Connected even factors in $\{K_{1,\ell}, K_{1,\ell}+e\}$-free graphs*

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Abstract. A connected factor F of a graph G is a connected spanning subgraph of G. If the degree of each vertex in F is an even number between 2 and $2s$, then F is a connected even $[2, 2s]$-factor of G, where s is an integer. In this paper, we show that every supereulerian $\{K_{1,\ell+1}, K_{1,\ell+1}+e\}$-free graph ($\ell \geq 2$) contains a connected even $[2, 2\ell-2]$-factor.

Key words: $\{K_{1,\ell}, K_{1,\ell}+e\}$-free graph; even factor

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1 Introduction

All graphs considered here are finite, undirected and simple. Suppose that S is a subset of the vertex set $V(G)$ of a graph G. Then we denote by $G[S]$ the subgraph of G induced by S; in particular, $G[S]$ will be written for $G[x_1, x_2, \cdots, x_n]$ if $S = \{x_1, x_2, \cdots, x_n\}$. A graph is $\{H_1, H_2, \cdots, H_k\}$*-free* if it contains no induced subgraph isomorphic to any H_i ($1 \leq i \leq k$). If $k = 1$ and H_1 is $K_{1,3}$, then it is *claw-free*. $K_{1,\ell}+e$ is the graph obtained from $K_{1,\ell}$ by joining a pair of nonadjacent vertices. Obviously, every claw-free graph is $\{K_{1,4}, K_{1,4}+e\}$-free. R. Li and R. Schelp [6, 7] and F. Duan [2] obtained some results on $\{K_{1,4}, K_{1,4}+e\}$-free graphs.

A *connected factor* F of a graph G is a connected spanning subgraph of G. If the degree $d(v)$ of each vertex v in F is an even number, then F is a *connected even factor* of G, and if $2 \leq d(v) \leq 2s$, then F is a *connected even* $[2, 2s]$*-factor* of G, where s is an integer. Obviously a Hamiltonian

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cycle of a graph is one of its connected even [2,2]-factors. H. Broersma et al. [1] proved the following

Theorem 1 *Every 4-connected claw-free graph has a connected [2,4]-factor.*

A *trail* of length k in G is an alternating sequence $v_0e_0v_1e_1\cdots e_{k-1}v_k$ of vertices and edges such that $e_i = v_iv_{i+1}$ for all $i < k$ and $e_i \neq e_j$ if $i \neq j$. A graph G is *supereulerian* if G has a closed trail containing every vertex (not necessarily containing every edge). M. Li et al. in [5] obtained the following

Theorem 2 *Every supereulerian $K_{1,\ell}$-free ($\ell \geq 2$) graph contains a connected even $[2, 2\ell - 2]$-factor.*

We denote by $K_{1,\ell} + e$ the graph obtained from $K_{1,\ell}$ by joining a pair of nonadjacent vertices. Obviously, if a graph is $K_{1,\ell}$-free, then it is also $\{K_{1,\ell+1}, K_{1,\ell+1} + e\}$-free. The graph in Figure 1 shows that the supereulerian $K_{1,\ell+1}$-free graph can contain no connected even $[2, 2\ell - 2]$-factor.

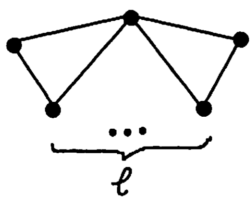


Fig. 1.

But in this paper we obtain the following result.

Theorem 3 *Every supereulerian $\{K_{1,\ell+1}, K_{1,\ell+1} + e\}$-free ($\ell \geq 2$) graph contains a connected even $[2, 2\ell - 2]$-factor.*

2 Proof of Theorem 3

We denote by $\Delta(G)$ the maximum degree of G.
Proof of Theorem 3. Let G be a supereulerian $\{K_{1,\ell+1}, K_{1,\ell+1} + e\}$-free graph ($\ell \geq 2$). Since G is supereulerian, G contains some connected even factors. It suffices to show that among these connected even factors there exists one whose maximum degree is smaller than 2ℓ. Let

$$\Delta = min\{\Delta(F) : \text{F is a connected even factor of G}\}$$

and $m(F,\Delta)$ be the number of vertices of F whose degrees in F are Δ. We assume without loss of generality that $m(F,\Delta)$ is minimum among all connected even factors of G whose maximum degrees are Δ. Now we verify that $\Delta(F) \le 2\ell - 2$. If $\Delta(F) \ge 2\ell \ge 4$ and w is the vertex whose degree is $\Delta(F)$, then there are at least ℓ edge-disjoint cycles $C_1, C_2, \cdots, C_\ell$ in F with a common vertex w. Let u_i and v_i be two neighbors of w on C_i $(1 \le i \le \ell)$.

Claim 1. *Let $x_i \in \{u_i, v_i\}$ and $x_j \in \{u_j, v_j\}$ $(i \ne j)$. Suppose that $x_i x_j \in E(G)$. Then we have*

(i) $x_i x_j \in E(F)$;

(ii) *exactly one of* $\{x_i x_j, w x_i\}$ *and* $\{x_i x_j, x_j w\}$ *is an edge-cut of* F;

(iii) *if* $u_i v_i \in E(G)$, *then* $u_i v_i \in E(F)$.

Proof. (i) If $x_i x_j \in E(G) \backslash E(F)$, then we can get another connected even factor F' from F by deleting wx_i and wx_j and adding $x_i x_j$. Clearly, $m(F',\Delta) = m(F,\Delta) - 1$, a contradiction.

(ii) If $F' = F - \{x_i x_j, x_j w, w x_i\}$ is connected, then F' is clearly another connected even factor of G. But $m(F',\Delta) = m(F,\Delta) - 1$, a contradiction. Hence $\{x_i x_j, x_j w, w x_i\}$ is an edge-cut of F. Since none of $x_i x_j$, $x_j w$ and $w x_i$ is a cut-edge, $F - \{x_i x_j, x_j w, w x_i\}$ contains exactly two components, that is, exactly one of x_i and x_j is in the same component as w, say x_i. This implies that x_i is also on C_j. Thus $\{x_i x_j, x_j w\}$ is an edge-cut of F but $\{x_i x_j, w x_i\}$ is not.

(iii) If $u_i v_i \in E(G) \backslash E(F)$, then we can get another connected even factor F' of G from F by deleting wu_i and wv_i and adding $u_i v_i$. But $m(F',\Delta) = m(F,\Delta) - 1$, a contradiction.

Claim 2. *If $n \le \ell$, then we can choose a vertex set X of $n+1$ order from $\{u_1, v_1, \cdots, u_n, v_n\}$ such that $G[X]$ contains at most one edge.*

Proof. Let $U = \{u_1, v_1, \cdots, u_n, v_n\}$ $(n \le \ell)$. We prove this Claim by induction on n. If there is no edge between $\{u_1, v_1\}$ and $\{u_2, v_2\}$ in G, then let $X = \{u_1, u_2, v_1\}$. So we assume that there is one edge between $\{u_1, v_1\}$ and $\{u_2, v_2\}$ in G, say $u_1 v_2 \in E(G)$. By Claim 1 (i), $u_1 v_2 \in E(F)$, and further, by Claim 1 (ii) we can assume without loss of generality that $\{u_1 v_2, u_1 w\}$ is an edge-cut of F, which implies that $u_1 u_2, u_1 v_1 \notin E(F)$. Hence, by Claim 1 (i), $u_1 u_2 \notin E(G)$, and by Claim 1 (iii), $u_1 v_1 \notin E(G)$. In this case let $X = \{u_1, v_1, u_2\}$.

Next suppose that $n \ge 3$. If for any $i \ne j$, there is no edge between $\{u_i, v_i\}$ and $\{u_j, v_j\}$ in G, then let $X = \{x_1, x_2, \cdots, x_n, x_{n+1}\}$, where $x_i \in \{u_i, v_i\}$ $(i = 1, 2, \cdots, n)$ and $x_{n+1} \in U \setminus \{x_1, x_2, \cdots, x_n\}$. Now we suppose that there exists one edge between $\{u_i, v_i\}$ and $\{u_j, v_j\}$ in G, say $u_1 u_2 \in E(G)$. Then $u_1 u_2 \in E(F)$ by Claim 1 (i). Thus, by Claim 1 (ii), we can assume without loss of generality that $\{u_1u_2, u_1w\}$ is an edge-cut of F, which implies that u_1u_2 is a unique edge of F between u_1 and U. By Claim 1 (i) and (iii) we know that u_1u_2 is a unique edge of G between u_1 and U.

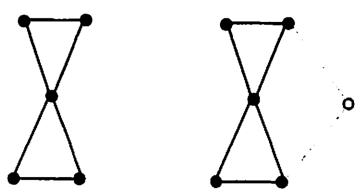
Let $C_2' = wv_2 \cdots u_2 \cdots v_1w$. Then $C_2', C_3, \cdots, C_n$ are $n-1$ edge-disjoint cycles contained in F having w on common. By the inductive hypothesis we can choose a set X_1 of n vertices from $U \backslash \{u_1, u_2\}$ such that $|E(G[X_1])| \leq 1$. In this case, let $X = X_1 \cup \{u_1\}$. The proof of Claim 2 is completed.

Therefore, by Claim 2, we can find a set Y of $\ell + 1$ vertices from $\{u_1, v_1, \cdots, u_\ell, v_\ell\}$ such that $G[Y \cup \{w\}]$ is isomorphic to either $K_{1,\ell+1}$ or $K_{1,\ell+1} + e$. This contradicts with the assumption of Theorem 3. Up to now we prove that $\Delta(F) \leq 2\ell - 2$, as required. □

The following well-known conjecture made by M. Matthews and D. Sumner [8] is still wide open.

Conjecture 4 *Every 4-connected claw-free graph is Hamiltonian.*

H. Broersma et al. [1] and T. Kaiser et al. [4] obtained its positive results for two special cases as follows. An *induced hourglasses S* of a graph G is an induced subgraph of G isomorphic to the graph in Fig. 2 (a). The graph G has the *hourglass property* if in every induced hourglass S, there are two non-adjacent vertices which have a common neighbor in $G - V(S)$ as in Fig. 2 (b).



(a) (b)

Fig. 2.

Theorem 5 *Every 4-connected claw-free hourglass-free graph is hamiltonian.*

Theorem 6 *Every 4-connected claw free graph with the hourglass property is hamiltonian.*

As one consequence of Theorem 3, we obtained positive result for the special case of conjecture 4 too.

Corollary 7 *Every supereulerian* $\{K_{1,3}, K_{1,3}+e\}$*-free graph is Hamiltonian.*

Since 4 edge connected graph is supereulerian [3], it follows that

Corollary 8 *Every* 4 *edge connected* $\{K_{1,3}, K_{1,3}+e\}$*-free graph is Hamiltonian.*

3 Acknowledgment

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