and, i = 0 and j $\neq$ 1) to represent the trees in the real case. Notice that $T_{0,1}$ may share vertices with $G_{(c-1)/4+1,j}$, $G_{(c-1)/4+2,j}$, ⋯, $G_{3(c-1)/4,j}$ for some 1$\leq$ j $\leq$ k−2, whereas does not intersect with $G_{1,j}$, $G_{2,j}$, ⋯, $G_{(c-1)/4,j}$, $G_{3(c-1)/4+1,j}$, $G_{3(c-1)/4+2,j}$, ⋯, $G_{c-1,j}$ for any 1$\leq$ j $\leq$ k−2 and $G_{0,j}$ for j $\neq$1.

Let $b \in N_{(c-1)/4+1}(a_{0,1})\backslash C$, $N_1(b)\backslash G_{0,1} = \{b\}$, $N_r(b)\backslash G_{0,1} = \{u \mid u \in V(G)\backslash V(G_{0,1}), d(u, b) = r-1, \text{ and } \exists x \in N_{r-1}(b)\backslash G_{0,1}, xu \in E(G)\}$ (r $\geq$ 2). Let $a \in N_{(c-1)/4}(a_{0,1})\backslash C$ such that $ab \in E(T_{0,1})$. Let $M_{(c-1)/4,r}(b)=\{a\} \bigcup_{k=1}^{r} N_k(b)\backslash G_{0,1}$ and $T_r(a,b)=G[M_{(c-1)/4,r}(b)]$. Then $T_r(a,b) \cap D$ is a subtree of $T_{0,1} \cap D$ for 1 $\leq$ r $\leq$ (c−1)/4. We also use T_r(a, b) to represent the subtree of $T_{0,1}$ in the case that $T_{0,1}$ is contained in D, and $T_r(a, b) \cap D$ to represent the subtree of $T_{0,1} \cap D$ in the real case. Notice that $T_{(c-1)/4}$(a, b) is a branch of $T_{0,1}$ above $G_{0,1}$ and $G_{0,1}$ is contained in $T_{0,1}$.

By similar argument to Case 1, we can obtain an edge cutset S′ with

$|S'| = |S|$ and the remaining subgraph of $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$ after cutted by S' is contained in D. For each G_{ij}, if we cut the edge $a_i a_{ij}$, we shall delete at most all vertices of G_{ij} except a_i, that is, to delete $(k-1)^0 + (k-1)^1 + \cdots + (k-1)^{(c-1)/4-1} = [(k-1)^{(c-1)/4}-1]/(k-2)$ vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$. To cut an edge ab in $T_{(c-1)/4}(a, b)$ for some $a \in N_{(c-1)/4}(a_{0,1})\backslash C$ and $b \in N_{(c-1)/4+1}(a_{0,1})\backslash C$ such that $ab \in E(T_{0,1})$, we can delete at most all vertices of $T_{(c-1)/4}(a, b)$ except a from $T_{0,1}$, that is, to delete $(k-1)^0 + (k-1)^1 + \cdots + (k-1)^{(c-1)/4-1} = [(k-1)^{(c-1)/4}-1]/(k-2)$ vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$. Since $V(T_{0,1}) \cap V(G_{ij}) = \emptyset$ ($i = 1, 2, \cdots, (c-1)/4, 3(c-1)/4+1, 3(c-1)/4+2, \cdots, c-1$; $1 \le j \le k-2$; and, $i = 0$ and $j \ne 1$), to cut an edge $a_i a_{ij}$ will delete at least as many vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$ as to cut an edge ab in $T_{(c-1)/4}(a, b)$.

By similar argument to Case 1, to find a lower bound of $\nu(D)$, we only need to consider the case that each G_{ij} ($i = 1, 2, \cdots, (c-1)/4, 3(c-1)/4+1, 3(c-1)/4+2, \cdots, c-1$; $j = 1, 2, \cdots, k-2$; and, $i = 0$ and $j \ne 1$) contains a cut edge $a_i a_{ij}$ of S', and there are n G_{ij}'s ($(c-1)/4+1 \le i \le 3(c-1)/4$, $1 \le j \le k-2$) not containing any cut edge of S', and there are at most n $T_{(c-1)/4}(a, b)$'s containing cut edges ab of S'.

Let a_0 be the root of $T_{0,1}$. Notice that $T_{0,1}$ originally has $(c-1)/2+1$ layers, G_{ij} ($(c-1)/4+1 \le i \le 3(c-1)/4$, $1 \le j \le k-2$) can provide at most $n \le (k-2)(c-1)/2$ cut edges to $T_{(c-1)/4}(a, b)$'s, and $N_{(c-1)/4}(a_{0,1})\backslash C$ has $(k-1)^{(c-1)/4-1}$ vertices. By k-regularity of G, if $(k-2)(c-1)/2 \le (k-2)(k-1)^{(c-1)/4-1}$, then we cut at most $(k-2)(k-1)^{(c-1)/4-1}$ subtrees $T_{(c-1)/4}(a, b)-a$ from $T_{0,1}$, the resulting tree of $T_{0,1}$ equivalently has at least the number of vertices of $(c-1)/2$ layers plus a layer of $(k-1)^{(c-1)/4-1}$ vertices, which is a lower bound of $\nu(D)$. So

$$\begin{aligned}\nu(D) &\ge (k-1)^0 + (k-1)^1 + \cdots + (k-1)^{(c-1)/2-2} + c + (k-1)^{(c-1)/4-1} \\ &= [(k-1)^{(c-1)/2-1}-1]/(k-2) + c + (k-1)^{(c-1)/4-1} \\ &\ge [(k-1)^{(c-1)/2-1}-1]/(k-2).\end{aligned}$$

However, $(k-2)(c-1)/2 \le (k-2)(k-1)^{(c-1)/4-1}$ is equivalent to $(c-1)/4 - 1 \ge \log_{k-1}((c-1)/2) = \log_{k-1}2 + \log_{k-1}((c-1)/4)$. Since $x - \log_{k-1}x \ge 1 + \log_{k-1}2$ is satisfied when $x \ge 4$ and $k \ge 3$, when $(c-1)/4 \ge 4$, i.e. $c \ge 17$, the above inequality holds.

Now we discuss the special cases of c = 5, 9, 13.

Case (2.1): c = 5.

$$\nu(D) \ge \nu(C) = 5 \ge [(k-1)^{(c-1)/2-1}-1]/(k-2) = 1.$$

Case (2.2): c = 9.

Similar to Case (1.3). Let a_0 be the root of $G_{0,1}$. Now $G_{0,1}$ has 3 layers $N_0(a_0)$, $N_1(a_{0,1})\backslash C$ and $N_2(a_{0,1})\backslash C$. Let $T'_{0,1} = G[M_{(c-1)/4+1}(a_{0,1})] = G[N_0(a_0) \cup N_1(a_{0,1})\backslash C \cup N_2(a_{0,1})\backslash C \cup N_3(a_{0,1})\backslash C]$. We shall prove that

D has at least as many vertices as $T'_{0,1}$.

The cut edges of S′ in $T'_{0,1}$ are of the form ab such that $a \in N_2(a_{0,1})\backslash C$ and $b \in N_3(a_{0,1})\backslash C$ in which $ab \in E(T'_{0,1})$. Each cut edge in $T'_{0,1}$ deletes one vertex from $T'_{0,1}$. The cut edges of S′ in G_{ij} (i = (c−1)/4+1, (c−1)/4+2, ⋯, 3(c−1)/4; $1 \le j \le k-2$) are of the form $a_i a_{ij}$ (i = 3, 4, 5, 6; $1 \le j \le k-2$). Notice that $T'_{0,1}$ may share at most one vertex with $G_{4,p}$ and $G_{5,q}$ respectively for one p and q such that $1 \le p, q \le k-2$ but does not intersect with the other G_{ij}'s (i = 3, 4, 5, 6; $1 \le j \le k-2$) as each cycle in D has length at least c and $T'_{0,1}$ does not intersect with G_{ij}'s (i = 1, 2, 7, 8; $1 \le j \le k-2$; and, i = 0 and $j \ne 1$). To obtain a lower bound of $\nu(D)$, we only need to consider the case that each G_{ij} (i = 1, 2, 7, 8; $1 \le j \le k-2$; and, i = 0 and $j \ne 1$) contains a cut edge $a_i a_{ij}$ and there are n G_{ij}'s which do not contain any cut edge of S′ (i = 3, 4, 5, 6; $1 \le j \le k-2$), whereas $T'_{0,1}$ contains at most n cut edges of S′. Then

$$\begin{aligned}\nu(D) &\ge \nu(T'_{0,1}) - n + n\nu(G_{ij}) - 2 + c\\ &\ge [(k-1)^0 + (k-1)^1 + (k-1)^2 - n] + n[(k-1)^0 + (k-1)^1] - 2 + c\\ &\ge (k-1)^0 + (k-1)^1 + (k-1)^2 + n[((k-1)^2-1)/(k-2)-1]\\ &\ge [(k-1)^3-1]/(k-2)\\ &= [(k-1)^{(c-1)/2-1}-1]/(k-2).\end{aligned}$$

Case (2.3): c = 13.

Similar to Case (1.3). Let a_0 be the root of $G_{0,1}$. Now $G_{0,1}$ has 4 layers $N_0(a_0)$, $N_1(a_{0,1})\backslash C$, $N_2(a_{0,1})\backslash C$ and $N_3(a_{0,1})\backslash C$. Let $T'_{0,1} = G[M_{(c-1)/4+2}(a_{0,1})] = G[N_0(a_0) \cup N_1(a_{0,1})\backslash C \cup N_2(a_{0,1})\backslash C \cup N_3(a_{0,1})\backslash C \cup N_4(a_{0,1})\backslash C \cup N_5(a_{0,1})\backslash C]$. We shall prove that D has at least as many vertices as $T'_{0,1}$.

Let $T_{(c-1)/4-1}(a, b) = G[M_{(c-1)/4,(c-1)/4-1}(b)]$ for some $a \in N_{(c-1)/4}(a_{0,1})\backslash C$ and $b \in N_{(c-1)/4+1}(a_{0,1})\backslash C$ such that $ab \in E(T_{0,1})$ and let a be the root of $T_{(c-1)/4-1}(a, b)$. Since c = 13, $T_{(c-1)/4-1}(a, b)$ has 3 layers and (c−1)/4−1 = 2 and $T_{(c-1)/4-1}(a, b)$ is a branch of $T'_{0,1}$ above $G_{0,1}$.

The cut edges of S′ in $T'_{0,1}$ are of the form ab such that $a \in N_3(a_{0,1})\backslash C$ and $b \in N_4(a_{0,1})\backslash C$, where $ab \in E(T_{0,1})$. Each cut edge in $T'_{0,1}$ deletes at most all vertices of one $T_{(c-1)/4-1}(a, b)$ except a. The cut edges of S′ in G_{ij} (i = (c−1)/4+1, (c−1)/4+2, ⋯, 3(c−1)/4; $1 \le j \le k-2$) are of the form $a_i a_{ij}$ (i = 4, 5, ⋯, 9; $1 \le j \le k-2$). Notice that $T'_{0,1}$ may share at most two vertices with $G_{6,p}$ and $G_{7,q}$ respectively for one p and q such that $1 \le p, q \le k-2$ but for the other G_{ij}'s (i = 4, 5, ⋯, 9; $1 \le j \le k-2$), $T'_{0,1}$ may share at most one vertex with G_{ij}, and $T'_{0,1}$ does not intersect with G_{ij} (i = 1, 2, 3, 10, 11, 12; $1 \le j \le k-2$; and i = 0 and $j \ne 1$) as each cycle in D has length at least c. To obtain a lower bound of $\nu(D)$, we only need to consider the case that G_{ij} (i = 1, 2, 3, 10, 11, 12; $1 \le j \le k-2$; and i = 0 and $j \ne 1$) contains a cut edge $a_i a_{ij}$ of S′ and there are n G_{ij}'s (i = 4, 5,

$\cdots$, 9; $1 \leq j \leq k-2$) which do not contain any cut edge of S′, whereas $T'_{0,1}$ contains at most n cut edges of S′. Then

$\nu(D) \geq \nu(T'_{0,1}) + n[\nu(G_{ij})-1] - n\,\nu(T_{(c-1)/4-1}(a, b)) - 2 + c$

$= (k-1)^0 + (k-1)^1 + \cdots + (k-1)^4 + n[(k-1)^0 + (k-1)^1 + (k-1)^2 - 1] - n[(k-1)^0 + (k-1)^1] - 2 + c$

$\geq [(k-1)^5-1]/(k-2) + n[((k-1)^3-1)/(k-2) - 1 - ((k-1)^2-1)/(k-2)]$

$\geq [(k-1)^5-1]/(k-2)$

$= [(k-1)^{(c-1)/2-1}-1]/(k-2)$.

So in all subcases of Case 2, $\nu(D) \geq [(k-1)^{(c-1)/2-1}-1]/(k-2)$.

Case 3: $c = 4m+2$ and m is a positive integer.

The main idea of proof is similar to that of Case 1.

Let $G_{ij} = G[M_{(c-2)/4}(a_{ij})]$ ($i = 0, 1, \cdots, c-1$; $j = 1, 2, \cdots, k-2$). Then $G_{ij} \cap D$ is a tree, and $E(G_{ij} \cap D) \cap E(G_{pq} \cap D) = \emptyset$, in fact, $V(G_{ij} \cap D) \cap V(G_{pq} \cap D) = \emptyset$ ($i \neq p$ or $j \neq q$) as every cycle in D has length at least c. Since $|S| \leq (k-2)g-1$ and $c \geq g$, without loss of generality, assume $G_{0,1}$ does not contain any edge in S.

Let $T_{0,1} = G[M_{(c-2)/2}(a_{0,1})] = G[M_{c/2-1}(a_{0,1})]$. Then $T_{0,1} \cap D$ is a tree as every cycle in D has length at least c. We use G_{ij} and $T_{0,1}$ ($i = 1, 2, \cdots, c-1$; $j = 1, 2, \cdots, k-2$; and, $i = 0$ and $j \neq 1$) to represent the trees in the case that G_{ij} and $T_{0,1}$ are contained in D. Then we use $G_{ij} \cap D$ and $T_{0,1} \cap D$ ($i = 1, 2, \cdots, c-1$; $j = 1, 2, \cdots, k-2$; and, $i = 0$ and $j \neq 1$) to represent the trees in the real case. Notice that $T_{0,1}$ may share vertices with $G_{(c-2)/4+2,j}, G_{(c-2)/4+3,j}, \cdots, G_{3(c-2)/4,j}$ for some $1 \leq j \leq k-2$, whereas does not intersect with $G_{1,j}, G_{2,j}, \cdots, G_{(c-2)/4+1,j}, G_{3(c-2)/4+1,j}, G_{3(c-2)/4+2,j}, \cdots, G_{c-1,j}$ for any $1 \leq j \leq k-2$ and $G_{0,j}$ for $j \neq 1$. Let $b \in N_{(c-2)/4+1}(a_{0,1}) \backslash C$, $N_1(b) \backslash G_{0,1} = \{b\}$, $N_r(b) \backslash G_{0,1} = \{u \mid u \in V(G) \backslash V(G_{0,1}),\ d(u, b) = r-1, \text{ and } \exists x \in N_{r-1}(b) \backslash G_{0,1},\ xu \in E(G)\}$ ($r \geq 2$). Let $a \in N_{(c-2)/4}(a_{0,1}) \backslash C$ such that $ab \in E(T_{0,1})$. Let $M_{(c-2)/4,r}(b) = \{a\} \bigcup_{k=1}^{r} N_k(b) \backslash G_{0,1}$ and $T_r(a, b) = G[M_{(c-2)/4,r}(b)]$. Then $T_r(a, b) \cap D$ is a subtree of $T_{0,1} \cap D$ for all r such that $1 \leq r \leq (c-2)/4$, and we use $T_r(a, b)$ to represent the subtree of $T_{0,1}$ in the case that $T_{0,1}$ is contained in D. Notice that $T_{(c-2)/4}(a, b)$ is a branch of $T_{0,1}$ above $G_{0,1}$ and $G_{0,1}$ is contained in $T_{0,1}$.

By similar argument to Case 1, we can obtain an edge cutset S′ with $|S'| = |S|$ and the remaining subgraph of $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$ after cutted by S′ is contained in D.

For each G_{ij}, if we cut the edge $a_i a_{ij}$ of S′, we shall delete at most all vertices of G_{ij} except a_i, that is, to delete $(k-1)^0 + (k-1)^1 + \cdots + (k-1)^{(c-2)/4-1} = [(k-1)^{(c-2)/4}-1]/(k-2)$ vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$. For each $T_{(c-2)/4}(a, b)$ for some $a \in N_{(c-2)/4}(a_{0,1}) \backslash C$ and $b \in N_{(c-2)/4+1}($

$a_{0,1})\backslash C$ such that $ab \in E(T_{0,1})$, if we delete an edge ab of S′, we can delete at most all vertices of $T_{(c-2)/4}(a, b)$ except a from $T_{0,1}$, that is, to delete $(k-1)^0 + (k-1)^1 + \cdots + (k-1)^{(c-2)/4-1} = [(k-1)^{(c-2)/4}-1]/(k-2)$ vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$. Since $V(T_{0,1}) \cap V(G_{ij}) = \emptyset$, $(i = 1, 2, \cdots, (c-2)/4+1, 3(c-2)/4+1, 3(c-2)/4+2, \cdots, c-1; 1 \le j \le k-2$; and, $i = 0$ and $j \neq 1)$, to cut $a_i a_{ij}$ will delete at least as many vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$ as to cut an edge ab in $T_{(c-2)/4}(a, b)$.

By similar argument to Case 1, to find a lower bound of $\nu(D)$, we only need to consider the case that each G_{ij} ($i = 1, 2, \cdots, (c-2)/4+1, 3(c-2)/4+1, 3(c-2)/4+2, \cdots, c-1; 1 \le j \le k-2$; and, $i = 0$ and $j \neq 1$) contains a cut edge $a_i a_{ij}$, and there are n G_{ij}'s ($(c-2)/4+2 \le i \le 3(c-2)/4$, $1 \le j \le k-2$) which do not contain any cut edge of S′, whereas there are at most n $T_{(c-2)/4}(a, b)$'s containing cut edges ab for some $a \in N_{(c-2)/4}(a_{0,1})\backslash C$ and $b \in N_{(c-2)/4+1}(a_{0,1})\backslash C$ such that $ab \in E(T_{0,1})$.

Let a_0 be the root of $T_{0,1}$. Notice that $T_{0,1}$ originally has $(c-2)/2+1 = c/2$ layers, G_{ij}'s ($(c-2)/4+2 \le i \le 3(c-2)/4$, $1 \le j \le k-2$) can provide at most $n \le (k-2)[(c-2)/2-1]$ cut edges to $T_{(c-2)/4}(a, b)$'s, and $N_{(c-2)/4}(a_{0,1})\backslash C$ has $(k-1)^{(c-2)/4-1}$ vertices. By k-regularity of G, if $(k-2)[(c-2)/2-1] \le (k-2)(k-1)^{(c-2)/4-1}$, then we cut at most $(k-2)(k-1)^{(c-2)/4-1}$ subtrees $T_{(c-2)/4}(a, b)-a$ from $T_{0,1}$, the resulting tree of $T_{0,1}$ equivalently has at least the number of vertices of $(c-2)/2 = c/2-1$ layers plus a layer of $(k-1)^{(c-2)/4-1}$ vertices, which is a lower bound of $\nu(D)$. So

$$\nu(D) \ge (k-1)^0 + (k-1)^1 + \cdots + (k-1)^{c/2-3} + c + (k-1)^{(c-2)/4-1}$$
$$= [(k-1)^{c/2-2}-1]/(k-2) + c + (k-1)^{(c-2)/4-1}$$
$$\ge [(k-1)^{c/2-2}-1]/(k-2).$$

However, $(k-2)[(c-2)/2-1] \le (k-2)(k-1)^{(c-2)/4-1}$ is equivalent to $(c-2)/4-1 = [(c-2)/4 - 1/2] - 1/2 \ge \log_{k-1}[(c-2)/2 - 1] = \log_{k-1}2 + \log_{k-1}[(c-2)/4 - 1/2]$. Since $x - \log_{k-1}x \ge \log_{k-1}2 + 1/2$ is satisfied when $x \ge 7/2$ and $k \ge 3$, when $(c-2)/4-1/2 \ge 7/2$, i.e. $c \ge 18$, the above inequality holds.

Now we discuss the special cases of c = 6, 10 and 14.

Case (3.1): c = 6.

$$\nu(D) \ge \nu(C) = 6 \ge [(k-1)^{6/2-2}-1]/(k-2) = 1 = [(k-1)^{c/2-2}-1]/(k-2).$$

Case (3.2): c = 10.

Similar to Case (1.3). Let a_0 be the root of $G_{0,1}$. Then $G_{0,1}$ has 3 layers $N_0(a_0)$, $N_1(a_{0,1})\backslash C$ and $N_2(a_{0,1})\backslash C$. Let $T'_{0,1} = G[M_{(c-2)/4+1}(a_{0,1})] = G[N_0(a_0) \cup N_1(a_{0,1})\backslash C \cup N_2(a_{0,1})\backslash C \cup N_3(a_{0,1})\backslash C]$. We shall prove that D has at least as many vertices as $T'_{0,1}$.

The cut edges of S′ in $T'_{0,1}$ are of the form ab such that $a \in N_2(a_{0,1})\backslash C$ and $b \in N_3(a_{0,1})\backslash C$ in which $ab \in E(T_{0,1})$. Each cut edge in $T'_{0,1}$ deletes one vertex from $T'_{0,1}$. The cut edges of S' in G_{ij} ($i = (c-2)/4+2, (c-2)/4+3, \cdots, 3(c-2)/4$; $1 \le j \le k-2$) are of the form $a_i a_{ij}$ ($i = 4, 5, 6$; $1 \le j \le k-2$). Notice that $T'_{0,1}$ may share at most one vertex with $G_{5,j}$ for one j such that $1 \le j \le k-2$ but does not intersect with other G_{ij}'s ($i = 4, 5, 6$; $1 \le j \le k-2$; and $i = 1, 2, 3, 7, 8, 9$; $1 \le j \le k-2$; and, $i = 0$ and $j \ne 1$) as each cycle in D has length at least c. By similar argument to Case (1.3), to find a lower bound of $\nu(D)$, we only need to consider the case that each G_{ij} ($i = 1, 2, 3, 7, 8, 9$; $1 \le j \le k-2$; and, $i = 0$ and $j \ne 1$) contains a cut edge $a_i a_{ij}$ and there are n G_{ij}'s ($i = 4, 5, 6$; $1 \le j \le k-2$) which do not contain any cut edge of S', and $T'_{0,1}$ contains at most n cut edges of S'. Then

$$\begin{aligned}\nu(D) &\ge \nu(T'_{0,1}) - n + n\nu(G_{ij}) - 1 + c\\ &\ge [(k-1)^0 + (k-1)^1 + (k-1)^2 - n] + n[(k-1)^0 + (k-1)^1] - 1 + c\\ &\ge [(k-1)^3 - 1]/(k-2) + n[((k-1)^2-1)/(k-2)-1]\\ &\ge [(k-1)^3-1]/(k-2)\\ &= [(k-1)^{c/2-2}-1]/(k-2).\end{aligned}$$

Case (3.3): $c = 14$.

Similar to Case (1.3). Let a_0 be the root of $G_{0,1}$. Now $G_{0,1}$ has 4 layers $N_0(a_0)$, $N_1(a_{0,1})\backslash C$, $N_2(a_{0,1})\backslash C$ and $N_3(a_{0,1})\backslash C$. Let $T'_{0,1} = G[M_{(c-2)/4+2}(a_{0,1})] = G[N_0(a_0) \cup N_1(a_{0,1})\backslash C \cup N_2(a_{0,1})\backslash C \cup \cdots \cup N_5(a_{0,1})\backslash C]$. We shall prove that D has at least as many vertices as $T'_{0,1}$.

Let $T_{(c-2)/4-1}(a, b) = G[M_{(c-2)/4,(c-2)/4-1}(b)]$ for some $a \in N_{(c-2)/4}(a_{0,1})\backslash C$ and $b \in N_{(c-2)/4+1}(a_{0,1})\backslash C$ such that $ab \in E(T_{0,1})$ and let a be the root of $T_{(c-2)/4-1}(a, b)$. Since $c = 14$, $T_{(c-2)/4-1}(a, b)$ has 3 layers and $(c-2)/4-1 = 2$. Let $G'_{ij} = G[M_{(c-2)/4-1}(a_{ij})]$ ($i = 1, 2, \cdots, c-1$; $1 \le j \le k-2$; and, $i = 0$ and $j \ne 1$). Let a_i be the root of G'_{ij}. Then G'_{ij} also has 3 layers.

The cut edges of S' in $T_{(c-2)/4-1}(a, b)$'s are of the form ab such that $a \in N_3(a_{0,1})\backslash C$ and $b \in N_4(a_{0,1})\backslash C$ in which $ab \in E(T_{0,1})$. Each cut edge of S' in $T'_{0,1}$ can delete at most all vertices of $T_{(c-2)/4-1}(a, b)$ except a, that is, to delete $(k-1)^0 + (k-1)^1 = [(k-1)^2-1]/(k-2)$ vertices from $\bigcup_{i,j} G'_{ij} \cup T'_{0,1} \cup C$. The cut edges of S' in G'_{ij} are of the form $a_i a_{ij}$ ($i = 1, 2, \cdots, 13$; $1 \le j \le k-2$; and, $i = 0$ and $j \ne 1$). Each cut edge in G'_{ij} can delete at most all vertices of G'_{ij} except a_i, that is, to delete $(k-1)^0 + (k-1)^1 = [(k-1)^2-1]/(k-2)$ vertices from $\bigcup_{i,j} G'_{ij} \cup T'_{0,1} \cup C$. Notice that $T'_{0,1}$ may share at most one vertex with $G'_{7,j}$ for one j such that $1 \le j \le k-2$, but does not intersect with the other G'_{ij}'s ($5 \le i \le 9$, $1 \le j \le k-2$; and, $i = 1, 2, 3, 4, 10, 11, 12, 13$; $1 \le j \le k-2$; and $i = 0$ and $j \ne 1$) as each cycle in D has length at least c.

Similar to Case 1, to find a lower bound of $\nu(D)$, we only need to consider the case that each G'_{ij} ($i = 1, 2, 3, 4, 10, 11, 12, 13$; $1 \leq j \leq k-2$; and, $i = 0$ and $j \neq 1$) contains a cut edge $a_i a_{ij}$ and, there are n G'_{ij}'s ($5 \leq i \leq 9$, $1 \leq j \leq k-2$) not containing any cut edge of S' and there are at most n $T_{(c-2)/4-1}(a, b)$'s containing cut edges ab of S'. Then

$$\nu(D) \geq \nu(T'_{0,1}) - n\nu(T_{(c-2)/4-1}(a, b)) + n\nu(G'_{ij}) - 1 + c$$
$$\geq [((k-1)^0 + (k-1)^1 + \cdots + (k-1)^4) - n((k-1)^0 + (k-1)^1)] + n[(k-1)^0 + (k-1)^1] - 1 + c$$
$$= [(k-1)^5 - 1]/(k-2) + c - 1$$
$$\geq [(k-1)^5 - 1]/(k-2)$$
$$= [(k-1)^{c/2-2} - 1]/(k-2).$$

So, in all subcases of Case 3, $\nu(D) \geq [(k-1)^{c/2-2} - 1]/(k-2)$.

Case 4: $c = 4m-1$ and m is a positive integer.

The main idea of proof is similar to that of Case 1.

Let $G_{ij} = G[M_{(c+1)/4}(a_{ij})]$ ($i = 0, 1, \cdots, c-1$; $j = 1, 2, \cdots, k-2$). Then $G_{ij} \cap D$ is a tree, and $E(G_{ij} \cap D) \cap E(G_{pq} \cap D) = \emptyset$ ($i \neq p$ or $j \neq q$). Since $|S| \leq (k-2)g-1$ and $c \geq g$, without loss of generality, assume that $G_{0,1}$ does not contain any edge in S. Since every cycle in D has length at least c, $G_{0,1} \cap D$ and $G_{(c-1)/2,j} \cap D$ (or $G_{(c+1)/2,p} \cap D$) may share at most one vertex in $N_{(c+1)/4}(a_{0,1})\backslash C \cap N_{(c+1)/4}(a_{(c-1)/2,j})\backslash C$ (or $N_{(c+1)/4}(a_{0,1})\backslash C \cap N_{(c+1)/4}(a_{(c+1)/2,p})\backslash C$) for one j (and p) such that $1 \leq j, p \leq k-2$.

Let $T_{0,1} = G[M_{(c-1)/2}(a_{0,1})]$. Then $T_{0,1} \cap D$ is a tree as every cycle in D has length at least c. We use G_{ij} and $T_{0,1}$ ($i = 1, 2, \cdots, c-1$; $j = 1, 2, \cdots, k-2$; and, $i = 0$ and $j \neq 1$) to represent the trees in the case that G_{ij} and $T_{0,1}$ are contained in D, and we use $G_{ij} \cap D$ and $T_{0,1} \cap D$ to represent the trees in the real case. Notice that $T_{0,1}$ may share vertices with $G_{(c+1)/4,j}$, $G_{(c+1)/4+1,j}, \cdots, G_{3(c+1)/4-1,j}$ for some $1 \leq j \leq k-2$, whereas does not intersect with $G_{1,j}, G_{2,j}, \cdots, G_{(c+1)/4-1,j}, G_{3(c+1)/4,j}, G_{3(c+1)/4+1,j}, \cdots, G_{c-1,j}$ for any $1 \leq j \leq k-2$ and $G_{0,j}$ for $j \neq 1$.

Let $b \in N_{(c+1)/4+1}(a_{0,1})\backslash C$, $N_1(b)\backslash G_{0,1} = \{b\}$, $N_r(b)\backslash G_{0,1} = \{u \mid u \in V(G)\backslash V(G_{0,1}), d(u, b) = r-1, \text{ and } \exists x \in N_{r-1}(b)\backslash G_{0,1}, xu \in E(G)\}$ ($r \geq 2$). Let $a \in N_{(c+1)/4}(a_{0,1})\backslash C$ such that $ab \in E(T_{0,1})$. Let $M_{(c+1)/4,r}(b) = \{a\} \bigcup_{k=1}^{r} N_k(b)\backslash G_{0,1}$ and $T_r(a, b) = G[M_{(c+1)/4,r}(b)]$. Then $T_r(a, b) \cap D$ is a subtree of $T_{0,1} \cap D$ for all r such that $1 \leq r \leq (c-3)/4$. We use $T_r(a, b)$ to represent the subtrees of $T_{0,1}$ in the case that $T_{0,1}$ is contained in D, and we use $T_r(a, b) \cap D$ to represent the subtrees of $T_{0,1} \cap D$ in the real case for each r such that $1 \leq r \leq (c-3)/4$.

By similar argument to Case 1, we can obtain an edge cutset S' with $|S'| = |S|$ and the remaining subgraph of $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$ after cutted by S' is contained in D.

For each G_{ij}, if we cut the edge a_ia_{ij} of S′, we shall delete at most all vertices of G_{ij} except a_i from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$, that is, to delete $(k-1)^0 + (k-1)^1 + \cdots + (k-1)^{(c+1)/4-1} = [(k-1)^{(c+1)/4}-1]/(k-2)$ vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$. For each $T_{(c-3)/4}(a, b)$, where $a \in N_{(c+1)/4}(a_{0,1})\backslash C$ and $b \in N_{(c+1)/4+1}(a_{0,1})\backslash C$ such that $ab \in E(T_{0,1})$, if we delete a cut edge ab of S′, we shall delete at most all vertices of $T_{(c-3)/4}(a, b)$ except a from $T_{0,1}$, that is, to delete $(k-1)^0 + (k-1)^1 + \cdots + (k-1)^{(c-3)/4-1} = [(k-1)^{(c-3)/4}-1]/(k-2)$ vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$. Since $V(T_{0,1}) \cap V(G_{ij}) = \emptyset$, to cut an edge a_ia_{ij} will delete more vertices from $\bigcup_{i,j} G_{ij} \cup T_{0,1} \cup C$ than to cut an edge ab in $T_{(c-3)/4}(a, b)$ $(i = 1, 2, \cdots, (c+1)/4-1, 3(c+1)/4, 3(c+1)/4+1, \cdots, c-1;\ 1 \le j \le k-2;$ and, $i = 0$ and $j \ne 1)$.

By similar argument to Case 1, to find a lower bound of $\nu(D)$, we only need to consider the case that each G_{ij} $(i = 1, 2, \cdots, (c+1)/4-1, 3(c+1)/4, 3(c+1)/4+1, \cdots, c-1;\ 1 \le j \le k-2;$ and, $i = 0$ and $j \ne 1)$ contains one cut edge a_ia_{ij}, and there are n G_{ij}'s $((c+1)/4 \le i \le 3(c+1)/4-1,\ 1 \le j \le k-2)$ not containing any cut edge of S′ but sharing vertices with $T_{0,1}$, whereas there are at most n $T_{(c-3)/4}(a, b)$'s containing cut edge ab of S′ for some $a \in N_{(c+1)/4}(a_{0,1})\backslash C$ and $b \in N_{(c+1)/4+1}(a_{0,1})\backslash C$ such that $ab \in E(T_{0,1})$.

Let a_0 be the root of $T_{0,1}$. Notice that $T_{0,1}$ originally has $(c-1)/2+1 = (c+1)/2$ layers, G_{ij} $((c+1)/4 \le i \le 3(c+1)/4-1,\ 1 \le j \le k-2)$ can provide at most $n \le (k-2)(c+1)/2$ cut edges to $T_{(c-3)/4}(a, b)$'s, and $N_{(c+1)/4}(a_{0,1})\backslash C$ has $(k-1)^{(c+1)/4-1}$ vertices. By k-regularity of G, if $(k-2)(c+1)/2 \le (k-2)(k-1)^{(c+1)/4-1}$, then we cut at most $(k-2)(k-1)^{(c+1)/4-1}$ subtrees $T_{(c-3)/4}(a, b)$ except a from $T_{0,1}$, the resulting tree of $T_{0,1}$ equivalently has at least the number of vertices of $(c-1)/2$ layers plus a layer of $(k-1)^{(c+1)/4-1}$ vertices, which is a lower bound of $\nu(D)$. So

$$\nu(D) \ge (k-1)^0 + (k-1)^1 + \cdots + (k-1)^{(c-1)/2-2} + (k-1)^{(c+1)/4-1} + c$$
$$= [(k-1)^{(c-1)/2-1}-1]/(k-2) + (k-1)^{(c+1)/4-1} + c$$
$$\ge [(k-1)^{(c-1)/2-1}-1]/(k-2).$$

However, $(k-2)(c+1)/2 \le (k-2)(k-1)^{(c+1)/4-1}$ is equivalent to $\log_{k-1}2 + \log_{k-1}(c+1)/4 \le (c+1)/4 - 1$. Since $x - \log_{k-1}x \ge 1 + \log_{k-1}2$ is satisfied when $x \ge 4$ and $k \ge 3$, when $(c+1)/4 \ge 4$, i.e. $c \ge 15$, the above inequality holds.

Now we discuss the special cases of c = 3, 7 and 11.

Case (4.1): c = 3.

Then c satisfies the requirement of Lemma 2.

Case (4.2): c = 7.

$\nu(D) \ge \nu(G_{0,1}) + c-1 = 7 + k \ge [(k-1)^{(c-1)/2-1}-1]/(k-2) = [(k-1)^{(7-1)/2-1}-1]/(k-2) = k(k-2)/(k-2) = k$.

Case (4.3): c = 11.

Similar to Case (1.3). Let a_0 be the root of $G_{0,1}$. Then $G_{0,1}$ has 4 layers $N_0(a_0)$, $N_1(a_{0,1})\backslash C$, $N_2(a_{0,1})\backslash C$ and $N_3(a_{0,1})\backslash C$. Let $T'_{0,1} = G[M_{(c+1)/4+1}(a_{0,1})] = G[N_0(a_0) \cup N_1(a_{0,1})\backslash C \cup N_2(a_{0,1})\backslash C \cup N_3(a_{0,1})\backslash C \cup N_4(a_{0,1})\backslash C]$. We shall prove that D has at least as many vertices as $T'_{0,1}$.

We can find a cutset S′ similarly as in Case (1.3). The cut edges of S′ in $T'_{0,1}$ are of the form ab such that $a \in N_3(a_{0,1})\backslash C$ and $b \in N_4(a_{0,1})\backslash C$ in which $ab \in E(T_{0,1})$. Each cut edge ab in $T'_{0,1}$ deletes one vertex from $T'_{0,1}$. Let $G'_{ij} = G[M_{(c+1)/4-1}(a_{ij})]$ (i = (c+1)/4, (c+1)/4+1, ⋯, 3(c+1)/4−1; $1 \le j \le k-2$). The cut edges of S′ in G'_{ij} are of the form $a_i a_{ij}$ (i = 3, 4, ⋯, 8; $1 \le j \le k-2$). Notice that $T'_{0,1}$ may share at most one vertex with $G'_{5,j}$ (or $G'_{6,p}$) for one j (and one p) such that $1 \le j, p \le k-2$, but does not intersect with other G'_{ij} (i = 3, 4, ⋯, 8; $1 \le j \le k-2$), and does not intersect with G_{ij} (i = 1, 2, 9, 10; $1 \le j \le k-2$; and, i = 0 and $j \ne 1$) as each cycle in D has length at least c. Similar to Case (1.3), we only need to consider the case that G_{ij} (i = 1, 2, 9, 10; $1 \le j \le k-2$; and, i = 0 and $j \ne 1$) contains a cut edge $a_i a_{ij}$, and there are n G'_{ij}'s (i = 3, 4, ⋯, 8; $1 \le j \le k-2$) which do not contain any cut edge of S′ whereas $T'_{0,1}$ contains at most n cut edges ab of S′. Then

$$\begin{aligned}\nu(D) &\ge \nu(T'_{0,1}) - n + n\nu(G'_{ij}) - 2 + c\\ &\ge [(k-1)^0 + (k-1)^1 + (k-1)^2 + (k-1)^3 - n] + n[(k-1)^0 + (k-1)^1] - 2 + c\\ &\ge [(k-1)^4 - 1]/(k-2) + n[((k-1)^2-1)/(k-2)-1]\\ &\ge [(k-1)^4 - 1]/(k-2)\\ &= [(k-1)^{(c-1)/2-1} - 1]/(k-2).\end{aligned}$$

So, in all subcases of Case 4, $\nu(D) \ge [(k-1)^{(c-1)/2-1}-1]/(k-2)$.

In summary of Cases 1, 2, 3 and 4, if c = 2r, then $\nu(G) \ge \nu(D) \ge [(k-1)^{c/2-2} - 1]/(k-2)$; if c = 2r+1, then $\nu(G) \ge \nu(D) \ge [(k-1)^{(c-1)/2-1} - 1]/(k-2)$.

Suppose c = 2r. Then $c \le 2[\log_{k-1}[(k-2)\nu(G)+1] + 2]$. Since $\nu(G) \ge 1$, $(k-2)\nu(G)+\nu(G) \ge (k-2)\nu(G)+1$. So $(k-1)\nu(G) \ge (k-2)\nu(G)+1$. Then $c \le 2[\log_{k-1}[(k-1)\nu(G)] + 2] = 2(\log_{k-1}\nu(G) + 3)$. Let $\log_{k-1}\nu(G) = \lfloor \log_{k-1}\nu(G) \rfloor + s$. Since $0 \le s < 1$, then $2(\log_{k-1}\nu(G) + 3) < (2\lfloor \log_{k-1}\nu(G) \rfloor + 6) + 2$. But c = 2r is a positive even integer, so $c \le 2\lfloor \log_{k-1}\nu(G) \rfloor + 6$.

Suppose c = 2r+1. Then $c \le 2[\log_{k-1}[(k-2)\nu(G) + 1] + 1] + 1 \le 2[\log_{k-1}[(k-1)\nu(G)] + 1] + 1 = 2[\log_{k-1}\nu(G) + 2] + 1$. By the same reason as above, $c \le 2\lfloor \log_{k-1}\nu(G) \rfloor + 5$.

Then the proof of Lemma 2 is complete. □

In the following, we shall prove the correctness of Algorithm 1.

Theorem 3: Algorithm 1 can find a minimum cyclic edge cutset and hence can determine $c\lambda(G)$.

Proof. By Lemma 1, if $g > \nu/2$, then Algorithm 1 will give answer that $c\lambda = \infty$ in Step 2.

Now suppose $g \leq \nu/2$. Let C be a shortest cycle in G such that $|V(C)| = g \leq \nu/2$. By the proof of Lemma 1, the co-cycle S of C is a cyclic edge cutset of G. So $c\lambda(G) \leq |S| = (k-2)g$. If $c\lambda(G) = (k-2)g$, by Steps 4, 8 and 9, Algorithm 1 will determine that $c\lambda(G) = (k-2)g$.

Now suppose that $c\lambda(G) \leq (k-2)g-1$. Let S be a minimum cyclic edge cutset of G, D_1 and D_2 be components of G−S such that D_1 and D_2 both contain a cycle. Let C_i be a shortest cycle in D_i (i = 1, 2). By Lemma 2, $|V(C_i)| \leq 2\lfloor \log_{k-1}\nu(G)\rfloor + 6$ if C_i is an even cycle; and $|V(C_i)| \leq 2\lfloor \log_{k-1}\nu(G)\rfloor + 5$ if C_i is an odd cycle (i = 1, 2).

In Step 3 of Algorithm 1, we find all minimal cycles of length at most $2\lfloor \log_{k-1}\nu(G)\rfloor + 6$ if the cycle is even and $2\lfloor \log_{k-1}\nu(G)\rfloor + 5$ if the cycle is odd. In Steps 5, 6 and 7, we find a minimum edge cutset which separates C_1 in D_1 and C_2 in D_2. Hence we can find a cyclic edge cutset of size |S|. By Step 8, we can determine the cyclic edge connectivity of G. □

Now we analyse the time complexity of Algorithm 1.

Theorem 4: The time complexity of Algorithm 1 is bounded by $O(k^9V^6)$.

Proof. In Step 1, we use breadth first search strategy to find a shortest cycle containing a vertex v. For a given vertex v, it takes $O(|E|)$ time. For all vertices v in V(G), Step 1 takes $O(|V||E|)$ time. Since G is a k-regular graph, $|E| = k|V|/2$. Hence $O(|V||E|) = O(k|V|^2)$.

Step 2 takes O(1) time.

Now we analyse Step 3. Let e = xy be an edge in E(G). We define $N_0(x) = \{x\}$, $N_0(y) = \{y\}$, $N_1(x) = \{u \mid \exists v \in N_0(x), uv \in E(G), u \notin N_0(y), u \neq x\}$. $N_1(y) = \{u \mid \exists v \in N_0(y), uv \in E(G), u \notin N_0(x) \cup N_1(x), u \neq y\}$. $N_r(x) = \{u \mid \exists v \in N_{r-1}(x), uv \in E(G), u \notin \bigcup_{i=0}^{r-1} N_i(y), u \neq x\}$, $N_r(y) = \{u \mid \exists v \in N_{r-1}(y), uv \in E(G), u \notin \bigcup_{i=0}^{r} N_i(x), u \neq y\}$ $(r \geq 2)$. Then $|N_0(x)| = (k-1)^0$, $|N_0(y)| = (k-1)^0$, $|N_1(x)| = (k-1)^1$, $|N_1(y)| \leq (k-1)^1$, $|N_r(x)| \leq (k-1)^r$, $|N_r(y)| \leq (k-1)^r$.

Since, in Step 3, we find all minimal cycles C containing e for a given edge e such that $|V(C)| \leq 2\lfloor \log_{k-1}\nu(G)\rfloor + 6$ if C is an even cycle or $|V(C)| \leq 2\lfloor \log_{k-1}\nu(G)\rfloor + 5$ if C is an odd cycle. Notice that C does not contain any chord and every edge from $N_r(x)$ to $N_r(y)$ corresponds to a different even cycle containing e for $1 \leq r \leq c/2-1$ and every edge from $N_{r-1}(y)$ to $N_r(x)$ corresponds to a different odd cycle containing e for $1 \leq r \leq (c-1)/2$. Notice also that, if there is an edge from $N_r(x)$ to $N_r(y)$ (or an edge from $N_{r-1}(y)$ to $N_r(x)$, then $|N_{r+1}(x)|$ (or $|N_r(y)|$) will be less than $(k-1)^{r+1}$ by one (or less than $(k-1)^r$ by one). Since G is k-regular, the number of such even cycles is at most $(k-1)(k-1)^{c/2-1} = (k-1)^{c/2} \leq (k-1)^{(2\lfloor \log_{k-1} \nu(G) \rfloor+6)/2} \leq (k-1)^{\log_{k-1} \nu(G)+3} = (k-1)^3(k-1)^{\log_{k-1} \nu(G)} = (k-1)^3\nu(G)$, and the number of such odd cycles is at most $(k-1)(k-1)^{(c-1)/2-1} = (k-1)^{(c-1)/2} \leq (k-1)^{(2\lfloor \log_{k-1}\nu(G)\rfloor+5-1)/2} \leq (k-1)^{\log_{k-1} \nu(G)+2} \leq (k-1)^2\nu(G)$. For all edges in E(G), there are at most $O((k-1)^3|V||E|) = O(k^4|V|^2)$ such even cycles and $O((k-1)^2|V||E|) = O(k^3|V|^2)$ such odd cycles. In general, we can say there are at most $O(k^4|V|^2)$ cycles of length at most $2\lfloor \log_{k-1}\nu(G)\rfloor + 6$ in F.

In Step 5, the FOR loop is for each combination of two different cycles in F. So the loop repeats $O(k^8|V|^4)$ times. Step 6 takes $O(|V|)$ time to test $V(C_1) \cap V(C_2) = \emptyset$ and to construct G′.

By [9], Step 7 takes $O(|V||E|) = O(k|V|^2)$ time to find a minimum edge cutset S_{xy} using the traditional maximum flow method. So the loop of Steps 5, 6, 7 and 8 takes totally $O(k^9|V|^6)$ time. The time complexity of Algorithm 1 is bounded by $O(k^9V^6)$. □

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