

5-factor-critical graphs on the torus*

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Abstract

A graph of order n is said to be *k-factor-critical* for non-negative integer $k \leq n$ if the removal of any k vertices results in a graph with a perfect matching. For a *k-factor-critical* graph of order n , it is called *trivial* if $k = n$ and *non-trivial* otherwise. It is known that the toroidal graphs are at most non-trivial 5-factor-critical. Motivated by this, we are to characterize all non-trivial 5-factor-critical graphs on the torus in this paper.

Keywords: Matching; *k*-Factor-critical; Graph on the torus.

1 Introduction

A *matching* M of G is a set of independent edges in it such that no two edges share a common endvertex. If it covers all the vertices of G , then it is called *perfect*. For $0 \leq k \leq n$, a graph G of order n is said to be *k-factor-critical* (*k-fc* for short) if the removal of any k vertices results in a graph with a perfect matching. 1-fc and 2-fc graphs are the usual factor-critical and bicritical graphs, respectively, which play important roles in characterizing the structure of graphs with respect to their matchings, especially in the canonical partition of an elementary graph and the procedure for constructing all elementary graphs from two family of basic building blocks, namely elementary bipartite graphs and bicritical graphs [9]. For a *k-fc* graph of order n , it is called *trivial* if $k = n$ and *non-trivial* otherwise. Some important properties of *k-fc* graphs are valid only for $k < n$; for example, a non-trivial *k-fc* graph is *k*-connected, and non-trivial *k-fc* graphs are also $(k - 2)$ -fc graphs for $k \geq 2$. A lot of research on non-trivial *k-fc* graphs

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is devoted to their vertex-connectivity, edge-connectivity, degree sum, relationship with matching extendability, construction, characterizations and so on. For more details, see [2, 5, 6, 12, 13, 15, 16, 17, 18, 19]. In particular, Refs. [12, 13] determined the factor-criticality of graphs on surfaces.

A *surface* Σ is a connected compact Hausdorff topological space which is locally homeomorphic to an open disc in the plane. According to the Classification Theorem of Surfaces, every surface Σ is homeomorphic to precisely one of the orientable surfaces S_g ($g \geq 0$) or the non-orientable surfaces N_k ($k > 0$), where S_g and N_k are obtained from the sphere by adding g handles and k cross-caps, respectively. Particularly, S_0 , S_1 and N_2 are the well-known sphere, torus and Klein bottle, respectively.

By a graph on a surface, we mean a drawing of the graph on the surface such that any pair of edges can only intersect at their endvertices. Such a drawing is called an embedding.

Su and Zhang [13] posed the concept of the factor-criticality of surfaces. For a given surface Σ , let $\rho(\Sigma)$ denote the smallest integer h such that no graph embedded on it is non-trivial h -fc. We call $\rho(\Sigma)$ the factor-criticality of surface Σ . In [13], Su and Zhang obtained that $\rho(\Sigma) = \lfloor \frac{5 + \sqrt{49 - 24\chi(\Sigma)}}{2} \rfloor$ for any surface Σ except the Klein bottle N_2 , where $\chi(\Sigma)$ is the Euler characteristic of Σ , i.e., $\chi(\Sigma) = 2 - 2g$ if $\Sigma = S_g$ and $\chi(\Sigma) = 2 - k$ if $\Sigma = N_k$. Plummer and Zha [12] showed that $\rho(N_2) = 6$ and completely solved the remaining problem of determining $\rho(\Sigma)$. By the result of Su and Zhang, $\rho(S_1) = 6$. Hence the graphs on S_1 are at most non-trivial 5-fc. So it is natural to pose the problem to characterize all non-trivial 5-fc graphs on S_1 . This is what we are going to do in this paper.

The structure of the paper can be sketched as follows. First we present some properties of simple 6-regular triangulations (i.e., every vertex is of degree 6 and every face is of length 3) on the torus. Next we show that a non-trivial 5-fc graph on the torus is necessarily a simple 6-regular triangulation. Then, based on the good properties of simple 6-regular triangulations on the torus obtained in Section 2, we show that a toroidal graph is non-trivial 5-fc if and only if it is a simple 6-regular triangulation on the torus other than one exceptional graph.

2 Some properties of simple 6-regular triangulations on the torus

In this section, we present some properties related to cyclic edge-connectivity and connectivity of simple 6-regular triangulations on the torus. In what follows, let Δ stand for a simple 6-regular triangulation on the torus.

Now we are to show that Δ is cyclically 12-edge-connected and further

cyclically optimal. Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. For $F \subseteq E(G)$ (resp. $S \subseteq V(G)$), by $G-F$ (resp. $G-S$) we mean the resulting graph by deleting the edges in F (resp. the vertices in S together with their incident edges) from G . For $\emptyset \neq X \subset V(G)$, we denote by $\partial(X)$ the set of edges of G with one end in X and the other end in $V(G) \setminus X$ and call it the *edge-cut* associated with X . Let $d(X) = |\partial(X)|$. For an edge-cut F of a graph G , if there are at least two components of $G - F$ containing cycles, then F is said to be a *cyclic edge-cut*. A connected graph G is called *cyclically k -edge-connected*, if G has a cyclic edge-cut and every such edge-cut is of size at least k . The maximum integer k such that G is cyclically k -edge-connected is called the *cyclic edge-connectivity* of G , denoted by $c\lambda(G)$. Let $\zeta(G) = \min\{d(X) | X \subseteq V(G) \text{ induces a shortest cycle in } G\}$. For simplicity, we also use $\partial(G')$ and $d(G')$ to substitute for $\partial(V(G'))$ and $d(V(G'))$, respectively, for a subgraph G' of G . Wang and Zhang [14] show that $c\lambda(G) \leq \zeta(G)$ for any graph with a cyclic edge-cut. If $c\lambda(G) = \zeta(G)$, then G is called *cyclically optimal*.

The following lemma, presenting the well-known *Euler's Formula* on the plane graph, is needed.

Lemma 2.1. *Let n, m and f denote the numbers of vertices, edges and faces of a connected plane graph G respectively. Then $n - m + f = 2$.*

The next theorem shows that Δ is cyclically optimal. Before proving the theorem, we introduce some facts related to the simple closed curves on the torus.

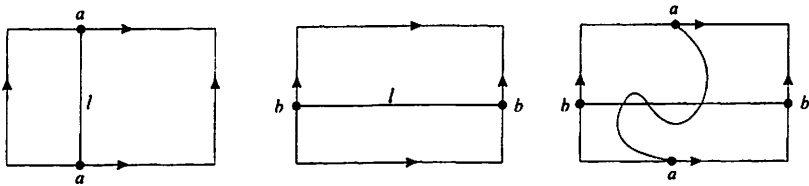


Figure 1. Illustration for essential simple closed curves on the torus.

Let l be a simple closed curve on the torus. If l bounds a 2-cell (i.e., a region that is homeomorphic to a disk), then it is said to be trivial, and essential otherwise. It is known that there are two types of essential simple closed curves [8], one is called a longitude and the other is called a parallel (see the left and middle ones in Figure 1). For an essential simple closed curve on the torus, the deletion of it from the torus results in a cylinder (not a 2-cell or a torus with a hole). Here are some facts that will be used in the sequel. For a graph on the torus, if it does not contain any essential cycle,

then it is a plane graph; If it contains only one type of essential cycles, then it is a cylindrical graph; If it contains two types of essential cycles, then the deletion of the graph from the torus results in several 2-cells.

Theorem 2.2. Δ is cyclically 12-edge-connected, and hence cyclically optimal.

Proof. First we show that Δ has a cyclic edge-cut of size 12. Let S be the vertices of a triangle in Δ . Then $d(S) = d(\Delta - S) = 12$. Since Δ is 6-regular and simple, there are at least three vertices in $\Delta - S$. Therefore, $|E(\Delta - S)| = \frac{6|V(\Delta - S)| - 12}{2} = 3|V(\Delta - S)| - 6 \geq |V(\Delta - S)|$. Consequently, $\Delta - S$ contains a cycle. Thus $\partial(S)$ is a cyclic edge-cut of size 12.

Now we are to show that $c\lambda(\Delta) \geq 12$. Let F be a cyclic edge-cut with $|F| = c\lambda(\Delta)$. Then $\Delta - F$ has exactly two components, denoted by G_1 and G_2 .

We claim that at least one of G_1 and G_2 is a cylindrical graph or a plane graph. If G_1 is cylindrical or plane, then we are done. Otherwise, G_1 contains the two types of essential cycles on the torus. The deletion of G_1 from the torus results in several 2-cells and G_2 must be bounded by some 2-cell. Hence G_2 is a plane graph. Without loss of generality, suppose that G_1 is a cylindrical graph or a plane graph.

Let $|V(G_1)| = n_1$ and $|E(G_1)| = m_1$. Then by the Handshake Lemma, which states that the degree sum of the vertices of a graph is equal to twice the size of the edge set, $6n_1 - |F| = 2m_1$.

If G_1 is a cylindrical graph, then it has two boundaries, the lengths of which are denoted by l_1 and l_2 respectively. By adding two 2-cells with their boundaries fit to the two boundaries of G_1 , we obtain a plane graph with all its faces triangles except at most two. By a simple computation, the number of faces of G_1 is $\frac{2m_1 - l_1 - l_2}{3} + 2$. By Lemma 2.1, $n_1 - m_1 + 2 + \frac{2m_1 - l_1 - l_2}{3} = 2$. Combining the equation with $6n_1 - |F| = 2m_1$, we obtain that $|F| = 2(l_1 + l_2) \geq 12$.

If G_1 is a plane graph, then it has one boundary, the length of which is denoted by l_1 . By adding a 2-cell with its boundary fit to the boundary of G_1 , we obtain a plane graph with all its faces triangles except at most one. Then the number of faces of G_1 is $\frac{2m_1 - l_1}{3} + 1$. By a similar calculation as the above case, we have that $|F| = 6 + 2l_1 \geq 12$.

By the above arguments, we have that $c\lambda(\Delta) = \zeta(\Delta) = 12$. \square

By applying the above result on the cyclic edge-connectivity of Δ , we are going to obtain that Δ is 6-connected and maximally connected. A graph G is said to be k -connected if at least k vertices must be deleted to disconnect the graph, $k < |V(G)|$. The maximum integer k such that G is k -connected is called the *connectivity* of G , denoted by $\kappa(G)$. It is

obvious that $\kappa(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G . If $\kappa(G) = \delta(G)$, then G is said to be *maximally connected*.

Theorem 2.3. Δ is 6-connected, and hence maximally connected.

Proof. Obviously Δ is connected. Let $S \subset V(G)$ be a minimum cut and let G_1 and G_2 be two components of $\Delta - S$. We are going to prove that $|S| \geq 6$. Since S is minimum, any vertex $v \in S$ has a neighbor x in G_1 and a neighbor y in G_2 . Since Δ is a simple triangulation, $\Delta[N(v)]$, the subgraph induced by the neighborhood $N(v)$ in Δ , contains a hamiltonian cycle. Hence there are two internally disjoint xy -paths in $\Delta[N(v)]$, which both pass through vertices in S . Then $|S \cap N(v)| \geq 2$. This implies that $|S| \geq 3$ and each vertex in S has at least two neighbors in S . Hence $\Delta[S]$ contains a cycle.

Suppose that $|S| \leq 4$. Since Δ is 6-regular and $|S| \leq 4$, each G_i ($i = 1, 2$) should have at least three vertices. If some G_i contains a cycle, then $\partial(G_i)$ is a cyclic edge-cut. Hence by Theorem 2.2, we have $d(G_i) \geq 12$. If some G_i contains no cycle, then it is a tree and $d(G_i) = 6|V(G_i)| - 2(|V(G_i)| - 1) = 4|V(G_i)| + 2 \geq 14$. Thus, in any case, G_1 and G_2 should totally receive at least 24 edges from S . But S sends at most $6|S| - 2|S| = 4|S| \leq 16$ edges to them, a contradiction.

Suppose that $|S| = 5$. Similarly we have that each G_i should have at least two vertices, and $d(S) \leq 4|S| = 20$. On the other hand, similarly we also have that if G_i has a cycle, then $d(G_i) \geq 12$, otherwise $d(G_i) = 4|V(G_i)| + 2 \geq 10$. Since $20 \leq d(G_1) + d(G_2) \leq d(S) \leq 20$, all equalities must hold. That implies that $|V(G_1)| = |V(G_2)| = 2$, and each vertex in both G_1 and G_2 must be adjacent to every vertex in S . Consequently, Δ contains $K_{4,5}$ as a subgraph. But $K_{4,5}$ cannot be embedded on the torus by Theorem 4.5.3 in [7], a contradiction.

By the above arguments, $|S| \geq 6$. Since $|S| \leq \delta(\Delta) = 6$, $|S| = 6$. So $\kappa(\Delta) = 6$. □

By the 6-connectivity of Δ , we obtain the following theorem, which characterizes the embedding property of one component obtained from the deletion of a minimum cyclic edge-cut.

Theorem 2.4. Let F be a cyclic edge-cut of Δ with $|F| = 12$. Then $\Delta - F$ has exactly two components. If one contains the two types of essential cycles on the torus, then the other is a plane triangle.

Proof. Let G_1 and G_2 be the two components of $\Delta - F$. If G_1 contains the two types of essential cycles on the torus, then by the similar arguments as those in Theorem 2.2, G_2 is a plane graph. Let l_2 be the length of the boundary walk W of G_2 . By the arguments in Theorem 2.2, $6 + 2l_2 =$

$|F| = 12$, so $l_2 = 3$. Since Δ is simple and 6-connected, W is a cycle and there are no vertices inside W , we are done. \square

3 The characterization of non-trivial 5-factor-critical graphs on the torus

In this section, we first show that a non-trivial 5-fc graph on the torus is necessarily a simple 6-regular triangulation.

A graph with at least two vertices is said to be *k-edge-connected* if each edge-cut has at least k edges.

Lemma 3.1. ([6]) *For $k \geq 1$, every non-trivial k -fc graph is k -connected and $(k + 1)$ -edge-connected.*

We define the *mean degree* of a graph G as the arithmetic mean value of degrees taken over all vertices of G , and denote it by $\bar{\delta}(G)$. The following result presents the upper bound of the mean degree of a toroidal graph and the embedding characteristic when the upper bound holds.

Lemma 3.2. ([11]) *For any toroidal graph G , $\bar{\delta}(G) \leq 6$ and equality holds if and only if G is a 6-regular triangulation.*

Lemma 3.3. *If G is a non-trivial 5-fc toroidal graph, then it is necessarily a simple 6-regular triangulation.*

Proof. Since G is 6-edge connected by Lemma 3.1, $\bar{\delta}(G) = 6$ and G is a 6-regular triangulation by Lemma 3.2. Since G is 5-connected, each vertex has at least five neighbors. If a vertex has exactly five neighbors, then their deletion from G results in a subgraph with an isolated vertex. This contradicts that G is 5-fc. Therefore G is a simple graph. \square

We now present a strengthened version of Tutte's 1-factor Theorem. It is less used than Tutte's 1-factor Theorem but helpful in our proof as we will see.

We call a vertex set $S \subseteq V(G)$ *matchable* to $G - S$ if the (bipartite) graph H_s , which arises from G by contracting each component $c \in \mathcal{C}_{G-S}$ to a singleton and deleting all the edges inside S , contains a matching covering the vertices of S , where \mathcal{C}_{G-S} denotes the set of the components of $G - S$.

Theorem 3.4. ([4], pp 41) *Every graph G contains a set $S \subseteq V(G)$ with the following two properties:*

- (i) *S is matchable to $G - S$;*
- (ii) *Every component of $G - S$ is factor-critical.*

Given any such set S , G has a perfect matching if and only if $|S| = |\mathcal{C}_{G-S}|$.

The next lemma gives an estimation of the number of edges in a bipartite graph embedded on the torus, which is obtained by Euler's formula of graphs on the torus ($n - m + f = 0$) and the property that the length of each face in a bipartite graph is at least four.

Lemma 3.5. *Let G be a bipartite graph embedded on the torus with n vertices and m edges. Then $m \leq 2n$.*

The following theorem characterizes all non-trivial 5-fc graphs in simple 6-regular triangulations.

Theorem 3.6. *Δ is non-trivial 5-fc if and only if the order of Δ is odd and $\Delta \neq \Delta_9$ (Δ_9 is shown in Figure 2).*

Proof. For necessity, by the definition, a non-trivial 5-fc graph must have an odd number of vertices. Meanwhile, Δ_9 is not non-trivial 5-fc from the illustration of Figure 2.

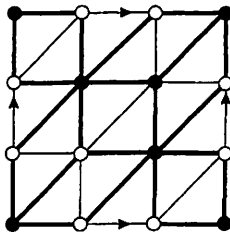


Figure 2. Δ_9 and the five white vertices preventing it to be 5-fc.

For sufficiency, suppose by the contrary that Δ is not non-trivial 5-fc. Then there exists a vertex set $S' \subseteq V(\Delta)$ of size 5 such that $\Delta - S'$ has no perfect matching. Set $G = \Delta - S'$. Then, by Theorem 3.4, G contains a vertex set S'' with S'' matchable to $G - S''$, and every component of $G - S''$ is 1-fc, while $|S''| < |C_{G-S''}|$ since G has no perfect matching. Denote such 1-fc components by G_1, G_2, \dots, G_t respectively. Then $t = |C_{G-S''}| \geq |S''| + 1$. Moreover, since the order of G is even, $|S''|$ and t have the same parity and $t \geq |S''| + 2$.

Let $S = S' \cup S''$. We contract each component G_i ($1 \leq i \leq t$) into a singleton, and delete the edges in $\Delta[S]$ and the multiple edges or loops produced by the contraction. Denote the resulting bipartite graph by G' . So G' can also be embedded on the torus. Let m' be the number of edges of G' . By Theorem 2.3, each component G_i has at least six neighbors in S . On the other hand, since Δ is 6-regular, each vertex in S receives at most 6 edges from the components. Then $6(|S''| + 2) \leq 6t \leq m' \leq 6(|S''| + 5)$. Since t and $|S''|$ have the same parity, $t = |S''| + 2$ or $t = |S''| + 4$.

Since G' has $5 + |S''| + t$ vertices and m' edges, by Lemma 3.5, $m' \leq 2(5 + |S''| + t)$. Together with $m' \geq 6t$, we have $2t \leq |S''| + 5$.

If $t = |S''| + 4$, then $2t \leq |S''| + 5$ implies that $|S''| \leq -3$, a contradiction.

Thus $t = |S''| + 2$. By substituting this into the inequality $2t \leq |S''| + 5$, we obtain that $|S''| = 0$ or 1 .

If $|S''| = 0$, then S' is a cut of Δ of size 5, since $t = 2$. But Δ is 6-connected by Theorem 2.3, a contradiction. Hence $|S''| = 1$, and $|S| = 6$ and $t = 3$.

We now claim that if any G_i is a singleton, then all the other components are singletons, too. Without loss of generality, assume that G_1 is a singleton, denoted by w . Since w is adjacent to every vertex in S , $\Delta[S]$ contains the six edges in the triangular faces at w that are not incident with w . It follows that S sends at most $36 - 12 - 6 = 18$ edges to G_2 and G_3 . If G_i ($i=2$ or 3) is not a singleton, then it contains a cycle since it is 2-edge-connected by Lemma 3.1, and $\partial(G_i)$ is a cyclic edge-cut. By Theorem 2.2, $d(G_i) \geq 12$. Hence at least one of G_2 and G_3 is a singleton, say $G_2 = u$. Suppose to the contrary that G_3 is not a singleton. Then $d(G_3) \geq 12$. So there are exactly 6 edges in $\Delta[S]$. Recall that $S = N(u)$ and $\Delta[N(u)]$ contains a 6-cycle. Thus $\Delta[S] = \Delta[N(u)]$ is a 6-cycle, that is, every vertex in S is of degree 2 in $\Delta[S]$. For any fixed $s \in S$, it has two neighbors in S , denoted by s_0 and s_1 , and two neighbors in G_3 , denoted by t_1 and t_2 , besides u and w . So $N(s) = \{s_0, s_1, w, u, t_1, t_2\}$. Note that $\Delta[N(s)]$ contains a 6-cycle. However, u and w can only be adjacent to s_0 and s_1 in $N(s)$ and therefore the vertices in $N(s)$ cannot form a 6-cycle, a contradiction. So the claim holds.

We first consider that case that each G_i is a singleton. To meet the condition that every vertex is of degree 6, each G_i is adjacent to every vertex in S . Therefore, Δ contains $K_{3,6}$ as a subgraph. It is known that $K_{3,6}$ has a unique embedding on S_1 (see the bold edges in Figure 2). So Δ is obtained from the unique embedding by triangulating all quadrangular faces. It is Δ_9 (see Figure 2).

Next we consider the case that each G_i ($i = 1, 2, 3$) is not a singleton. By the above arguments, $d(G_i) \geq 12$ for each $1 \leq i \leq 3$. Since $d(S) \leq 36$, we have $d(G_i) = 12$ for each $1 \leq i \leq 3$ and S is an independent set. Further $\partial(G_i)$ is a minimum cyclic edge-cut. Therefore each component G_i is a plane graph or a cylindrical graph by Theorem 2.4. By the proof of Theorem 2.2, if it is a cylindrical graph, then the two boundaries are two triangles; If it is a plane graph, then its boundary is a triangle. For each vertex $v \in S$, in order to fulfill the 6-regular triangulation at it, we can only use the edges on the boundaries of the three components. Recall that the 6 edges lying in the triangular faces at a vertex which are not incident with the vertex form a 6-cycle and S is independent. Hence every vertex s in S must be connected to a 6-cycle on the boundary in some component

G_i . However, the boundaries of the components are triangulations. That is a contradiction. \square

Combining Lemma 3.3 with Theorem 3.6, we conclude the main result as follows.

Theorem 3.7. *All non-trivial 5-fc graphs on the torus are the simple 6-regular triangulations with odd number of vertices except Δ_9 .*

Remark 3.8. *The simple 6-regular triangulations on the torus have been classified in 1973 [1].*

Acknowledgments

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