Defensive alliances in regular graphs

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Abstract

In this paper we study defensive alliances in some regular graphs. We determine which subgraphs could a critical defensive alliance of a graph G induce, if G is 6-regular and the cardinality of the alliance is at most 8.

Keywords: Alliance, induced subgraph.

1 Introduction

An alliance in a graph is a kind of *community*, in the sense that nodes in the alliance either protect each other from attacks of other nodes, in the case of defensive alliances, or are able to collaborate to attack other nodes, in the case of offensive alliances.

Alliances, which where introduced in [9], can be defined as follows. A *defensive alliance* is a set of vertices satisfying that each vertex has at least as many neighbors in the alliance (including itself) than neighbors not belonging to the alliance. A defensive alliance is *strong* if each vertex has more neighbors in the alliance than outside, and it is *critical* if it doesn't include other defensive alliances. An *offensive alliance* [2] is a set of vertices satisfying that each vertex in its boundary has at least as many neighbors in the alliance than neighbors not belonging to the alliance (including itself). Strong and critical offensive alliances are defined similarly to the strong and critical defensive ones.

An alliance is called *global* if it is also a dominating set. Global defensive alliance and global offensive alliances were first studied in [7] and [14], respectively.

Though the concept of alliance is relatively new, it is related with some other well known concepts and problems. Moreover, it has given rise to new concepts and problems that are worth to mention. In the context of complex networks, the definition of web community, as in [5], coincides with the definition of offensive alliance. Some works relate alliances with community detection and partitioning [5, 8]. Other related concepts are modules [11] and, in the context of distributed computing, coalition and monopolies [6, 10, 12]. From an algorithmic point of view, the clustering coefficient is defined in terms of small alliances in [1], and a study of algorithms for global alliances is given in [19]. Some of the works reated with alliances in the context of graph theory are [4, 17], where the concept of k-alliance is defined and studied, and [13, 16], in which the authors focus on the spectral properties of alliances. The questions about complexity and alliances are studied in [3].

In this paper we study defensive alliances in regular graphs. In a d-regular graph, a defensive alliance is a set of vertices that induces a subgraph with minimum degree at least $\lfloor \frac{d}{2} \rfloor$ and maximum degree at most d. We are interested in the following problem: Which graphs can a critical defensive alliance induce?

The answer is known for degree $d \leq 5$. For 6-regular graphs, it turns out to be a difficult question. We study alliances in graphs of degree 6, and of given cardinality $k \leq 8$. Even in these restricted cases, there is not an easy description of such alliances. Because of the complexity of the problem, we also restrict the question to a family of very symmetric graphs, the well known circulant graphs.

The paper is organized as follows. Basic definitions and properties are given in Section 2. Section 3 deals with alliances in regular graphs of small degrees. We finish with some conclusions and open problems.

2 Definition and basic properties

First, we introduce some notation and basic definitions. Given a graph $G = (V, E)$ we denote by n and m its order and size, respectively. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) := \{u \in V : u \sim v\}$, and the *closed neighborhood* of v is the set $N[v] := N(v) \cup \{v\}$. The degree of v is $d(v) := |N(v)|$. We denote by δ_G the minimum degree of G.

Given a non-empty set of vertices S, the neighborhood of v in S is $N_S(v) := \{u \in S : u \sim v\} = N(v) \cap S$. Denoting by $\overline{S}$ the complement in V of S, we have $N(v) = N_S(v) \cup N_{\overline{S}}(v)$. The *boundary* of S is the set $\partial(S) = \cup_{v \in S} N(v) - S$ and we denote by $\langle S \rangle$ the subgraph of G induced by S.

2.1 Alliances

The following definitions are taken from [9].

Definition 2.1 (Defensive alliance) *A non-empty set $S \subseteq V$ is a* defensive alliance *of G if, for every $v \in S$,*

$$|N_S[v]| \geq |N_{\overline{S}}(v)|. \tag{1}$$

We say that the alliance is strong *if, for every $v \in S$, the inequality is strict.*

The inequality (1) is called the (defensive) boundary condition.

Definition 2.2 (Offensive alliance) *A non-empty set $S \subseteq V$ is an* offensive alliance *of G if, for every $v \in \partial(S)$,*

$$|N_S(v)| \geq |N_{\overline{S}}[v]|. \tag{2}$$

We say that the alliance is strong *if, for every $v \in \partial(S)$, the inequality is strict.*

The inequality (2) is called the (offensive) boundary condition.

An alliance (of any type) is said to be *global* if it is also a dominating set of the graph. (Recall that S is a *dominating set* if every vertex of G is in S or has a neighbor in S, that is, $N[S] = V$.) An alliance (of any type) is said to be *critical* if none of its proper subsets is an alliance (of the same type). A *dual (or powerful) alliance* is a set that is both a defensive and an offensive alliance.

In the remaining of the paper we will focus on defensive alliances. Notice that, the whole graph G is a defensive alliance in G. Moreover, if S is a critical (strong) defensive alliance in G, then $\langle S \rangle$ is connected.

2.2 Alliance numbers

From the definition of alliance, some problems naturally arise. The first studied problem is to find the minimum cardinality of a defensive alliance of given a graph G. The problem we are interested in is which subsets of V, or the induced subgraphs of G, are critical defensive alliances and, among them, which are the minimal ones.

For a graph G, we can consider the following classes.

- $\mathcal{A}(G)$, the class of critical defensive alliances.
- $\hat{\mathcal{A}}(G)$, the class of critical strong defensive alliances.

Associated with this classes, the following invariants are defined.

- The defensive alliance number of G,

$$a(G) := \min\{|S| \, : \, S \in \mathcal{A}(G)\}.$$

  The upper defensive alliance number of G,

$$A(G) := \max\{|S| \, : \, S \in \mathcal{A}(G)\}.$$

- The strong defensive alliance number of G,

$$\hat{a}(G) := \min\{|S| \, : \, S \in \hat{\mathcal{A}}(G)\}.$$

  The upper strong defensive alliance number of G,

$$\hat{A}(G) := \max\{|S| \, : \, S \in \hat{\mathcal{A}}(G)\}.$$

For the defensive alliance number of a graph, or alliance number from now, it is easy to find tight lower bounds in terms of the minimum degree of the graph, as well as tight upper bounds in terms of the order:

$$\left\lfloor \frac{\delta_G}{2} \right\rfloor + 1 \leq \; a(G) \; \leq \left\lceil \frac{n}{2} \right\rceil, \tag{3}$$

$$\left\lceil \frac{\delta_G}{2} \right\rceil + 1 \leq \; \hat{a}(G) \; \leq \left\lfloor \frac{n}{2} \right\rfloor + 1. \tag{4}$$

The alliance number of a graph G is also related with its girth $g(G)$, i.e., the length of the shortest cycle of the graph (if any): If $\delta_G \geq 4$ then

$$g(G) \leq a(G).$$

The classes of critical offensive alliances and critical strong offensive alliances, with their corresponding alliance numbers can be analogously defined. Also, we can define the classes and alliance numbers for global alliances of any type.

It is worth mentioning that the decision problems associated to the different variation of alliances are all NP-complete (see [3] and the references therein). Therefore, it makes sense to study both the properties of the different types of alliance numbers and the alliance number of restricted classes of graphs.

3 Defensive alliances in regular graphs

The alliance numbers of regular graphs are known only for small degrees [9, 15].

We denote by $g(G)$ the girth of G and by $lc(G)$ the maximum length of an induced cycle in G. If G is d-regular, then it is known that:

- $d = 1 \Rightarrow a(G) = A(G) = 1$, $\hat{a}(G) = \hat{A}(G) = 2$;
- $d = 2 \Rightarrow a(G) = A(G) = \hat{a}(G) = \hat{A}(G) = 2$;
- $d = 3 \Rightarrow a(G) = A(G) = 2$, $\hat{a}(G) = g(G)$, and $\hat{A}(G) = lc(G)$;
- $d = 4 \Rightarrow a(G) = \hat{a}(G) = g(G)$, $A(G) = \hat{A}(G) = lc(G)$; and
- $d = 5 \Rightarrow a(G) = g(G)$, $A(G) = lc(G)$.

If $G = (V, E)$ is a graph, we say that a vertex $v \in S \subset V$ is defended in S if and only if it satisfies the boundary condition with respect to S. Similarly, if v satisfies the strong boundary condition with respect to S we say that v is strongly defended in S. Let $G = (V, E)$ a graph, and $v \in S \subset V$. The following properties are direct consequences of the definition of alliance and strong alliance.

Property 3.1 *If $d(v) = 2k$, v is defended in S if and only if $d_S(v) \geq k$. Moreover, the strong boundary condition is equivalent to the boundary condition, i.e., v is defended in S if and only if it is strongly defended in S.*

Property 3.2 *If $d(v) = 2k + 1$, v is defended in S if and only if $d_S(v) \geq k$; v is strongly defended in S if and only if $d_S(v) \geq k + 1$.*

Property 3.3 *If G is d-regular, then S is an alliance in G if and only if S induces a subgraph of minimum degree $\delta_S \geq \lfloor \frac{d}{2} \rfloor$; S is a strong alliance in G if and only if it induces a subgraph of minimum degree $\delta_S \geq \lceil \frac{d}{2} \rceil$.*

In fact, the known results for regular graphs of degree $d \leq 5$ allow us to completely characterize critical alliances for these graphs:

- If G is 1-regular, the critical alliances are exactly the singletons.
- The strong critical alliances in a 1-regular or 2-regular graph and the critical alliances in a 2-regular or 3-regular graph are exactly the edges.
- The strong critical alliances in a 3-regular or 4-regular graph and the critical alliances in a 4-regular or 5-regular graph are exactly the induced cycles.

Given a d-regular graph, G, we are concerned with two basic problems: determine $a(G)$, $\hat{a}(G)$, $A(G)$, and $\hat{A}(G)$, and characterize critical alliances in G, i.e., if S is a critical alliance in G, which graphs could $\langle S \rangle$ be isomorphic to?

Unfortunately, there is no simple characterization of the alliances, respectively strong alliances, of d-regular graphs if $d > 5$, respectively $d > 4$. So, we will concentrate on alliances of given cardinality. For that purpose, we give the following definition.

Definition 3.4 (Induced alliances set) *The* (k,d)-induced alliances set *is the set of graphs H of order k, minimum degree $\delta_H \geq \lfloor \frac{d}{2} \rfloor$, and maximum degree $\Delta_H \leq d$, with no proper subgraph of minimum degree greater than $\lfloor \frac{d}{2} \rfloor$. We denote this set by $\mathcal{S}_{(k,d)}$.*

Similarly, the (k,d)-induced strong alliances set *is the set of graphs H of order k, minimum degree $\delta_H \geq \lceil \frac{d}{2} \rceil$, and maximum degree $\Delta_H \leq d$, with no proper subgraph of minimum degree greater than $\lceil \frac{d}{2} \rceil$. We denote this set by $\hat{\mathcal{S}}_{(k,d)}$.*

For instance, $\mathcal{S}_{(2,2)} = \mathcal{S}_{(2,3)} = \{K_2\}$, and $\mathcal{S}_{(k,2)} = \mathcal{S}_{(k,3)} = \varnothing$, if $k \geq 3$; $\mathcal{S}_{(5,4)} = \mathcal{S}_{(5,5)} = \{C_5\}$, and $\mathcal{S}_{(k,4)} = \mathcal{S}_{(k,5)} = \{C_k\}$, if $k \geq 6$.

The following result is a consequence of the definitions of defensive alliance and (k,d)-induced alliances set, or (k,d)-ias for short.

Proposition 3.5 *If G is d-regular, then S is a critical alliance of G of cardinality k, if and only if $\langle S \rangle \in \mathcal{S}_{(k,d)}$.*

Proof. It follows straightforward from Property 3.3. ■

Notice that Proposition 3.5 says that alliances in regular graphs are defined by induced subgraphs of given minimum degree. The family of graphs that can be induced by a critical alliance can be described by its degree sequence.

Definition 3.6 (Admittable sequence) *A sequence* $\mathrm{s} = (d_1, d_2, \ldots, d_k)$ *is a* (k,d)-admittable sequence, *or an admittable sequence, if there is a graph G_s in $\mathcal{S}_{(k,d)}$ with degree sequence* s.

3.1 Defensive alliances in 6-regular graphs

In this section we pay attention to 6-regular graphs. Our study is based on determining all $(k,6)$-admittable sequences and then describing the corresponding $(k,6)$-induced alliance sets.

- If $|S| = 4$ then $\langle S \rangle = K_4$ and its associated degree sequence is $(3,3,3,3)$. That is, $\mathcal{S}_{(4,6)} = \{K_4\}$.
- If $|S| = 5$ then $\langle S \rangle = W_4$ and its associated degree sequence is $(4,3,3,3,3)$. That is, $\mathcal{S}_{(5,6)} = \{W_4\}$.

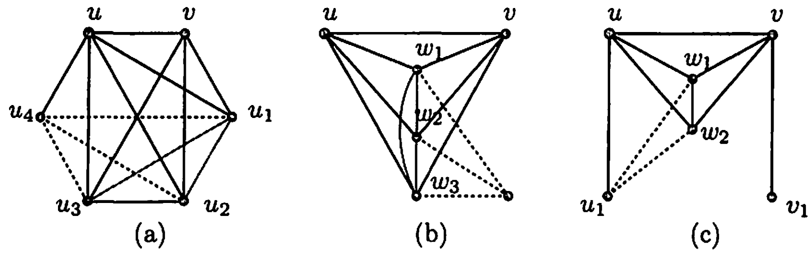

Figure 1: Proof of Lemma 3.7.

Lemma 3.7 *If G is 6-regular and contains a critical alliance S of cardinality 6, then the associated degree sequence of $\langle S \rangle$ is one of the following:*

$$(3,3,3,3,3,3),\ (4,4,3,3,3,3),\ \textit{or}\ (5,3,3,3,3,3)$$

Any other degree sequence with minimum degree 3 gives graphs containing K_4 or W_4.

Proof. Notice that if S is a critical alliance of G of cardinality 6 then any vertex v in $\langle S \rangle$ satisfies $3 \leq d_S(v) \leq 5$. Moreover, there must be at least one vertex of degree 3 in $\langle S \rangle$.

First, we prove that if S is an alliance of cardinality 6 then $\langle S \rangle$ cannot have two vertices, u and v, with $d_S(u) = 5$ and $d_S(v) \geq 4$. For that purpose, assume $d(u) = 5$ and $d(v) \geq 4$. We can assume, w.l.o.g., that $N(u) = \{v, u_1, u_2, u_3, u_4\}$ and $\{u, u_1, u_2, u_3\} \subseteq N(v)$ (see Figure 1 (a)). Now, there is no edge between the vertices u_1, u_2, and u_3, that is, none of the grey edges in Figure 1 (a) is in $\langle S \rangle$, otherwise there is an induced K_4. But every vertex has degree at least 3. So these three vertices must be all adjacent to u_4, that is, the dotted edges in Figure 1 (a) must be in $\langle S \rangle$, and then there is at least one induced W_4.

Notice that W_5 does not contain K_4, neither W_4 as a subgraph. Its degree sequence is $(5,3,3,3,3,3)$. So, this is the only admittable sequence with one vertex of degree 5.

Let us consider now degree sequences with only vertices of degree 3 and 4, i.e., the sequences $(3,3,3,3,3,3)$,$(4,4,3,3,3,3)$,$(4,4,4,4,3,3)$, and $(4,4,4,4,4,4)$. The graph $K_{3,3}$ has degree sequence $(3,3,3,3,3,3)$ and contains no K_4 nor W_5. The graph $K_{3,3}+e$ has degree sequence $(4,4,3,3,3,3)$ and contains no K_4 nor W_5. Thus, both sequences $(3,3,3,3,3,3)$ and $(4,4,3,3,3,3)$ are admittable.

We only need to show that any graph H of order 6 with at least four vertices of degree 4 contains either K_4 or W_4. The graph H must contain two adjacent vertices of degree 4, say u and v. There are two possibilities: u and v have three common neighbors, w_1, w_2 and w_3 (see Figure 1 (b)), or u and v share only two neighbors, w_1 and w_2 (see Figure 1 (c)).

In the first case, there is no edge between the vertices w_1, w_2 and w_3, that is, none of the grey edges in Figure 1 (b) is in $\langle S\rangle$, otherwise there is an induced K_4. But, then, none of them can have degree 4, a contradiction. In the second case, assume that u_1 is adjacent to u but not to v, and v_1 adjacent to v but not to u. Now, w_1 and w_2 cannot be adjacent, that is, the grey edge in Figure 1 (c) cannot be in $\langle S\rangle$, otherwise, there is a K_4, induced by $\{u,v,w_1,w_2\}$. Since there are at least four vertices of degree 4, at least one of the vertices u_1 or v_1, say u_1, is adjacent to w_1 and w_2, that is, the dotted edges in Figure 1 (c) must be in $\langle S\rangle$. Then, $\{u,v,u_1,w_1,w_2\}$ induce a subgraph isomorphic to W_4.

This completes the proof. ■

Notice that, by using Definition 3.6, this lemma can be reformulated as: the only $(6,6)$-admittable sequences are $(3,3,3,3,3,3)$, $(4,4,3,3,3,3)$ and $(5,3,3,3,3,3)$.

Proposition 3.8 *The* $(6,6)$*-ias are:*

$$\mathcal{S}_{(6,6)} = \{C_3\square K_2,\ K_{3,3},\ (C_3\square K_2)+e,\ K_{3,3}+e,\ \overline{C_4}+\overline{K_2},\ W_5\}$$

This set contains exactly the six graphs in Figure 2.

Proof. Let H be a graph in $\mathcal{S}_{(6,6)}$. Its degree sequence is one of the sequences in Lemma 3.7, i.e., $(3,3,3,3,3,3)$, $(4,4,3,3,3,3)$, and $(5,3,3,3,3,3)$.

If the degree sequence of H is $(5,3,3,3,3,3)$, then $H\cong W_5$. If the degree sequence of H is $(3,3,3,3,3,3)$, then we consider two cases:

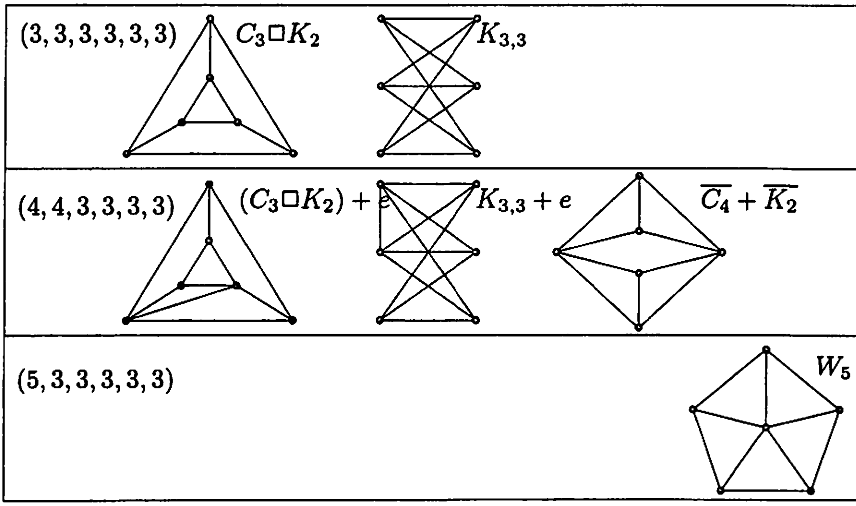

Figure 2: The (6, 6)-induced alliances set, with their associated degree sequence.

if H is triangle free, then $H \cong K_{3,3}$; otherwise H contains a triangle and then $H \cong C_3 \square K_2$.

Finally, if the degree sequence of H is $(4, 4, 3, 3, 3, 3)$, and $d(u) = d(v) = 4$, we consider the following two cases: if $u \sim v$ then the graph $H - e$, with $e = \{u, v\}$, has degree sequence $(3, 3, 3, 3, 3, 3)$ and this implies that either $H \cong K_{3,3} + e$, or $H \cong (C_3 \square K_2) + e$; otherwise $u \nsim v$, then $H - \{u, v\} \cong \overline{C_4}$ and thus, $H \cong \overline{C_4} + \overline{K_2}$. ■

Lemma 3.9 *If G is 6-regular and contains a critical alliance S of cardinality 7, then the associated degree sequence of $\langle S \rangle$ is one of the following:*

$(4, 3, 3, 3, 3, 3, 3)$,$(4, 4, 4, 3, 3, 3, 3)$,$(5, 4, 3, 3, 3, 3, 3)$ *or* $(6, 3, 3, 3, 3, 3, 3)$

Any other degree sequence with minimum degree 3 gives graphs containing K_4, W_4, or some graph in $\mathcal{S}_{(6,6)}$.

Proof. Notice that if S is a critical alliance of G of cardinality 7 then any vertex v in $\langle S \rangle$ satisfies $3 \leq d_S(v) \leq 6$. Moreover, in $\langle S \rangle$

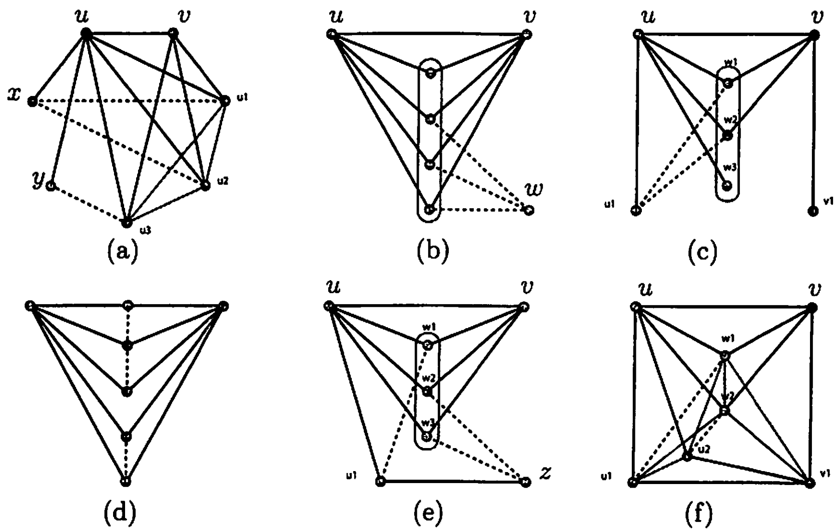

Figure 3: Proof of Lemma 3.9.

there must be at least two vertices of degree 3. Assume that there are only two vertices of degree 3, u and v. Then, if $u \sim v$, there is a vertex w adjacent only to vertices of degree greater than 3. By removing w, we obtain an induced subgraph of $\langle S \rangle$ with minimum degree at least 3. On the other hand, if u and v are not adjacent, then u is adjacent only to vertices of degree greater than 3. By removing u, we obtain an induced subgraph of $\langle S \rangle$ with minimum degree at least 3. But this is a contradiction, because S is a critical alliance. Thus, $\langle S \rangle$ has at least three vertices of degree 3.

To find all the $(7, 6)$-admittable sequences, we first prove that $\langle S \rangle$ cannot have two vertices, u and v, with $d_S(u) = 6$ and $d_S(v) \geq 4$. Assume that $d(u) = 6$ and $d(v) \geq 4$, and let u_1, u_2, and u_3 be three common neighbors of u and v (see Figure 3 (a)). There are two more vertices x, y in $\langle S \rangle$, which must be adjacent to (at least) u. Since there is no induced K_4, u_1, u_2, and u_3 are independent. That is, none of the grey edges in Figure 3 (a) is in $\langle S \rangle$. Since the minimun degree of $\langle S \rangle$ is 3, each of the vertices u_1, u_2, and u_3 has to be adjacent to one of the vertices x, y. We can assume, w.l.o.g., that $u_1 \sim x$, $u_2 \sim x$, and $u_3 \sim y$. That is, the dotted edges in Figure 3 (a) are in $\langle S \rangle$. Then, $\{u, u_1, v, u_2, x\}$ induce W_4, a contradiction.

Since W_6 is clearly in $\mathcal{S}_{(6,7)}$, we have that the only admitted degree sequence for $\langle S \rangle$ with maximum degree 6 is $(6,3,3,3,3,3)$.

Now we prove that, if $3 \leq d_S(v) \leq 5$, at most one vertex can have degree 5. Moreover, if there is one vertex u with degree 5, only one vertex v has degree 4.

- If there are two adjacent vertices of degree 5, say u and v, with four common neighbors (which have to be pairwise independent) then there is one vertex, w, not adjacent to u nor v (see Figure 3 (b)). In this case, w has to be adjacent to at least 3 of the common neighbors of u and v and then there is an induced $K_{3,3} + e$.

- If there are two adjacent vertices of degree 5, say u and v, with three common neighbors (which have to be pairwise independent) then there is one vertex, u_1 adjacent to u but not to v, and a vertex v_1 adjacent to v but not to u (see Figure 3 (c)). Since the minimum degree is 3, we can assume, w.l.o.g., that u_1 is adjacent to two of the common neighbors of u and v. That is, the dotted edges in Figure 3 (c) have to be in $\langle S \rangle$. But then, there is an induced W_4, a contradiction.

- If there are two non adjacent vertices of degree 5, then there are at least three edges between their five common neighbors (see Figure 3 (d)). Two of these three edges must be incident and thus, $\langle S \rangle$ contains a W_4.

Assume now that there is exactly one vertex u of degree 5. Assume also that there is more than one vertex of degree 4. Then, one of them is adjacent to u, say v.

- If u and v share three neighbors, we have: w_1, w_2 and w_3 the common neighbors of u and v, one vertex u_1 adjacent to u and not to v, and one vertex z not adjacent to u neither to v (see Figure 3 (e)). To avoid the existance of induced $K_{3,3} + e$, z can only be adjacent to two of the common neighbors. So, z is adjacent to u_1, w_2 and w_3. Since w_1, w_2 and w_3 have to be independent, and the minimum degree in $\langle S \rangle$ is 3, w_1 is adjacent to u_1. But now, we cannot add more edges, without introducing one of the forbidden induced subgraphs. So there are no more vertices of degree 4, a contradiction.

- If u and v share only two neighbors, we have: two vertices u_1 and u_2, adjacent to u and not to v, one vertex v_1, adjacent to v and not to u, and w_1 and w_2 the common neighbors to u and v (see Figure 3 (f)). Now, u_1 cannot be adjacent to both w_1 and w_2, and the same is true for u_2. We can assume, w.l.o.g., that $u_1 \sim w_1$ and $u_2 \sim w_2$. That is, the dotted edges in Figure 3 (f) are in $\langle S \rangle$. Since the minimum degree is 3, both u_1 and u_2 have to be adjacent to v_1. We also have that v_1 cannot be adjacent to w_1 neither w_2, because this would induce a W_4. That is, none of the grey edges in Figure 3 (f) is in $\langle S \rangle$. This implies that the maximum degree of w_1, w_2 and v_1 is 3. Now, the only way to obtain two vertices of degree 4 is adding an edge between u_1 and u_2. But then, there is an induced W_5.

The graph G_1 obtained from $P_5 + \overline{K_2}$ by removing two edges linking the same vertex of $\overline{K_2}$ with any two internal vertices of P_5 has degree squence $(5, 4, 3, 3, 3, 3, 3)$. Moreover, G_1 does not contain K_4, nor W_4, nor a graph of $\mathcal{S}_{(6,6)}$, as induced subgraphs.

Finally, if $3 \leq d_S(v) \leq 4$, since we have seen that there are at least three vertices of degree 3, the degree sequence is either $(4, 4, 4, 3, 3, 3, 3)$ or $(4, 3, 3, 3, 3, 3, 3)$. The graph G_2 obtained identifying an arbitrary pair of adjacent vertices of the cube Q_3 in a single vertex v, has degree sequence $(4, 3, 3, 3, 3, 3, 3)$. The graph G_3 obtained by adding one edge to G_2, between two of the vertices adjacent to v, has degree sequence $(4, 4, 4, 3, 3, 3, 3)$. None of these graphs contains K_4, nor W_4, nor a graph of $\mathcal{S}_{(6,6)}$. ■

Proposition 3.10 *The set $\mathcal{S}_{(7,6)}$ of the $(7, 6)$-ias contains exactly the* 15 *graphs in Figure 4.*

The proof of this proposition, which is omitted, is similar to but longer than that of Proposition 3.8. For every admittable sequence s in Lemma 3.9 we can constructively find every graph in $\mathcal{S}_{(7,6)}$ with degree sequence s. The obtained graphs are exactly the 15 graphs in Figure 4.

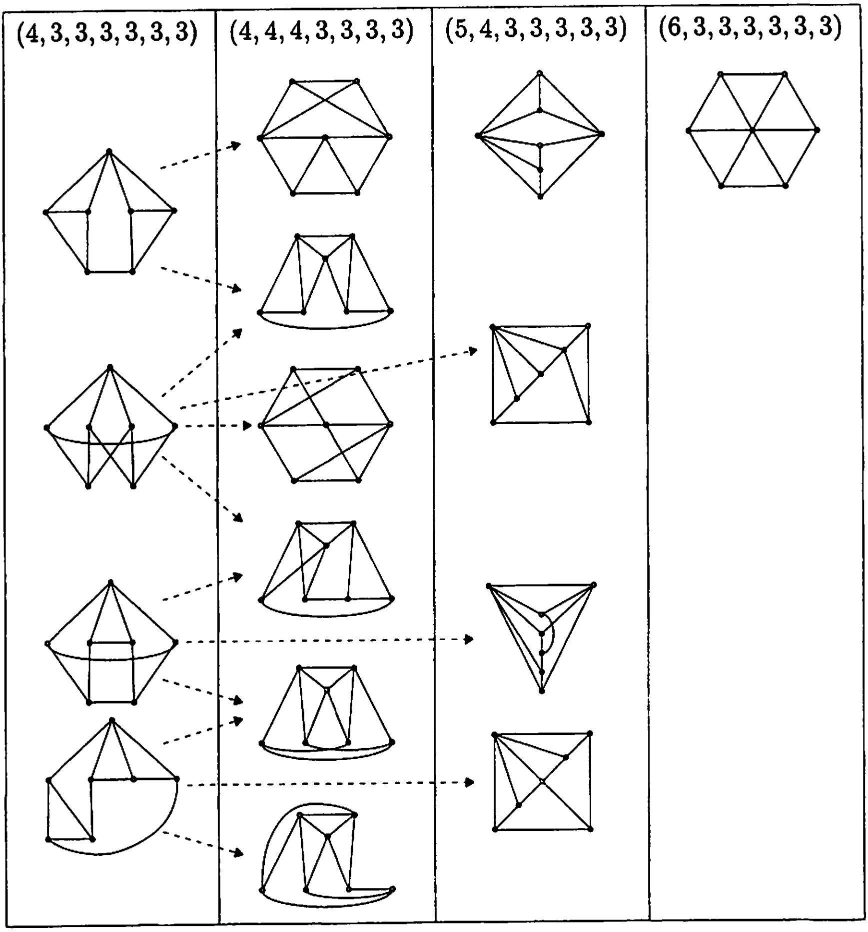

Figure 4: The (7, 6)-induced alliances set, with their associated degree sequence. The arrows indicate the subgraph relation.

Corollary 3.11 *Let $G = (V, E)$ be a 6-regular graph.*

- $a(G) = 4 \Leftrightarrow K_4$ *is an induced subgraph of* G*;*
- $a(G) = 5 \Leftrightarrow W_4$ *is an induced subgraph of* G *and* K_4 *is not; and*
- $a(G) = 6 \Leftrightarrow$ *some graph in* $\mathcal{S}_{(6,6)}$ *is an induced subgraph of* G*, and neither* K_4 *nor* W_4 *are.*

- *$a(G) = 7 \Leftrightarrow$ some graph in $\mathcal{S}_{(7,6)}$ is an induced subgraph of G, and neither K_4 nor W_4, nor any of the graphs in $\mathcal{S}_{(6,6)}$ are.*

The number of $(m, 6)$-admittable sequences and, consequently, the number of graphs in $\mathcal{S}_{(m,6)}$, increases with the cardinality, m. A similar but longer reasoning gives the set of degree sequences associated to $(8, 6)$-induced alliances. In this case, the number of graphs is significantly larger. However, an exhaustive search allows us to give the following two claims.

Claim 3.12 *The $(8, 6)$-admittable sequences are*

$$(6,4,3,3,3,3,3,3),\ (5,5,4,4,3,3,3,3),\ (5,5,3,3,3,3,3,3),$$
$$(5,4,4,4,4,3,3,3),\ (5,4,4,3,3,3,3,3),\ (4,4,4,4,3,3,3,3),$$
$$(4,4,3,3,3,3,3,3),\ \textit{and}\ (3,3,3,3,3,3,3,3).$$

Claim 3.13 *The set $\mathcal{S}_{(8,6)}$ of the $(8, 6)$-ias contains exactly the 65 graphs in Figure 5.*

3.2 Defensive alliances in 7-regular graphs

We can easily extend the results in the previous section to 7-regular graphs.

Indeed, we only need to notice that, if $m \leq 7$, then the $(m, 7)$-admittable sequences coincide with the $(m, 6)$-admittable sequences, and the $(m, 7)$-ias are the same as the $(m, 6)$-ias. Moreover, the set of $(8, 7)$-admittable sequences is exactly the set of $(8, 6)$-admittable sequences, adding the sequence $(7, 3, 3, 3, 3, 3, 3, 3)$. This implies that the set of $(8, 7)$-ias, $\mathcal{S}_{(8,7)}$, contains exactly the graphs in $\mathcal{S}_{(8,6)}$, plus W_7, which corresponds to the degree sequence $(7, 3, 3, 3, 3, 3, 3, 3)$. To summarize, we have

$$\mathcal{S}_{(6,7)} = \mathcal{S}_{(6,6)}, \quad \mathcal{S}_{(7,7)} = \mathcal{S}_{(7,6)}, \quad \mathcal{S}_{(8,7)} = \mathcal{S}_{(8,6)} \cup \{W_7\}.$$

(See Figures 2 and 4.)

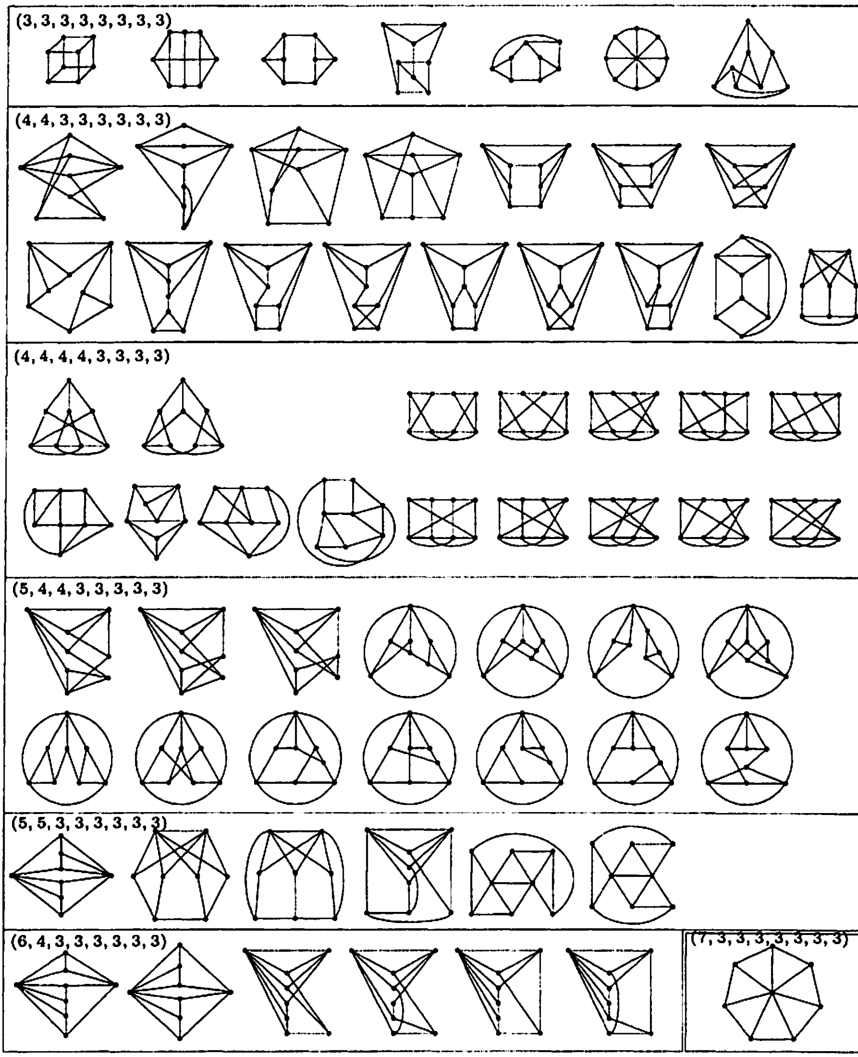

Figure 5: The (8, 6)- and the (8, 7)-induced alliances set, with their associated degree sequence. The first 65 graphs are the graphs in $\mathcal{S}_{(8,6)}$, which are also in $\mathcal{S}_{(8,7)}$. The set $\mathcal{S}_{(8,7)} \setminus \mathcal{S}_{(8,6)}$ contains only the graph W_7.

3.3 Strong defensive alliances in regular graphs

Defensive alliances and strong defensive alliances coincide if G is d-regular, with d even. For d odd, a defensive alliance is a set of vertices that induces a subgraph with minimum degree at least $\frac{d-1}{2}$ and maximum degree at most d, while a strong defensive alliance is a set of vertices that induces a subgraph with minimum degree at least $\frac{d+1}{2}$ and maximum degree at most d. (See Property 3.3.)

5-regular graphs. We have that $\hat{\mathcal{S}}_{(m,5)}$, defined in Definition 3.4, is the set of graphs of minimum degree at least 3 and maximum degree at most 5. Thus, if $m \leq 6$, $\hat{\mathcal{S}}_{(m,5)} = \mathcal{S}_{(m,6)}$. For $m = 7, 8$, we have to remove from $\mathcal{S}_{(m,6)}$ the graphs with maximum degree 6. To be precise, $\hat{\mathcal{S}}_{(7,5)} = \mathcal{S}_{(7,6)} \setminus \{W_6\}$ and $\hat{\mathcal{S}}_{(8,5)}$ contains the 59 graphs in $\mathcal{S}_{(8,6)}$ (see Figure 5) corresponding to the degree sequences

$$(5,5,4,4,3,3,3,3),\ (5,5,3,3,3,3,3,3),\ (5,4,4,4,4,3,3,3),$$

$$(5,4,4,3,3,3,3,3),\ (4,4,4,4,3,3,3,3),\ (4,4,3,3,3,3,3,3),$$

$$\text{and } (3,3,3,3,3,3,3,3).$$

6-regular graphs. We have that $\hat{\mathcal{S}}_{(m,6)} = \mathcal{S}_{(m,6)}$.

7-regular graphs. A graph is in $\hat{\mathcal{S}}_{(m,7)}$ if it has minimum degree 4 and maximum degree at most 7, and it contains no subgraph isomorphic to a graph in $\hat{\mathcal{S}}_{(m',7)}$, for any $4 \leq m' < m$.

In this case, we cannot derive any result about the $(m, 7)$-induced strong alliances set from the $(m, 7)$-induced alliances set.

4 Conclusions and open problems

We have studied defensive alliances of cardinality $k \leq 8$ in regular graphs of degree 6.

Open problems. We let some open problems about defensive alliances in regular graphs

- Can we describe, in some constructive way, the graphs in $\mathcal{S}_{(m,6)}$?
- We think that the number of graphs in $\mathcal{S}_{(m,6)}$ exponentially increases with m.

- We have seen that the study of defensive alliances in regular graphs becomes more and more complex as the degree increases. Therefore, we propose to restrict the study of alliances to more symmetric graphs. In particular, we propose the study of alliances in the well known (undirected) circulant graphs.

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Lattices generated by partial injective maps of finite sets

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Abstract Let n be a positive integer with $n \geq 2$ and $[n] := \{1, 2, \cdots, n\}$. An m-partial injective map of $[n]$ is a pair (A, f) where A is an m-subset of $[n]$ and $f : A \to [n]$ is an injective map. Let $P = L \cup \{1\}$, where L is the set of all the partial injective maps of $[n]$. Partially ordered P by ordinary or reverse inclusion, two families of finite posets are obtained. This article proves that these posets are atomic lattices, discusses their geometricity, and computes their characteristic polynomials.

AMS classification : 20G40, 05B35

Key words: Partial injective map; Atomic lattice; Characteristic polynomial

1 Introduction

The results on the lattices generated by transitive sets of subspaces under finite classical groups may be found in Huo, Liu and Wan [4, 5, 6]. In [1], Guo discussed the lattices associated with finite vector spaces and finite affine spaces. The lattices generated by the orbits of subspaces under finite classical groups have been obtained in a series of papers by Huo and Wan [7], Guo, Li and Wang [3], Wang and Feng [9], Wang and Guo [10, 11], Guo and Nan [2, 8], Wang and Li [12], Xu et al. [13] studied the lattices generated by partial maps of finite sets. In this paper, we contiune this research, and consider the similar problem for partial injective maps of finite sets.

Let $(P, \leq)$ be a poset. We write $a < b$ whenever $a \leq b$ and $a \neq b$. For any two elements $a, b \in P$, we say a *covers* b, denoted by $b \lessdot a$, if $b < a$ and there exists

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