A surprising regularity in the number of Hamilton paths in polygonal bigraphs

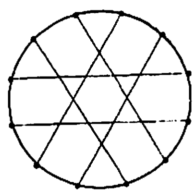
Gustavus J. Simmons

Abstract:

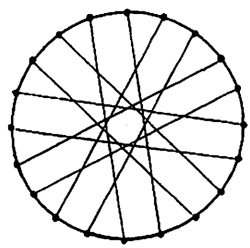
The smallest bigraph which is edge critical but not edge minimal with respect to Hamilton laceability is the Franklin graph. Polygonal bigraphs*, P_m, which generalize one of the many symmetries of the Franklin graph, share this property of being edge critical but not edge minimal [1]. An enumeration of Hamilton paths in P_m for small m reveals surprising regularities: there are 2^m Hamilton paths between every pair of adjacent vertices, $3x2^{m-1}$ between every vertex and a unique companion vertex and $3x2^{m-2}$ between all other pairs. Hamilton laceability only requires there be at least one Hamilton path between every pair of vertices in different parts; this says there are exponentially many.

Introduction:

Figure 1 shows the most common graphical representation of the Franklin graph and the same graphical representation for P_5.



Franklin graph (P_3)



P_5

Figure 1

This representation, which lends itself to proving P_m to be edge critical with respect to Hamilton laceability, does not lend itself to the investigation of Hamilton paths in P_m. However an easily described alternate representation which maps P_m onto an annulus does. The 6m edges in P_m have a natural partition into the 2m edges not on a quadrilateral, R_m, and the 4m that are, Q_m.

* Extend the sides of a regular polygon on 2m vertices, $m \geq 2$, to define the $2m(m - 1)$ finite points of intersection. Circumscribe a centrally symmetric circle large enough all of the points of intersection are in its interior. The 4m points of intersection of the extended edges of the polygon with the circle are the vertices of the polygonal bigraph, P_m. The edges are the 4m arcs of the circle between the vertices and the 2m diagonals defined by the extended edges. P_2 is the edge skeleton of the 3-cube, Q_3. P_3 is the Franklin graph.

The graphical representation in Figure 1 further partitions the 4m edges in Q_m into 2m on the outer cycle, C_m, and the 2m diagonals, D_m. This is an artificial partition since there are mappings of P_m that carry any edge in Q_m and it's edge incidences into any other edge in Q_m, but is useful in describing the annular mapping. Instead of a single cycle on 4m vertices, define two cycles on 2m vertices each. Start with an arbitrary edge in R_m and form a path using only edges from R_m and D_m. If m is odd, the path will close to form a cycle with 2m edges. Form another cycle on the remaining 2m edges in R_m and D_m. Construct a new graphical representation of P_m with one of the cycles symmetrically enclosing the other – it is immaterial which is the outer cycle and which the inner. If the pairs of edges that were in a quadrilateral in the Figure 1 representation of P_m are rotated to be in the same relative position in the two cycles, the edges in C_m will cross connect to the edges in D_m to form m twisted quadrilaterals lying on the annulus defined by the two concentric cycles. The same annular representation is formally possible when m is even – even though the path constructed using only edges from R_m and D_m lies on all 4m edges instead of breaking up into two cycles on 2m. Figure 2 shows the annular representation of P_m.

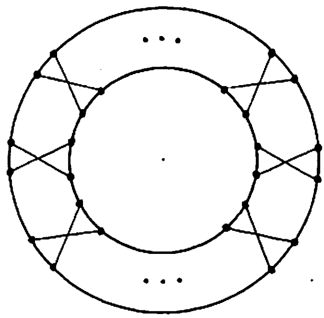


Figure 2

The construction in Figure 2 suggests a general method for constructing and/or counting Hamilton paths in P_m. While the number of quadrilaterals, m, can be arbitrarily large, at most two can host endpoints for a Hamilton path, which means that for all $m \geq 3$ there will be runs of quadrilaterals not containing an endpoint. If the endpoints of the Hamilton path are in the same or adjacent quadrilaterals in P_m there will be only one such run, while if the endpoints are in non-adjacent quadrilaterals there will be two. Simple parity says that either all four edges from R_m are used to connect a run to the rest of P_m or else just one on each end of the run. In either case the path(s) in the run must connect from one end of the run to the other. Since all of the quadrilaterals are twisted, for a path to reverse direction it would have to lie on three of the vertices in some quadrilateral, leaving the fourth vertex isolated from being in a Hamilton path.

There are a couple of simple observations. A path through a run of n quadrilaterals can start on either cycle and end on either the same cycle or the other. If there is only one path it must span all 4n vertices. If there are two paths they must each lie on 2n vertices. The number of Hamilton consistent paths through a run grows exponentially with n but a simple technique allows them to be succinctly described and enumerated.

There are only six ways paths can lie on all four vertices in a quadrilateral, i.e for them to be Hamilton consistent; see Figure 3. The quadrilateral path(s) represented by symbols A, C and C' switch from one cycle to the other. The path(s) represented by symbols B, D and D' do not.

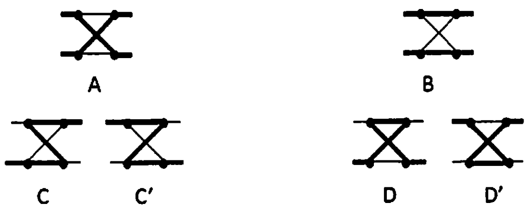


Figure 3

The edge(s) from R_m leaving one quadrilateral must match those entering the next, so there are always exactly two compatible choices for the paths in the next quadrilateral. Table 1 gives the syntactical rules for constructing Hamilton consistent symbol sequences.

Symbol	Successor symbol
A	A or B
B	A or B
C	C' or D'
C'	C or D
D	C or D
D'	C' or D'

Table 1

The important thing to notice in Table 1 is that all symbols have one successor that switches cycles for the path(s) and one that doesn't. Therefore the 2^{n-1} syntactically correct sequences, starting from an arbitrary symbol, partition into 2^{n-2} that switch cycles and 2^{n-2} that do not depending on whether an odd or even number of the symbols A, C or C' occur in the sequence. All that matters, so far as Hamilton paths are concerned, is which of the edges from R_m connect the run to the rest of P_m and how. There are six ways this can be done. If the connection uses both edges on each end of the run, the paths through the run can either switch cycles or not. If only a single edge is used at each end, it can enter on either cycle and exit on either that same cycle or the other. The argument just given shows that there are 2^{n-2} path(s) in each of these six cases. The proofs that follow depend critically on this result.

As an illustration of the power of this result, it is little more than a remark now to show that there are $3x2^{m-1}$ Hamilton cycles in P_m. As noted earlier the path cannot double back on itself so must encircle the annulus. This says an m symbol sequence representing a Hamilton cycle must also close on itself to form a cycle. A cyclical AB sequence must have an odd number of occurrences of A so there are 2^{m-1} such cycles. A CD sequence can start with any one of the four symbols from which 2^{m-1} syntactically correct m symbol sequences originate. But only half of these end in a symbol that can precede the starting symbol. Therefore there are $4x2^{m-1}x½ = 2^m$ CD cycles and hence a total of $3x2^{m-1}$ Hamilton cycles in P_m.

Hamilton paths in P_m:

The problem of counting Hamilton paths between specified endpoints in P_m reduces to characterizing the ways in which initial paths from the endpoints can connect to runs of quadrilaterals – and then using the properties of runs just developed to describe all possible Hamilton paths. There are three basic cases which must be considered, each of which has further subdivisions depending on in which cycles the endpoints are located.

Case 1. The endpoints are in the same quadrilateral
- i. The endpoints are on the same cycle
- ii. The endpoints are on different cycles

Case 2. The endpoints are in adjacent quadrilaterals.
- i. The endpoints are on the same edge in R_m connecting the host quadrilaterals.
- ii. The endpoints are on distinct edges in Rm each of which connects the host quadrilaterals.
- iii. The endpoints are on distinct edges in R_m, neither of which connects the two host quadrilaterals.

Case 3. The endpoints are in non-adjacent quadrilaterals.

Case 1.
Figure 4 shows the eight possible quadrilateral paths; the upper four for sub-case 1i and the lower four for sub-case 1ii. The left hand pair of paths must be connected by an AB sequence which must switch cycles in subcase 1i and not in sub-case 1ii. The remaining quadrilateral paths must be connected by CD sequences. Consequently each of the eight quadrilateral paths in Figure 4 contributes 2^{m-2} Hamilton paths for a total of 2^m in each sub-case.

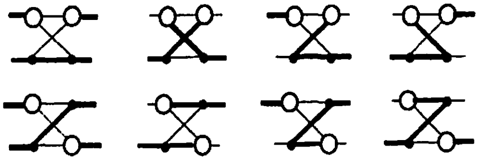


Figure 4

Case 2.
The subcases must be treated separately since the paths connecting the endpoints to runs differ so greatly. Figure 5 shows the eight possible quadrilateral paths for subcase 2i. Four must be connected by AB sequences and four by CD sequences. Two of the AB sequences require the cycles to be switched and two do not, therefore each of the eight quadrilateral paths contributes 2^{m-3} Hamilton paths for a total of 2^m.

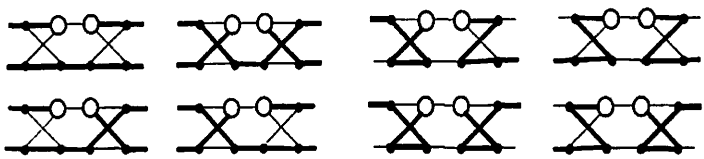


Figure 5

Since the only way a pair of endpoints can be adjacent in P_m is to either be in the same quadrilateral or else to be endpoints of an edge in R_m this completes the proof that there are 2^m Hamilton paths between every pair of adjacent vertices in P_m.

Figure 6 shows the eight possible quadrilateral paths for subcase 2ii. Just as in subcase 2i, four are connected by AB sequences and four by CD sequences. However the four connected by AB sequences this time all have a path reversal so all AB sequences will work. Therefore the total number of Hamilton paths between the endpoints in subcase 2ii is $4x2^{m-2} + 4x2^{m-3} = 3x2^{m-1}$. Figure 2 shows that the 2m edges in R_m are paired through connecting the same pair of quadrilaterals. Since all 4m vertices are in R_m, the companion vertex to any vertex is defined by the conditions for subcase 2ii.

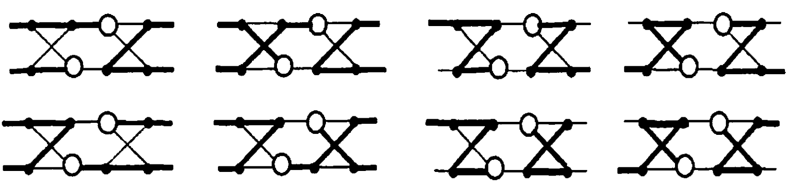


Figure 6

Figure 7 shows the six possible quadrilateral paths for subcase 2iii when the endpoints are on the same cycle and Figure 8 for when they are not. The proof argument is the same for both subcases. All twelve paths must be connected by CD sequences, each of which contributes 2^{m-3} Hamilton paths. Therefore the total number of Hamilton paths for either vertex pair is $3x2^{m-2}$.

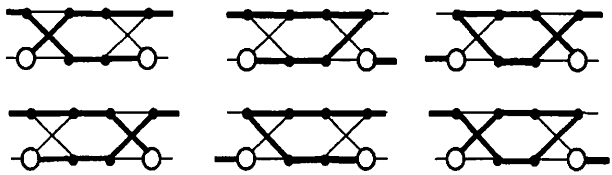


Figure 7

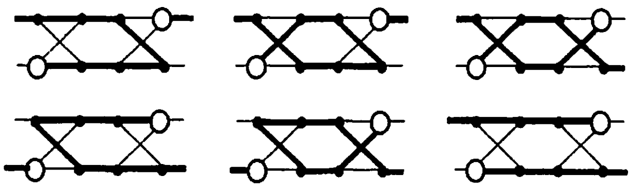


Figure 8

Case 3. The endpoints are in non-adjacent quadrilaterals.

The endpoints can be in the same cycle or in different cycles. With no loss of generality assume one is in the outer cycle. There are only three Hamilton consistent paths through the host quadrilateral for that choice of an endpoint; see Figure 9.

Figure 9

There will also be three equivalent Hamilton consistent paths through the other host quadrilateral irrespective of the cycle on which the endpoint lies. It is easy to see from Figure 9 and the fact that the endpoints are by parity necessarily proximate in the annulus that the run connecting the endpoint sides of the two host quadrilaterals will be a CD sequence and the other run an AB sequence. This observation is crucial to the proof argument.

Let X', Y' and Z' denote the mapping of X, Y and Z into the other host quadrilateral. There are nine pairings of these quadrilateral paths. The five pairings that include either X or X' must be treated separately due to the path reversal in X and X'. The pair X-X' cannot have a Hamilton path since the two path reversals form a closed loop. The other four pairings do. Assume there are $i \geq 1$ quadrilaterals in the AB run. The path reversal at X (or X') forms a loop. Each occurrence of A in the sequence switches cycles, but the endpoints remain connected by the loop. Therefore for Hamilton path considerations it doesn't matter how many A cycle switches occur. As remarked earlier, in this case there are 2^i paths through the AB sequence. There are m - 2 - i quadrilaterals in the CD run, and consequently $2^{(m-2-i)-1}$ Hamilton consistent paths associated with each pair of the quadrilateral paths. The paths in the two runs are independent, so the total number is multiplicative; $2^i x\ 2^{(m-2-i)-1}$ which when multiplied by four, the number of pairs, yields the result that there are 2^{m-1} Hamilton paths in cases in which one of the quadrilateral paths is either X or X'. The cases in which neither X nor X' appear require the AB sequence to either switch cycles or else to not switch them which results in only 2^{i-2} paths through the AB run and 2^{m-2} paths in all. Summing these two values shows there are $3x2^{m-2}$ Hamilton paths between any pair of vertices in different parts in non-adjacent quadrilaterals.

In summary, there are 2^m Hamilton paths between any of the 6m pairs of adjacent vertices, $3x2^{m-1}$ between any of the 2m companion vertex pairs defined in subcase 2ii and $3x2^{m-2}$ between all of the other $4m^2$-8m vertex pairs in P_m.

Concluding remark:

P_m is edge critical with respect to Hamilton laceability [1]. It is truly surprising, given that there are exponentially many Hamilton paths between every pair of vertices from different parts in P_m, that deleting an arbitrary edge results in at least one pair having none.

References:

1. G. J. Simmons, A family of edge critical, but not edge minimal, Hamilton laceable bigraphs, pending publication Bulletin of the Institute of Combinatorics and its Applications