Completely Regular Endomorphisms of Split Graphs

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Abstract

In [8], Weimin Li and Jianfei Chen studied split graphs such that the monoid of all endomorphisms is regular. In this paper, we extend the study of [11]. We find conditions such that regular endomorphism monoids of split graphs are completely regular. Moreover, we find completely regular subsemigroups contained in the monoid $End(G)$.

0 Introduction

In 1987 Knauer and Wilkeit posed the question, for which graph G is the endomorphism monoid of G regular [9]. After this question was posed, there are many results, which investigate this situation for different types of graphs, were studied [1, 2, 3, 6, 8, 12]. Now some extentions of this question have been studied. In this paper we consider completely regular endomorphisms. We focus the discussion to split graphs, i.e., graphs which have a specially simple combinatorial structure. Split graphs were first introduced and studied by Földes and Hammer in 1977 [4]. Endo-regularity of split graphs first appeared in [3]. But Weimin Li and Jianfei Chen found that the main result in [3] is wrong. They studied endo-regularity of split graphs again [8]. Endo-completely regularity of split graphs appeared in [7, 11]. Their main results are equivalent. This paper continues the investigation from [11].

1 Preliminaries

All graphs will be finite undirected without loops and multiple edges. If G is a graph, we denote by $V(G)$ (or simply G) and $E(G)$ its vertex set and edge set, respectively. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, if for any $a, b \in V(H)$, $\{a,b\} \in E(H)$ if and only if $\{a,b\} \in E(G)$, then we call H an *strong or induced subgraph* of G. We denote by K_n the complete graph with n vertices. Let G be a graph and let H be a clique (i.e., a complete subgraph) of G. If H has the maximal order of all the cliques of G, i.e., $|V(A)| \leq |V(H)|$ for any clique A of G, then H is called a maximal clique of G. An *independent* set of G is a set of pairwise non-adjacent vertices and a *complete* set is a set which induces a clique. Let H be

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a subgraph of G and $v \in H$. Denote $N_H(v) := \{x \in H | \{x,v\} \in E(H)\}$, called it the *neighborhood* of v in H; use $N(v)$ for $N_H(v)$ if it is clear which graph H is referred to.

A graph $G(V,E)$ is called a *split graph* if its vertex-set can be partitioned into disjoint (non-empty) sets I and K, i.e., $V = K \cup I$, such that I is an independent set and K is a complete set. In this paper a split graph G is always written as $K_n \cup I_r$ where K_n is a maximal complete subgraph of G and $I_r = \overline{K}_r$.

Definition 1.1. Let $G = K_n \cup I_r$ be a split graph where K_n is a (may be not maximal) complete subgraph of G. We call $K_n \cup I_r$ be a *unique decomposition* of G with the clique size n if for every complete subgraph K_n' and every independent set I_r' such that $G = K_n' \cup I_r'$ one has $K_n' = K_n$ and $I_r' = I_r$.

Example 1.2. Let G be the graph as follows.

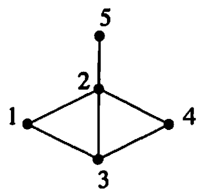

Figure 1: Split graph which has no a unique decomposition with the clique size 3.

We see that there are 2 complete subgraphs size 3, $K_3 = \{1,2,3\}$ and $K_3' = \{2,3,4\}$. And G can be partitioned to both of $K_3 \cup \{4,5\}$ and $K_3' \cup \{1,5\}$. So this is no unique decompositions of G with the clique size 3. We have one complete subgraph $K_2 = \{2,3\}$ of G with $K_2 \cup \{1,4,5\}$, i.e., a unique decomposition of G with the clique size 2.

This can be formulated in general as follows.

Proposition 1.3. *If K_n is a maximal complete subgraph of a split graph G and $K_n \cup I_r$ is not a unique decomposition with the clique size n, then $K_{n-1} \cup I_{r+1}$ is a unique decomposition with the clique size $n-1$.*

Let G and H be graphs. A *homomorphism* $f : G \longrightarrow H$ is a vertex-mapping $V(G) \longrightarrow V(H)$ which preserves adjacency, i.e., for any $a,b \in V(G)$, $\{a,b\} \in E(G)$ implies that $\{f(a), f(b)\} \in E(H)$. Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an *isomorphism* from G to H. A homomorphism from G to itself is called an *endomorphism* of G. An isomorphism of a graph G to itself is called an *automorphism* of G. $End(G)$ and $Aut(G)$ denote the sets of endomorphisms and automorphisms of the graph G, respectively. It is well-known that $End(G)$ is a monoid (i.e., a semigroup with an identity element) and $Aut(G)$ is a group with respect to the composition of mappings. They are called the monoid of G and the group of G, respectively.

Let f be an endomorphism of a graph G. If H is a subgraph of G, by $f|_H$ we denote the restriction of f on H; and $f(H) := \{f(x) | x \in H\}$. Let $G(V,E)$ be a graph. Let $\rho \subseteq V \times V$ be an equivalence relation on V. Denote by $[a]_\rho$ the equivalence class of $a \in V$ under ρ. The graph, denoted by G/ρ, is called the *factor graph* of G under

ρ, if $V(G/\rho) = V/\rho$ and $\{[a]_\rho, [b]_\rho\} \in E(G/\rho)$ if and only if there exist $c \in [a]_\rho$ and $d \in [b]_\rho$ such that $\{c,d\} \in E(G)$. Let f be an endomorphism of G. By ρ_f we denote the equivalence relation on $V(G)$ induced by f, i.e., for $a,b \in V(G)$, $(a,b) \in \rho_f$ if $f(a) = f(b)$. The graph G/ρ_f is called *the factor graph by f*.

An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$. If every element of S is regular, S is called *regular*. A graph G with regular $End(G)$ is called *endomorphism-regular*, or simply *endo-regular*. An element a of a semigroup S is called *completely regular* if there exists $x \in S$ such that $axa = a$ and $ax = xa$. A semigroup S is called *completely regular* if all its elements are completely regular. A graph G with completely regular $End(G)$ is called *endomorphism-completely-regular*, or simply *endo-c.r.*

Theorem 1.4. ([10]) *The following conditions on a semigroup S are equivalent:*
(i) S is completely regular.
(ii) S is a union of (disjoint) groups.

An element z of a semigroup S is called a *right* (*left*) *zero* if $sz = z$ ($zs = z$) for all $s \in S$; z is called a *zero* of S if it is both a right and a left zero of S. A semigroup all of whose elements are right (left) zeros is called a *right* (*left*) *zero semigroup*. We denote by R_n (L_n) the n element right (left) zero semigroup. For more concepts about semigroups we refer to [10].

Right groups are of the form $G \times R_n$, i.e., they are the unions of n copies of an arbitrary (finite) group G, analogously $G \times L_n$ for left groups, with the multiplication as given by $G \times R_n$ or $G \times L_n$.

Definition 1.5. For any split graph $G = K_n \cup I_r$, let J be a subset of I_r. We call *J a split component of I_r*, if for any $a,b \in J$, $N(a) = N(b)$ (including the case whose $N(a)$ and $N(b)$ are empty) and there are no $c \in I_r \setminus J$ such that $N(c) = N(a)$. And we say that I has s split components if I_r contains s distinct split components, i.e., $I_r = \bigcup_{i=1}^{s} J_i$, J_i a split component of I_r for all $i = 1,2,...,s$.

We observe that a split component is a ν-class in the terminology of [5]. This means that the canonical strong factor graph of $K_n \cup I_r$ is the form $K_n \cup I_s$, if I_r has s split components.

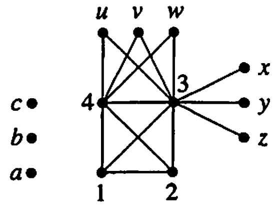

Figure 2: Split graph $K_4 \cup I_9$.

Example 1.6. Let G be the graph as in Figure 2. So we consider $G = K_4 \cup I_9$ where $K_4 = \{1,2,3,4\}$ and $I_9 = \{a,b,c,u,v,w,x,y,z\}$, the independent set I_9 has 3 split components, $J_1 = \{a,b,c\}$, $J_2 = \{u,v,w\}$ and $J_3 = \{x,y,z\}$. If we consider $G = K_3 \cup I_{10}$ where $K_3 = \{2,3,4\}$ and $I_{10} = I_9 \cup \{1\}$, we have that the independent set I_{10} has 4 split components, J_1, J_2, J_3 and $J_4 = \{1\}$.

Theorem 1.7. ([8]) *Let $G(V,E)$ be a connected split graph with $V = K_n \cup I_r$. Then G is endo-regular if and only if for all $a \in I_r$ one has $|N(a)| = d$, $d \in \{1,...,n-1\}$.*

Theorem 1.8. ([8]) *A non-connected split graph $K_n \cup I_r$ is endo-regular if and only if $N(a) = \emptyset$ for all $a \in I_r$.*

Lemma 1.9. ([11]) *For any split graph $G = K_n \cup I_r$, let f be an endomorphism of G. If $|N(a)| < n-1$ for all $a \in I_r$, then $f(V(K_n)) = V(K_n)$.*

Lemma 1.10. ([11]) *Let $G = K_n \cup I_r$ be an endo-regular split graph. If End(G) is completely regular, then $r < 2$.*

Lemma 1.11. ([5]) *Let X be a graph, $x_1, x_2 \in X$. There exists a strong endomorphism $f \in SEnd(X)$ with $f(x_1) = f(x_2)$ if and only if $N(x_1) = N(x_2)$.*

Remark 1.12. 1) If an endo-regular split graphs $G = K_n \cup I_r$ with I_r has exactly one split component and $|N(a)| = n-1$ for all $a \in I_r$, they are of the form $K_n \cup I_r = K_2[\overline{K}_{r+1}, K_{n-1}]$ (generalized lexicographic product see [5]). In this case we have by Proposition 1.3 that $K_{n-1} \cup I_{r+1}$ is a unique decomposition of G with the clique size $n-1$, and the canonical strong factor graph of $K_{n-1} \cup I_{r+1}$ is K_n. Then by Theorem 3.4 in [5], we have that $SEnd(K_{n-1} \cup I_{r+1}) \cong Aut(K_n)$ *wr* $\mathscr{K}$ where $\mathscr{K} = \{\{u\} \mid u \in K_{n-1}\} \cup \{I_{r+1}\}$ is a small category (for definitions and notation see [5]). This means that every strong endomorphism can be described by an automorphism φ of K_n followed by a family of mappings. For every element x of K_n we take a mapping from the class $[x]$ of x to the class $[\varphi(x)]$ of $\varphi(x)$. For all $x \in K_n$ we get the family of mappings. Here most classes are one element, except for the class corresponding to I_{r+1}.

2) For any endo-regular split graph $G = K_n \cup I_r$ with K_n is a maximal complete subgraph of G, if I_r has $s > 1$ split components, it is clear that $K_n \cup I_r$ is a unique decomposition of G with the clique size n.

2 Completely Regular Endomorphisms in Endo-regular Split Graphs

In this section we find conditions such that a regular endomorphism of any graph G is completely regular and specify this condition for split graphs G.

We begin this section by describing a property of a mapping f of a finite set G. We denote $T(G)$ the set of all mapping from G to itself.

Lemma 2.1. *Let G be a (finite) set, if $f \in T(G)$ and there exist $a,b \in G$ with $f(a) \neq f(b)$ and $f^2(a) = f^2(b)$, then f is not completely regular.*

Proof. Take f is a mapping of the set G. Let $a,b \in G$ with $f(a) \neq f(b)$ and $f^2(a) = f^2(b)$. Assume that f is completely regular, then there exists $g \in T(G)$ with $fgf = f$ and $fg = gf$. Consider at vertices a and b, we have

$$gf^2(a) = fgf(a) = f(a) \neq f(b) = fgf(b) = gf^2(b) = gf^2(a).$$

This is a contradiction. Then we get f is not completely regular. □

We call this property *square injective* since it is equivalent to saying $f^2(a) = f^2(b)$ implies $f(a) = f(b)$.

For the next theorem, we were inspired from Proposition 2.4 in [7]. It gives a condition such that any regular endomorphism f of any graph G is completely regular. Now we give another way to show which endomorphisms are completely regular.

Theorem 2.2. *Let G be a finite graph and f be an endomorphism of G. Then f is completely regular if and only if for all $a,b \in V(G)$, $f(a) \neq f(b)$ implies $f^2(a) \neq f^2(b)$, i.e., f is square injective. In this case, if f is not idempotent, we have $ff^{i-1}f = f$ and $ff^{i-1} = f^{i-1}f$ where f^i is the idempotent power of f.*

Proof. Necessity. This follows from Lemma 2.1.
Sufficiency. Let f be a square injective endomorphism of G. Since G is finite, there exists some $i \in \mathbb{N}$ such that f^i is an idempotent, i.e., $(f^i)^2 = f^i$.

If f is idempotent, it is clear that f is completely regular. Now we suppose that f is not idempotent. So there exists $2 \leq i \in \mathbb{N}$ such that f^i is idempotent.

First we show that $f(a) = f^{i+1}(a)$ for all $a \in V(G)$. Let $a \in V(G)$. Since f^i is an idempotent, we have $f^2(f^{2i-2}(a)) = f^{2i}(a) = (f^i)^2(a) = f^i(a) = f^2(f^{i-2}(a))$. Since f is square injective, we get that $f^{2i-1}(a) = f^{i-1}(a)$. By repeating this process for $i-1$ times, we get that $f^{i+1}(a) = f(a)$, i.e., $ff^{i-1}f = f$. It is clear that $ff^{i-1} = f^{i-1}f$. Now we have f is completely regular. □

Next, we will specify the condition in Theorem 2.2 for an endo-regular split graph G. We begin with a lemma which shows an additional property of a completely regular f of an endo-regular split graph G.

Lemma 2.3. *Let $G = K_n \cup I_r$ be an endo-regular split graph and let f be a completely regular endomorphism on G. If $|N(a)| < n-1$ for all $a \in I$, then for any $d \in I_r$, if $f(d) \in K_n$, then $d \notin Im(f)$.*

Proof. Let f be a completely regular endomorphism of G. Let $d \in I_r$ with $f(d) \in K_n$. Assume that $d \in Im(f)$. By Lemma 1.9, $f(K_n) = K_n$, there exists $c \in I_r$ such that $f(c) = d$. Now we have that $f^2(c) = f(d) =: x \in K_n$. Since $f(K_n) = K_n$, there exists $u \in K_n$ with $f(u) \in K_n$ and $f^2(u) = x$. Since $f^2(u) = f^2(c)$ and f is completely regular, by Theorem 2.2, we have that $f(u) = f(c) = d \in I_r$. This a contradiction. Then we get that $d \notin Im(f)$. □

We reprove one direction of the main theorem from [11] with the next lemma. For any $f \in End(G)$, define

$$End_f(G) := \{g \in End(G) | \rho_f = \rho_g\}$$

the set of all endomorphisms of G with congruence relation ρ_f. Note that $End_f(G)$ is Green's $\mathscr{L}$-class of f.

Lemma 2.4. *For any endo-regular split graph $G = K_n \bigcup I_r$, let f be an endomorphism of G. If f is a bijective or $f(G) \cong K_n$, then $End_f(G)$ is a group.*

Proof. If f is bijective, we see that $End_f(G) = Aut(G)$. Otherwise:

(a) If $|N(a)| = m < n-1$ for all $a \in I_r$, by Lemma 1.9, it is clear that $End_f(G) \cong End(K_n) \cong S_n$.

(b) If $|N(a)| = n-1$ for all $a \in I_r$, we have to consider the ways $Im(f) \cong K_n$ can be embedded into G. There are $r+1$ ways each followed by all permutation of the image. So we get $r+1$ times S_n. Moreover, it is clear that $End_f(G)$ altogether is isomorphic to the left group $S_n \times L_{r+1}$. □

Theorem 2.5. ([11]) *For any endo-regular split graph $G = K_n \cup I_r$, $End(G)$ is completely regular if and only if $r = 1$.*

Proof. Let $G = K_n \cup I_r$ be an endo-regular split graph. If $r = 1$, then by Lemma 2.4 and Theorem 1.4 we get $End(G)$ is completely regular. If $r > 1$, we get that $End(G)$ is not completely regular monoid by Lemma 1.10. □

Continuing the consideration from Remark 1.12 we get the following remark.

Remark 2.6. *For any endo-regular split graph $G = K_n \cup I_r$,*

1) if $|N(a)| < n-1$ for all $a \in I_r$, then $f \in End(G)$ is a strong endomorphism if and only if $f(c) \in I_r \ \forall c \in I_r$;

2) if $K_n \cup I_r$ is not a unique decomposition of G with the clique size n, then all $f \in End(G)$ are strong endomorphisms.

Proof. 1) Necessity. Let $f \in End(G)$ be a strong endomorphism. Assume that there exists $c \in I_r$ with $f(c) = u \in K_n$. By Lemma 1.9, we have that $f(K_n) = K_n$. Then there exist $x \in K_n$ such that $f(x) = u$, so $f(x) = f(c)$. Since $|N(c)| < n-1$ and $|N(x)| \geq n-1$, by Lemma 1.11 we get that f is not a strong endomorphism. This is a contradiction. Then $f(c) \in I_r$ for all $c \in I_r$.

Sufficiency. Let $f \in End(G)$ with $f(c) \in I_r$ for all $c \in I_r$. Let $\{f(u), f(v)\} \in E(G)$. If $f(u), f(v) \in K_n$, it is clear that $u, v \in K_n$, so $\{u, v\} \in E(G)$. It remains to consider $f(u) \in K_n$ and $f(v) \in I_r$. By Lemma 1.9 and hypothesis we have that $u \in K_n$ and $v \in I_r$. Since $v, f(v) \in I_r$, by hypothesis we have $|N(v)| = |N(f(v))|$. Since f is an endomorphism, then $f(N(v)) = N(f(v))$. Since $f(u) \in N(f(v))$ and $f(K_n) = K_n$, then $u \in N(v)$ so $\{u, v\} \in E(G)$. Then we get that f is a strong endomorphism.

2) This case is obvious, look for example to the graph in Example 1.2 without point 5. □

3 Completely Regular Subsemigroups - Exactly one split component

In this section, we characterize completely regular subsemigroups contained in $End(G)$. We will begin with the endo-regular split graph G whose independent set has exactly one split component. And then, in the next section, we consider the endo-regular split graph G whose independent set has $s > 1$ split components. First, we give some lemma which describes the image of any endomorphism and the composition of any two endomorphisms of an endo-regular split graph $G = K_n \cup I_r$ restricted to $K_n \setminus N(a)$ and to N(a).

Lemma 3.1. *Let $G = K_n \cup I_r$ be an endo-regular split graph such that I_r has exactly one split component, i.e., $N(a) = N(b)$ for all $a, b \in I_r$. If $f, g \in End(G)$ with $f(G) \ncong K_n$ and $g(G) \ncong K_n$, we have $f(N(a)) = N(a)$, and $(f \circ g)(N(a)) = N(a)$. If $|N(a)| < n-1$ for all $a \in I_r$, we have in addition $f(K_n \setminus N(a)) = K_n \setminus N(a)$, $(f \circ g)(K_n \setminus N(a)) = K_n \setminus N(a)$ and the statement is also true for $f(G) = K_n$.*

Proof. Let f be an endomorphism of G which $f(G) \ncong K_n$. Let $u \in N(a)$. Assume that $f(u) \notin N(a)$. Then $f(u) \in (K_n \setminus N(a)) \cup I_r$. We consider two cases.

Case 1. $N(a) < n-1$ for all $a \in I_r$. By Lemma 1.9, it is impossible that $f(u) \in I_r$, so $f(u) \in K_n \setminus N(a)$. Since $f(G) \ncong K_n$ and $f(K_n) = K_n$, there exists a vertex $v \in I_r$ such that $f(v) \in I_r$. Since $f(u) \notin N(a)$ for all $a \in I_r$, then $f(u) \notin N(f(v))$, i.e., $\{f(u), f(v)\} \notin E(G)$. But $\{u, v\} \in V(G)$ and f is an endomorphism, then this is a contradiction.

Case 2. $N(a) = n-1$ for all $a \in I_r$. Since I_r has exactly one split component and K_n is a maximal complete subgraph, there exists one vertex $x \in K_n$ such that $x \notin N(a)$ and $N(x) = N(a)$. For example, we consider the graph as in Figure 3 where $K_n = K_3 = \{1, 2, x\}$ and $I_r = I_5 = \{a, b, c, d, e\}$. It is clear that only vertex $x \in K_3$ is such that $x \notin N(a)$ and $N(x) = N(a)$. It is obvious that $I_r \cup \{x\}$ is an independent set

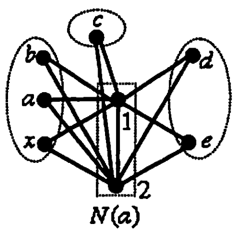

Figure 3: Endo-regular split graph $G = K_3 \cup I_5$ which $K_3 \cup I_5$ is not a unique decomposition of G with the clique size 3.

of G. Now we assume that $f(u) \in I_r \cup \{x\}$. Since $f(G) \ncong K_n$ and f preserves K_n, there exists $v \in I_r \cup \{x\}$ such that $f(v) \in I_r \cup \{x\}$. Since $I_r \cup \{x\}$ is an independent set, $\{f(u), f(v)\} \notin E(G)$. But $\{u, v\} \in E(G)$ and f is an endomorphism, we have a contradiction.

Moreover, if $|N(a)| < n-1$ for all $a \in I_r$, by Lemma 1.9, we have $f(K_n) = K_n$. So we get that $f(K_n \setminus N(a)) = K_n \setminus N(a)$. □

Remark 3.2. Lemma 3.1 is not true in the case when $|N(a)| = n-1$ for all $a \in I_r$ and $f \in End(G)$ with $f(G) \cong K_n$. For example, take G a graph as in Figure 3. We see that $K_3 = \{1, 2, x\}$ is a maximal complete subgraph of G, $I_5 = \{a, b, c, d, e\}$ is an independent set and $N(a) = \{1,2\}$. It is obvious that $f = \begin{pmatrix} 1 & 2 & x & a & b & c & d & e \\ a & 1 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$ is an endomorphism of G with $f(G) \cong K_3$. But $f(N(a)) = f(\{1,2\}) = \{1,a\} \neq N(a)$.

Note that if A is any set, then we denote by S_A the group of permutations of the elements in A. For examples, $S_{\{1,2,3\}}$, $S_{\{\{a,b\},\{c,d\}\}}$ are the symmetric group S_3 and S_2, respectively.

In Theorem 3.3 and Corollary 3.5, K_n is not necessary to be a maximal complete subgraph of the split graph $G = K_n \cup I_r$. Since for some $f \in End(G)$ with $f(G)$ isomorphic to a maximal complete subgraph of G, we may have the following situation. For example, we consider f as in Remark 3.2. We see that $f(\{a,b,c,d\}) = \{2\} \not\subseteq I_4 = \{a,b,c,d,\}$, so there is no congruence class whose a subset of I_4. Then we can not construct the set of representatives A as is defined in Theorem 3.3. This implies that we can not construct the set $CRE_f^A(G)$. Then in the next theorem and its corollary, we leave the case when $f(G)$ isomorphic to a maximal complete subgraph of G. Although, we have Lemma 2.4 whose shows $End_f(G)$ is a group, so $End_f(G)$ is completely regular monoid.

Theorem 3.3. *Let $G = K_n \cup I_r$ be an endo-regular split graph such that I_r has exactly one split component and $K_n \cup I_r$ is a unique decomposition of G with the clique size n. Suppose $f \in End(G)$ with $f(G)$ is not isomorphic to the maximal complete subgraph of G. Suppose that f has q congruence classes which are subsets of I_r for some $q \in \mathbb{N}$, namely, $[i_1]_{\rho_f}$, $[i_2]_{\rho_f}, \ldots, [i_q]_{\rho_f}$, $i_1, \ldots, i_q \in I_r$. For every $j = 1,2,\ldots,q$, choose a representative $a_j \in [i_j]_{\rho_f}$ for all $j = 1,2,\ldots,q$ and set $A := \{a_1,a_2,\ldots,a_q\}$. Set $I_r^f := \{i \in I_r \mid f(i) \in I_r\}$ and*

$$CRE_f^A(G) := \{h \in End_f(G) | h \text{ c.r.}, h(I_r^f) = A\}$$

the set of all completely regular endomorphisms in $End_f(G)$ such that their restrictions on I_r^f give the set A. Then we have that $CRE_f^A(G)$ is the group $S_{n-m} \times S_m \times S_q$.

Proof. Case 1. K_n is a maximal complete subgraph of G. To illustrate the situation in this case, i.e., $|N(a)| = m < n-1$ for all $a \in I_r$, we consider the graph as in Figure 4. In this graph we use $K_n = K_5$, $m = 2$ and $q = 3$. Take f such that the dotted ovals in the picture are the congruence classes induced by f which are subsets of I_r. Now take $A = \{a,d,e\}$. We get $CRE_f^A(G)$ is isomorphic to $S_3 \times S_2 \times S_3 = S_{\{1,2,3\}} \times S_{\{4,5\}} \times S_A$.

By the graph as in Figure 4 and Lemma 3.1, it is obvious that $CRE_f^A(G)|_{(K_n \setminus N(a))}$ and $CRE_f^A(G)|_{N(a)}$, the sets of restrictions of all endomorphisms in $CRE_f^A(G)$ to $K_n \setminus N(a)$ and to $N(a)$, are isomorphic to S_{n-m} and S_m, respectively. For any endomorphism h in $CRE_f^A(G)$, we get $h(u) = h(a_j)$ for all $u \in [i_j]_{\rho_f}$, $j = 1,2,\ldots,q$. So we have that $CRE_f^A(G)|_{I_r^f}$ is isomorphic to $CRE_f^A(G)|_A$. By inspection it is clear that $CRE_f^A(G)|_A$ is isomorphic to S_q. Then we have that $CRE_f^A(G)$ is isomorphic to $S_{n-m} \times S_m \times S_q$

Case 2. K_n is not a maximal complete subgraph of G. Consider the graph as in Figure 3. Here $K_n = K_2 = N(a)$ and $q = 3$. The three dotted ovals in the graph are the congruence classes induced by f which are subsets of I_r. Take now $A = \{x,c,d\}$. We get $CRE_f^A(G)$ is isomorphic to $S_2 \times S_3 = S_{\{1,2\}} \times S_A$.

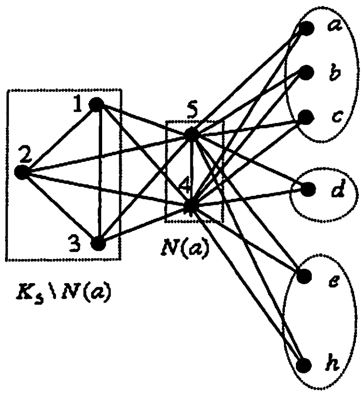

Figure 4: Endo-regular split graph $G = K_5 \cup I_6$ which $K_5 \cup I_6$ is a unique decomposition of G with the clique size 5.

Formally the result is the same as before since now $K_n \setminus N(a) = \emptyset$, then $m = n-1$ and $CRE_f^A(G) = S_{n-m} \times S_m \times S_q \cong S_{n-1} \times S_q$. □

Before we determine the maximal completely regular subsemigroup contained in $End_f(G)$ for an endo-regular split graph $G = K_n \cup I_r$ where I_r has exactly one split component, we give two examples which show the composition between the elements of two groups $CRE_f^A(G)$ and $CRE_f^B(G)$ which are contained in $End_f(G)$, where f is an endomorphism of an endo-regular split graph G.

Example 3.4. First, we consider $K_n \cup I_r$ with a unique decomposition of G with the clique size n and next we consider $K_n \cup I_r$ with a non-unique decomposition of G with the clique size n, where K_n is a maximal complete subgraph of G.
(1) Take G a graph as in Figure 5.

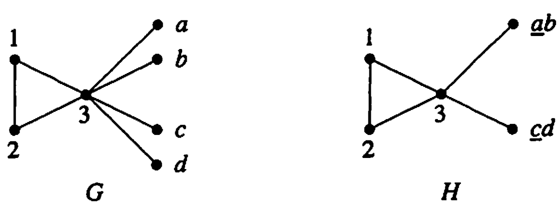

Figure 5: Endo-regular split graph $G = K_3 \cup I_4$ and H a factor graph induce by f in Example 3.4 (1).

Consider the mapping $f = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & a & a & c & c \end{pmatrix}$ from G to G. Note that $\underline{ab}$, $\underline{cd}$ in graph H (in Figure 5) mean $f(\{a,b\}) = \{a\}$ and $f(\{c,d\}) = \{c\}$. It is clear that f is an endomorphism. The graph H in Figure 5 is the factor graph of G induced by f. It is clear that f is idempotent, so it is completely regular. We have two congruence classes $\{a,b\}$ and $\{c,d\}$ which are subsets of the independent set $I_4 = \{a,b,c,d\}$. For every completely regular endomorphism $h \in End_f(G)$, it is impossible that $h(\{a,b\}) \cap h(\{c,d\}) \neq \emptyset$, since $h(\{a,b\}) \cap h(\{c,d\}) \neq \emptyset$, would imply that $h(a) \neq h(c)$ and $h^2(a) = h^2(c)$. This contradicts to Theorem 2.2. Now we get that for any completely regular endomorphism $h \in End_f(G)$,

(a) h sends $\{a,b\}$ to $\{a,b\}$ if and only if h sends $\{c,d\}$ to $\{c,d\}$

(b) h sends $\{a,b\}$ to $\{c,d\}$ if and only if h sends $\{c,d\}$ to $\{a,b\}$.

By Theorem 3.3, we know that $CRE_f^{\{a,c\}}(G)$ is isomorphic to $S_2 \times S_1 \times S_2 = S_2 \times S_2$. The 4 endomorphisms in $CRE_f^{\{a,c\}}(G)$ are

$f_1 = f$, $f_2 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & c & c & a & a \end{pmatrix}$, $f_3 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 2 & 1 & 3 & a & a & c & c \end{pmatrix}$ and $f_4 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 2 & 1 & 3 & c & c & a & a \end{pmatrix}$.

Similarly, we know that $CRE_f^{\{a,d\}}(G)$ is isomorphic to $S_2 \times S_2$. The 4 endomorphisms in $CRE_f^{\{a,d\}}(G)$ are

$g_1 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & a & a & d & d \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & d & d & a & a \end{pmatrix}$,

$g_3 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 2 & 1 & 3 & a & a & d & d \end{pmatrix}$ and $g_4 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 2 & 1 & 3 & d & d & a & a \end{pmatrix}$.

We will consider the composition between the elements of $CRE_f^{\{a,c\}}(G)$ and the elements of $CRE_f^{\{a,d\}}(G)$. For any $h \in CRE_f^{\{a,c\}}(G)$ and $k \in CRE_f^{\{a,d\}}(G)$, it is clear by inspection that $(h \circ k) \in CRE_f^{\{a,c\}}(G)$. The table in Table 1 shows the composition between the elements of these two groups.

From the Table 1, it is clear that we get the left group $(S_2 \times S_2) \times L_2$. Moreover, we

$\circ$	f_1	f_2	f_3	f_4	g_1	g_2	g_3	g_4
f_1	f_1	f_2	f_3	f_4	f_1	f_2	f_3	f_4
f_2	f_2	f_1	f_4	f_3	f_2	f_1	f_4	f_3
f_3	f_3	f_4	f_1	f_2	f_3	f_4	f_1	f_2
f_4	f_4	f_3	f_2	f_1	f_4	f_3	f_2	f_1
g_1	g_1	g_2	g_3	g_4	g_1	g_2	g_3	g_4
g_2	g_2	g_1	g_4	g_3	g_2	g_1	g_4	g_3
g_3	g_3	g_4	g_1	g_2	g_3	g_4	g_1	g_2
g_4	g_4	g_3	g_2	g_1	g_4	g_3	g_2	g_1

Table 1: Composition of two completely regular subsemigroups $CRE_f^{\{a,c\}}(G)$ and $CRE_f^{\{a,d\}}(G)$ in Example 3.4 (1).

have two more groups $CRE_f^{\{b,c\}}(G)$ and $CRE_f^{\{b,d\}}(G)$ contained in $End_f(G)$. Then we get $\bigcup_{i\in\{a,b\}} \bigcup_{j\in\{c,d\}} CRE_f^{\{i,j\}}(G)$ is isomorphic to the left group $(S_2 \times S_2) \times L_4$ and this is a maximal completely regular subsemigroup of $End_f(G)$.

(2) Take $G = K_2 \cup I_5$ the split graph as in Figure 6, with $K_2 = \{1,2\}$ and $I = \{a,b,c,d,e\}$.

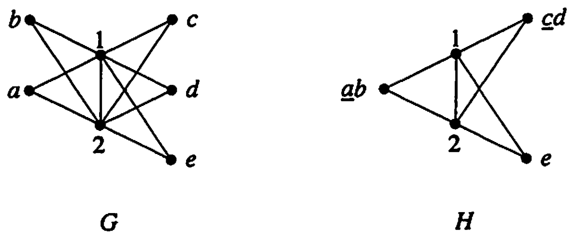

Figure 6: Endo-regular split graph $G = K_2 \cup I_5$ and H a factor graph induce by f in Example 3.4 (2).

Consider the mapping $f = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d & e \\ 1 & 2 & a & a & a & c & c & e \end{pmatrix}$ from G to G. It is clear that f is an endomorphism. The image graph $H = f(G)$ (in Figure 6) is a subgraph of G. Now we know that all endomorphisms in $End_f(G)$ are the embeddings of H into G. By Theorem 1.7, we have that f is regular. And we have three congruence classes $\{a,b\}$, $\{c,d\}$ and $\{e\}$ induced by f which are subsets of I_5. For every completely regular endomorphism $h \in End_f(G)$, it is impossible that $h(\{a,b\}) \cap h(\{c,d\}) \neq \emptyset$. Since $h(\{a,b\}) \cap h(\{c,d\}) \neq \emptyset$, then $h(a) \neq h(c)$ and $h^2(a) = h^2(c)$. This contradicts to Theorem 2.2. By the same ways, it is impossible that $h(\{a,b\}) \cap h(\{e\}) \neq \emptyset$ and $h(\{c,d\}) \cap h(\{e\}) \neq \emptyset$. This implies that for every completely regular endomorphism $h \in End_f(G)$, $h(I_5)$ is isomorphic to some element in the symmetric group $S_{\{\{a,b\},\{c,d\},\{e\}\}}$.

We have 4 different sets of representatives, $\{a,c,e\}$, $\{a,d,e\}$, $\{b,c,e\}$ and $\{b,d,e\}$. By Theorem 3.3, we know that $CRE_f^{\{i,j,e\}}(G)$ is isomorphic to $S_2 \times S_3 (= S_{\{1,2\}} \times S_{\{i,j,e\}})$ for all $i \in \{a,b\}$ and $j \in \{c,d\}$.

By inspection, it is clear that $\bigcup_{i\in\{a,b\}} \bigcup_{j\in\{c,d\}} CRE_f^{\{i,j,e\}}(G)$ is isomorphic to the left group $(S_2 \times S_3) \times L_4$.

Using Theorem 3.3 and Example 3.4, we get the next corollary.

Corollary 3.5. *Let $G = K_n \bigcup I_r$ be an endo-regular split graph such that I_r has exactly one split component and $K_n \cup I_r$ is a unique decomposition of G. Suppose $f \in End(G)$ with $f(G)$ is not isomorphic to maximal complete subgraph of G. Suppose that f has q congruence classes which are subsets of I_r for some $q \in \mathbb{N}$, namely, $[i_1]_{\rho_f}, [i_2]_{\rho_f}, \ldots, [i_q]_{\rho_f}$, $i_1, \ldots, i_q \in I_r$. Set $\mathscr{A} := \{\{a_1, a_2, \ldots, a_q\} \mid a_j \in [i_j]_{\rho_f}\}$ the set of sets of representatives. The maximal completely regular subsemigroup of $End_f(G)$ denote by $CRE_f(G)$*

is the union of $|\mathscr{A}|$ *groups* $CRE_f^A(G)$ *where* $A \in \mathscr{A}$. *And we have that* $CRE_f(G)$ *is the left group* $(S_{n-m} \times S_m \times S_q) \times L_{|\mathscr{A}|}$.

4 Completely Regular Subsemigroups - $s > 1$ split components and $|N(a)| = 1$

Now we turn to characterize completely regular subsemigroups of endo-regular split graphs $G = K_n \cup I_r$ where I_r has $s > 1$ split components $J_1, J_2, \ldots, J_s$ and $|N(a)| = 1$ for all $a \in I_r$. Let f be a completely regular endomorphism of G. This notation will be used everywhere in this section. To get the theorem which describes the structure of this completely regular subsemigroups, we need 3 lemmas.

The following lemma is the analogue of Lemma 3.1 for $s > 1$ and $|N(a)| = 1$.

Lemma 4.1. *With the above notation, suppose that* $J_1, J_2, \ldots, J_p$ *are the split components of* I_r *with* $f(J_j) \subseteq K_n$ *for* $j = 1, 2, .., p$. *Set* $J := J_1 \cup J_2 \cup \ldots \cup J_p$. *Then we have* $f(K_n \setminus \bigcup_{a \in I_r \setminus J} N(a)) = K_n \setminus \bigcup_{a \in I_r \setminus J} N(a)$ *and* $f(\bigcup_{a \in I_r \setminus J} N(a)) = \bigcup_{a \in I_r \setminus J} N(a)$.

Proof. Let $u \in K_n \setminus \bigcup_{a \in I_r \setminus J} N(a)$. Assume that $f(u) \in \bigcup_{a \in I_r \setminus J} N(a)$. Since $f(K_n) = K_n$ by Lemma 1.9, there exists $v \in \bigcup_{a \in I_r \setminus J} N(a)$ such that $f(v) \in K_n \setminus \bigcup_{a \in I_r \setminus J} N(a)$, i.e., $f(v) \notin N(I_r \setminus J)$. Suppose that $v \in N(J_l)$ for some $J_l \notin \{J_1, J_2, \ldots, J_p\}$. Since $|N(a)| = 1$ for all $a \in I_r$, by Lemma 2.3, we know that for all $d \in I_r \setminus J$ if $f(d) \in I_r$, then $f(d) \in I_r \setminus J$. Since $J_l \notin \{J_1, J_2, \ldots, J_p\}$, there exists $e \in J_l$ such that $f(e) \in I_r \setminus J$. Now we have $f(v) \notin N(f(e))$. Since $\{v, e\} \in E(G)$ and f is an endomorphism, we get that $\{f(v), f(e)\} \in E(G)$, i.e., $f(v) \in N(f(e))$. This is a contradiction. Thus we have $f(K_n \setminus \bigcup_{a \in I_r \setminus J} N(a)) = K_n \setminus \bigcup_{a \in I_r \setminus J} N(a)$. Consequently, since $f(K_n) = K_n$, we get that $f(\bigcup_{a \in I_r \setminus J} N(a)) = \bigcup_{a \in I_r \setminus J} N(a)$. □

Lemma 4.2. *With the above notation, set* $J_j^{\rho_f} := \{[i]_{\rho_f} \mid i \in J_j$ *and* $[i]_{\rho_f} \subseteq J_j\}$ *and* $J_j^f := \{i \in J_j \mid f(i) \in I\}$ *for all* $j = 1, 2, \ldots, s$. *Then we have for any* $\alpha, \beta \in \{1, 2, \ldots, s\}$ *that* $f(J_\alpha^f) \subseteq J_\beta$ *implies* $|J_\alpha^{\rho_f}| = |J_\beta^{\rho_f}|$.

Proof. Let f be a completely regular endomorphism of G and $f(I_\alpha^f) \subseteq J_\beta$ for some $\alpha, \beta \in \{1, 2, \ldots, s\}$, $\alpha \neq \beta$. Assume that $\ell_\alpha := |J_\alpha^{\rho_f}| \neq |J_\beta^{\rho_f}| =: \ell_\beta$.

First, we consider the case $\ell_\alpha > \ell_\beta$. Let $[a_1]_{\rho_f}$, $[a_2]_{\rho_f}$, ..., $[a_{\ell_\alpha}]_{\rho_f}$ be ℓ_α congruence classes in $J_\alpha^{\rho_f}$. Since $f(J_\alpha^f) \subseteq J_\beta$, then for any $l \in \{1, 2, \ldots, \ell_\alpha\}$, $f(a_l) = b_l$ for some b_l in J_β. By Lemma 2.3, we know that $b_l \in J_\beta^f$. Since $\ell_\alpha > \ell_\beta$, there exist $j \neq k \in \{1, 2, \ldots, \ell_\alpha\}$ such that $f(a_j) = b_j \neq b_k = f(a_k)$ and $[b_j]_{\rho_f} = [b_k]_{\rho_f}$, i.e., $f^2(a_j) = f^2(a_k)$. That means f is not square injective, contradicting to Theorem 2.2.

Next, we consider the case $\ell_\alpha < \ell_\beta$. Since I_r is finite, there exists some split components J_μ and J_ν of I_r with $f(J_\mu^f) \subseteq J_\nu$ and $|J_\mu^{\rho_f}| > |J_\nu^{\rho_f}|$. As in the first case we get a contradiction. Then we have that $|J_\alpha^{\rho_f}| = |J_\beta^{\rho_f}|$. □

Now we give an example which illustrates the next lemma.

Example 4.3. Take $G = K_4 \cup I_9$ an endo-regular split graph as in Figure 7.

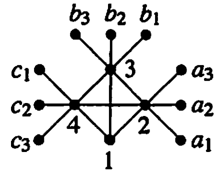

Figure 7: Split graph $G = K_4 \cup I_5$ with $Aut(G) = S_3 \times S_3 \times S_3 \times S_3$.

Here $J_1 = \{a_1, a_2, a_3\}$, $J_2 = \{b_1, b_2, b_3\}$ and $J_3 = \{c_1, c_2, c_3\}$ are the three split components of I_9. By Lemma 4.1, we have $f(1) = 1$ and $f(\{2,3,4\}) = \{2,3,4\}$ for all $f \in Aut(G)$. And by Lemma 4.2, we get that all automorphisms of G permute three split components J_1, J_2 and J_3. And in any split component, we can permute all vertices to get an automorphism. Then it is clear that $Aut(G) = S_1 \times S_3 \times (S_3 \times S_3 \times S_3)$.

Lemma 4.4. *With the above notation, if* $|J_1| = |J_2| = \ldots = |J_s| =: \ell$, *we have that* $Aut(G)$ *is isomorphic to* $S_{n-s} \times S_s \times \underbrace{S_\ell \times S_\ell \times \ldots \times S_\ell}_{s \text{ times}}$.

Theorem 4.5. *Take an endo-regular split graph* $G = K_n \cup I_r$ *where* $I_r = \bigcup_{k=1}^{s} J_k$ *with* $s > 1$ *split components* $J_1, J_2, \ldots, J_s$. *Suppose that for all* $a \in I_r$, $|N(a)| = 1$ *and* $|\bigcup_{a \in I_r} N(a)| = m$. *Take a regular endomorphism* f *of* G *with* q *congruence classes* $[i_1]_{\rho_f}, [i_2]_{\rho_f}, \ldots, [i_q]_{\rho_f}$ *each contained in* I_r. *Set* $I_r^f := \{i \in I_r | f(i) \in I_r\}$, $J_j^f := \{i \in J_j | f(i) \in I_r\}$ *and take the set of sets of representatives* $\mathscr{A} := \{\{a_1, a_2, \ldots, a_q\} \mid a_j \in [i_j]_{\rho_f},\ j = 1, 2, \ldots, q\}$. *Take* $A \in \mathscr{A}$ *and let* $CRE_f^A(G)$ *be the same as in Theorem 3.3. For any* $k = 1, 2, \ldots, s$, *if* $J_k^f \neq \emptyset$, *take* $u \in N(J_k^f)$ *and set* $M_A^f(u) := \{v \in N(J_l^f) \mid |J_k^f \cap A| = |J_l^f \cap A|,\ l \in \{1, \ldots, s\}\}$. *Suppose that there are* t *disjoint sets* $M_A^f(u_1)$, $M_A^f(u_2)$, ..., $M_A^f(u_t)$. *Then we have that* $CRE_f^A(G) = S_{n-m+p} \times \prod_{j=1}^{t} S_{M_A^f(u_j)} \times \prod_{k=1}^{s} S_{J_k^f \cap A}$. *Here* p *is the number of split components whose vertices are all sent to* K_n *by* f,

S_{n-m+p} *is the group of permutations of all vertices in* $(K_n \setminus N(I_r)) \cup \bigcup_{|J_j^f|=0} N(J_j^f)$,

$S_{M^f(u_j)}$ *is a the group of permutations of all vertices in* $M^f(u_j)$ *and*

$S_{J_k^f \cap A}$ *is the group of permutations of all vertices in* $J_k^f \cap A$.

The next example shows the idea how to prove the above theorem.

Example 4.6. Consider the split graph $G = K_8 \cup I_{11}$ as in Figure 8 and $f \in End(G)$ such that $H = Im(f) \cong G/\rho_f$, where notations $\underline{b_1}b_2$, $\underline{2}c$ and $d_1\underline{d_2}$ are as in Example 3.4.

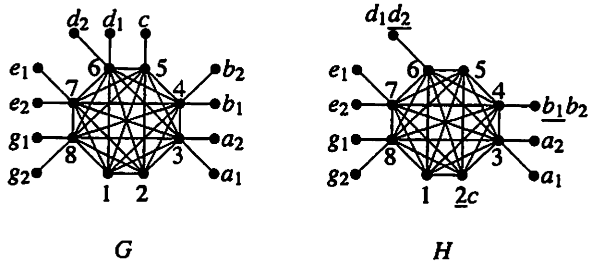

Figure 8: Endo-regular split graph $G = K_8 \cup I_{11}$ and H a factor graph induce by f in Example 4.6.

We have the 6 split components, $J_1 = \{a_1,a_2\}$, $J_2 = \{b_1,b_2\}$, $J_3 = \{c\}$, $J_4 = \{d_1,d_2\}$, $J_5 = \{e_1,e_2\}$ and $J_6 = \{g_1,g_2\}$. By Theorem 1.7, we know that all endomorphisms in $End(G)$ are regular. Take

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a_1 & a_2 & b_1 & b_2 & c & d_1 & d_2 & e_1 & e_2 & g_1 & g_2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a_1 & a_2 & b_1 & b_1 & 2 & d_2 & d_2 & e_1 & e_2 & g_1 & g_2 \end{pmatrix},$$

the image graph is H (in Figure 8) as a subgraph of G. We see that $f(G) \ncong K_8$ and we have 8 congruence classes induced by f which are subsets of I_{11}, namely, $\{a_1\}$, $\{a_2\}$, $\{b_1,b_2\}$, $\{d_1,d_2\}$, $\{e_1\}$, $\{e_2\}$, $\{g_1\}$ and $\{g_2\}$ only $\{c,2\} \nsubseteq I_{11}$, now we have for p from Theorem 4.5 that $p = 1$.

Choose the set of representatives $A = \{a_1,a_2,b_1,d_1,e_1,e_2,g_1,g_2\}$ then $I_{11}^f = \{i \in I_{11} \mid f(i) \in I_{11}\} = \{a_1,a_2,b_1,b_2,d_1,d_2,e_1,e_2,g_1,g_2\}$. We will show that $CRE_f^A(G)$ is isomorphic to $S_3 \times (S_3 \times S_2 \times S_2 \times S_2) \times S_2$. We have exactly one split component, J_3, such that $f(J_3) \subseteq K_8$. And the congruence relation for all endomorphisms in $End_f(G)$ is ρ_f. By definition, it is clear that $CRE_f^A(G)|_{(\{1,2,5\})}$, the set of restrictions of all endomorphisms in $CRE_f^A(G)$ to $\{1,2,5\}$, is isomorphic to $S_{\{1,2,5\}}$, the group S_3 of permutations of the set $\{1,2,5\}$.

Since $J_j^f = \{i \in J_j \mid f(i) \in I_{11}\}$ for all $j = 1,\ldots,6$, we see that $2 = |J_1^f \cap A| = |J_5^f \cap A| = |J_6^f \cap A| \neq |J_2^f \cap A| = |J_4^f \cap A| = 1$, then we get $t = 2$, t from Theorem 4.5 , and we have $M_A^f(3) = M_A^f(7) = M_A^f(8) = \{3,7,8\}$, $M_A^f(4) = M_A^f(6) = \{4,6\}$. By definition of $J_j^{\rho_f}$ in Lemma 4.2, we have $J_1^{\rho_f} = \{\{a_1\},\{a_2\}\}$, $J_2^{\rho_f} = \{\{b_1,b_2\}\}$, $J_4^{\rho_f} = \{\{d_1,d_2\}\}$, $J_5^{\rho_f} = \{\{e_1\},\{e_2\}\}$ and $J_6^{\rho_f} = \{\{g_1\},\{g_2\}\}$. Since $2 = |J_1^{\rho_f}| = |J_5^{\rho_f}| = |J_6^{\rho_f}| \neq |J_2^{\rho_f}| = |J_4^{\rho_f}| = 1$, by Lemma 4.2, we know that all endomorphisms in $CRE_f^A(G)$ do not send an element in $J_1^f \cup J_5^f \cup J_6^f$ to an element in $J_2^f \cup J_4^f$. Similarly, all endomorphisms in $CRE_f^A(G)$ do not send an element in $J_2^f \cup J_4^f$ to an element in $J_1^f \cup J_5^f \cup J_6^f$. This implies that all endomorphisms in $CRE_f^A(G)$ do not send any vertex in $M_A^f(4)$ to a vertex in

$M_A^f(3)$. Similarly, all endomorphisms in $CRE_f^A(G)$ do not send any vertex in $M_A^f(3)$ to a vertex in $M_A^f(4)$.

Now we consider $CRE_f^A(G)|_{(M_A^f(3)\cup J_1^f\cup J_5^f\cup J_6^f)}$ and $CRE_f^A(G)|_{(M_A^f(4)\cup J_2^f\cup J_4^f)}$, the set of restrictions of all endomorphisms in $CRE_f^A(G)$ to $M_A^f(3)\cup J_1^f\cup J_5^f\cup J_6^f$ and to $M_A^f(4)\cup J_2^f\cup J_4^f$, respectively.

It is clear that $CRE_f^A(G)|_{(M_A^f(3)\cup J_1^f\cup J_5^f\cup J_6^f)} \cong Aut(M_A^f(3)\cup \bigcup_{j\in\{1,5,6\}} (J_j^f\cap A))$. Since $(J_1^f\cap A)=\{a_1,a_2\}$, $(J_5^f\cap A)=\{e_1,e_2\}$ and $(J_6^f\cap A)=\{g_1,g_2\}$ are split components of the factor graph H and $|(J_1^f\cap A)|=|(J_5^f\cap A)|=|(J_6^f\cap A)|=2$, then by Lemma 4.4, we have that $CRE_f^A(G)|_{(M_A^f(3)\cup J_1^f\cup J_5^f\cup J_6^f)}$ is isomorphic to $S_{M_A^f(3)}\times S_{J_1^f\cap A}\times S_{J_5^f\cap A}\times S_{J_6^f\cap A}\cong S_3\times S_2\times S_2\times S_2$. Similarly, we get that $J_2^f\cap A=\{b_1\}$, $J_4^f\cap A=\{d_1\}$ and $|J_2^f\cap A|=|J_4^f\cap A|=1$, so $CRE_f^A(G)|_{(M_A^f(4)\cup J_2^f\cup J_4^f)}$ is isomorphic to $S_{M_A^f(4)}\times S_{J_2^f\cap A}\times S_{J_4^f\cap A}\cong S_2\times S_1\times S_1=S_2$.

Hence we get that $CRE_f^A(G)$ is isomorphic to $S_3\times(S_3\times S_2\times S_2\times S_2)\times S_2$.

Moreover, it is clear by inspection that for any $B,C\in\mathscr{A}$, $CRE_f^B(G)\cong CRE_f^C(G)$. In this example we have that

$\{a_1,a_2,b_1,d_1,e_1,e_2,g_1,g_2\}$, $\{a_1,a_2,b_1,d_2,e_1,e_2,g_1,g_2\}$,
$\{a_1,a_2,b_2,d_1,e_1,e_2,g_1,g_2\}$ and $\{a_1,a_2,b_2,d_2,e_1,e_2,g_1,g_2\}$

are 4 distinct sets in $\mathscr{A}$ so $|\mathscr{A}|=4$. Then it is clear that the maximal completely regular subsemigroup containing in $End_f(G)$ is

$$\bigcup_{B\in\mathscr{A}} CRE_f^B(G)\cong(S_3\times(S_3\times S_2\times S_2\times S_2)\times S_2)\times L_4.$$

Corollary 4.7. *Take G, f and $\mathscr{A}$ as in Theorem 4.5. For $A\in\mathscr{A}$, the maximal completely regular subsemigroup of $End_f(G)$ denoted by $CRE_f(G)$ is the left group $(S_{n-m+p}\times \prod_{j=1}^{t} S_{|M_A^f(u_j)|}\times\prod_{k=1}^{s} S_{|J_k^f\cap A|})\times L_{|\mathscr{A}|}$. Here $S_{|M_A^f(u_j)|}$ and $S_{|J_k^f\cap A|}$ are the symmetric groups on $|M_A^f(u_j)|$ and $|J_k^f\cap A|$ elements, respectively.*

5 Completely Regular Subsemigroups - $s>1$ split components and $|N(a)|\geq 2$

We can use the same idea from Sections 3 and 4 to find a completely regular subsemigroup of $End(G)$, where $G=K_n\cup I_r$ is an endo-regular split graph for which I_r has more than one split component and $|N(a)|\geq 2$ for all $a\in I_r$. But we can not generalize which group is isomorphic to $CRE_f^A(G)$ for any the set of representatives A. We give the reason as follows.

For any complete graph K_n and independent set $I_r=\overline{K}_r$, we can construct many non-isomorphic endo-regular split graphs whose I_r has $s>1$ split components and $|N(a)|=m\geq 2$ for all $a\in I_r$. Let G_1 and G_2 be two non-isomorphic endo-regular split graphs with the maximal complete subgraph K_n and the independent set I_r of both

G_1 and G_2. If f is an endomorphism of both G_1 and G_2, then $CRE_f^A(G_1)$ may be not isomorphic to $CRE_f^A(G_2)$ for some possible set of representatives A. The next example shows this fact.

Example 5.1. Consider two graphs G_1 and G_2 as in Figure 9.

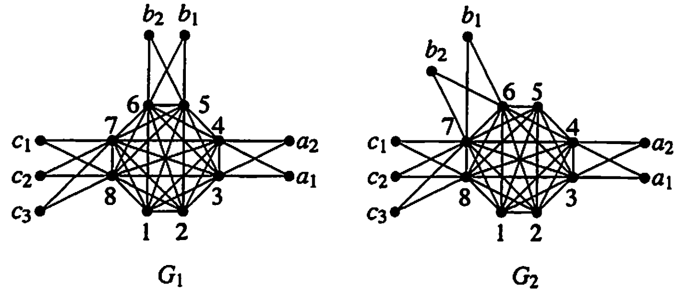

Figure 9: Split graph $G_1 = K_8 \cup I_7$ and $G_2 = K_8 \cup I_7$ with $G_1 \ncong G_2$.

The essential difference between the graph G_1 and the graph G_2 lie in the neighborhoods of b_2 and of c_1. The neighborhood of the split component $\{b_1, b_2\}$ and the neighborhood of the split component $\{c_1, c_2, c_3\}$ are disjoint in the graph G_1 but are not disjoint in the graph G_2. Consider the mapping as follows

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a_1 & a_2 & b_1 & b_2 & c_1 & c_2 & c_3 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a_1 & a_2 & b_1 & b_1 & c_1 & c_1 & c_3 \end{pmatrix}.$$

It is clear that f is an endomorphism of G_1 and G_2. By Lemma 1.7, we have that f is regular. And we have the congruence relation $\rho_f = \{\{i\} | i \notin \{b_1, b_2, c_1, c_2\}\} \cup \{\{b_1, b_2\}, \{c_1, c_2\}\}$ and we have 5 congruence classes contained in an independent set, that is $\{a_1\}$, $\{a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$ and $\{c_3\}$. The following pictures are the image graphs of G_1 and G_2 under f, notation as in Example 3.4.

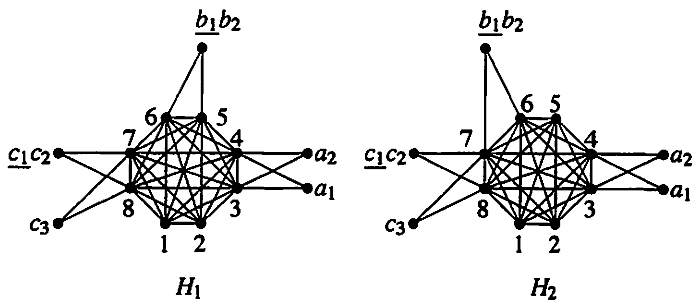

Figure 10: H_1 and H_2 factor graphs induce by f (in Example 5.1) of graphs G_1 and G_2 (in Figure 8), respectively.

We see that all endomorphisms in $End_f(G_1)$ and $End_f(G_2)$ are the embeddings from H_1 (in Figure 10) to G_1 and from H_2 (in Figure 10) to G_2, respectively.
Choose $A = \{a_1, a_2, b_1, c_1, c_3\}$. By inspection it is clear that $CRE_f^A(G_1)$ and $CRE_f^A(G_2)$ are isomorphic to $S_{\{1,2\}} \times (S_{\{\{3,4\},\{7,8\}\}} \times S_{\{3,4\}} \times S_{\{7,8\}} \times S_{\{\{a_1\},\{a_2\}\}} \times S_{\{\{c_1\},\{c_3\}\}}) \times S_{\{5,6\}}$ and $S_{\{1,2,5\}} \times (S_{\{3,4\}} \times S_{\{\{a_1\},\{a_2\}\}}) \times S_{\{\{c_1\},\{c_3\}\}}$, respectively. These are the groups $S_2 \times (S_2 \times S_2 \times S_2 \times S_2 \times S_2) \times S_2$ and $S_3 \times (S_2 \times S_2) \times S_2$, respectively.

Finally, we give an example to show that for any endo-regular split graph G, if $f, g \in End(G)$ with $\rho_f \neq \rho_g$, it is not necessary that the composition between two endomorphisms in $CRE_f(G)$ and $CRE_g(G)$ is completely regular. This means $CRE_f(G) \cup CRE_g(G)$ is not necessarily closed.

Example 5.2. Let G be the graph as in Example 3.4. It is clear that $f = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & a & a & d & c \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & b & b & b & d \end{pmatrix}$ are endomorphisms of G. Now we have the congruence relations

$\rho_f = \{\{1\}, \{2\}, \{3\}, \{a,b\}, \{c\}, \{d\}\}$

and

$\rho_g = \{\{1\}, \{2\}, \{3\}, \{a,b,c\}, \{d\}\}$.

It is clear that $\rho_f \subseteq \rho_g$. And we get that

$CRE_f(G) = CRE_f^{\{a,c,d\}}(G) \cup CRE_f^{\{b,c,d\}}(G)$

and

$CRE_g(G) = CRE_g^{\{a,d\}}(G) \cup CRE_g^{\{b,d\}}(G) \cup CRE_g^{\{c,d\}}(G)$

are isomorphic to $(S_2 \times S_3) \times L_2$ and $(S_2 \times S_2) \times L_3$, respectively. Since f and g are idempotents, it is clear that f and g are completely regular. Then $f \in CRE_f(G)$ and $g \in CRE_g(G)$. Consider the following composition

$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & a & a & a & c \end{pmatrix}.$$

We see that $a = (f \circ g)(c) \neq (f \circ g)(d) = c$ and $(f \circ g)^2(c) = a = (f \circ g)^2(d)$, i.e., $f \circ g$ is not square injective. By Theorem 2.2, we get that $f \circ g$ is not completely regular. This means $f \circ g$ is not in $CRE_f(G) \cup CRE_g(G)$.

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