The extremal primitive digraph with both Lewin index $n-2$ and girth 2 or 3*

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Abstract

Let D be a primitive digraph. Then there exists a nonnegative integer k such that there are walks of length k and $k+1$ from u to v for some $u, v \in V(D)$ (possibly u again). Such smallest k is called the Lewin index of the digraph D, denoted by $l(D)$. In this paper, the extremal primitive digraphs with both Lewin index $n-2$ and girth 2 or 3 are determined.

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1 Introduction

In this paper, we permit no loop and no multiple arcs for a digraph. Let $D = (V, E)$ be a digraph with order n. We call a digraph D is strongly connected if there exist both directed walks from u to v and from v to u for any $u, v \in V(D)$. Let $W = v_0e_1v_1e_2\cdots e_kv_k$ ($e_i = (v_{i-1}, v_i)$ for $1 \leq i \leq k$) be a directed walk of digraph D and we call a directed walk W directed circuit when $v_k = v_0$. If all the vertices of directed circuit W are different, W can be called a directed cycle. Sometimes a directed walk can be denoted simply by $W = v_0v_1\cdots v_k$ or $W = e_1e_2\cdots e_k$ if there is no ambiguity. Positive integer k is called the length of the directed walk W, denoted by $L(W)$. If all vertices of a directed walk W are different, W can be called directed path, denoted by P usually. The length of the shortest directed path form v_i to v_j is called the distance from v_i to v_j in S, denoted by $d(v_i, v_j)$. A directed cycle with length k is called k-cycle. The length of the shortest directed cycle in D is called the girth of D, denoted by g usually. In a strongly connected digraph D, let $d(C_1, C_2) = \min\{d(u, v) : u \in V(C_1), v \in V(C_2)\}$ denote the distance from directed cycle C_1 to directed cycle C_2 and $d^0(C_1, C_2) = \min\{d(C_1, C_2), d(C_2, C_1)\}$ denote the distance between directed cycle C_1 and directed cycle C_2. If p is a positive integer and C is a direct cycle, then pC denotes the direct walk obtained by traversing C p times. If a direct cycle C passes through the end vertex of W, $W \bigcup pC$ denotes the the direct walk obtained by going along W and then going around the cycle C p times. $pC \bigcup W$ is similarly defined. The *union* of two digraph S and H is always denoted by $S \bigcup H$.

Definition 1.1 *A digraph D is primitive if there exists an nonnegative integer k such that for each ordered pair of vertices $v_i, v_j \in V(D)$ (not necessarily distinct) there is a directed walk from v_i to v_j with length k. Such smallest k is called the exponent of the graph D, denoted by $exp(D)$.*

Definition 1.2 *Let D be a primitive digraph. Then there exists a nonnegative integer k such that there are directed walks of length k and $k+1$ from u to v for some $u, v \in V(D)$ (possibly u again). Such smallest k is called the Lewin index of the digraph D, denoted by $l(D)$.*

In a primitive digraph D, let $\mathcal{C}_k = \{C_k^1, C_k^2, \cdots, C_k^m\}$ ($m \in \mathbf{Z}^+$) denote the k-cycle set, $\mathcal{Q}_k = \{Q_k^1, Q_k^2, \cdots, Q_k^t\}$ ($t \in \mathbf{Z}^+$) denote the set of all cycles satisfying that $\gcd(k, L(Q_k^i)) = 1$ for $i = 1, 2, \cdots, t$, and $d^*(\mathcal{C}_k, \mathcal{Q}_k) = \min\{d^0(C_k^i, Q_k^j) : C_k^i \in \mathcal{C}_k, Q_k^j \in \mathcal{Q}_k\}$. We also let $u \overset{k,\ k+1}{\longrightarrow} v$ denote that there exist directed walks with length k and $k+1$ and let $u \overset{k,\ k+1}{\not\longrightarrow} v$

denote there exist no directed walk with length k or $k+1$ from vertex u to v.

Lewin proved that a strongly connected digraph is primitive if and only if there exists a nonnegative integer k such that there are directed walks of length k and $k+1$ from u to v for some u, $v \in V(D)$ (possibly u again) and so proposed the Lewin index about the primitive digraph in [1].

Definition 1.3 *Let D be a primitive digraph. For any u, $v \in V(D)$, let $l(u,v) = \min\{k|u \overset{k,\ k+1}{\longrightarrow} v\}$ denote the Lewin index from u to v and $l(u) = \min\{l(u,v)|v \in V(D)\}$ denote the Lewin index at u. It is easy to see that $l(D) = \min\{l(u)|u \in V(D)\} = \min\{l(u,v)|u,v \in V(D)\}$. Let $R_i(u)$ denote the set of vertices arrived by i steps from vertex u in primitive digraph D. Denote by $D_{n,g}$ the set of all primitive digraphs with girth g and order n.*

In [2], J. Shen proved that $l(D) \leq n-2$ for all primitive digraphs with girth $g = 2,3$ and order n. In [5], X.Q. Zhuang get the Lewin index set for all primitive digraphs with both girth 2 and order n. In [4], L.Q. Wang and Z.K. Miao get the Lewin index set for all primitive digraphs with both girth 3 and order n. In this paper, the extremal digraphs with both Lewin index $n-2$ and girth 2 or 3 are determined.

2 Preliminaries

Lemma 2.1 ([3]) *Let $\{r_1, r_2, \cdots, r_\lambda\}$ denote the cycle length set of digraph D. Then D is primitive if and only if D satisfies that D is strongly connected and* $\gcd(r_1, r_2, \cdots, r_\lambda) = 1$.

Lemma 2.2 ([2]) *Let D be a primitive digraph of order n. Then $l(D) \leq n-2$ if $2 \leq g \leq 3$.*

Suppose $n \equiv 0 \pmod 3$, $n \geq 6$, and $D^*_{3(0)}$ is a digraph consisting of $(n-1)$-cycle $C_{n-1} = (v_1, v_2, \cdots, v_{n-2}, v_{n-1}, v_1)$ and 3-cycle $C_3 = (v_1, v_n, v_{n-1}, v_1)$. Suppose $n \equiv 1 \pmod 3$, $n \geq 7$, and $D^*_{3(1)}$ is a digraph consisting of $(n-2)$-cycle $C_{n-2} = (v_1, v_2, \cdots, v_{n-3}, v_{n-2}, v_1)$ and 3-cycle $C_3 = (v_{n-2}, v_{n-1}, v_n, v_{n-2})$. Suppose $n \equiv 2 \pmod 3$, $n \geq 5$, and $D^*_{3(2)}$ is a digraph consisting of n-cycle $(v_1, v_2, \cdots, v_{n-1}, v_n, v_1)$ and 3-cycle $C_3 = (v_1, v_2, v_n, v_1)$. Then $\{D^*_{3(0)}, D^*_{3(1)}, D^*_{3(2)}\} \subseteq D_{n,3}$.

Lemma 2.3 ([4]) $l(D^*_{3(i)}) = n-2$ *for* $i = 0, 1, 2$.

Suppose $n \equiv 0 \pmod 2$, $n \geq 4$, and $D^*_{2(0)}$ is a digraph consisting of $(n-1)$-cycle $(v_1, v_2, \cdots, v_{n-2}, v_{n-1}, v_1)$ and 2-cycle $C_2 = (v_1, v_n, v_1)$. Suppose $n \equiv 1 \pmod 2$, $n \geq 3$, and $D^*_{2(1)}$ is a digraph consisting of n-cycle $(v_1, v_2, \cdots, v_{n-1}, v_n, v_1)$ and 2-cycle $C_2 = (v_1, v_2, v_1)$. Then $\{D^*_{2(0)}, D^*_{2(1)}\} \subseteq D_{n,2}$.

Lemma 2.4 ([5]) $l(D^*_{2(i)}) = n-2$ *for* $i = 0, 1$.

3 Main results of this paper

Lemma 3.1 *Let* $D \in D_{n,3}$. *If* $d^*(C_3, Q_3) > 0$, *then* $l(D) \leq n-3$.

Proof. Let $d^*(C_3, Q_3) = d^0(C_3^1, Q_3^1)$, $C_3^1 = (v_1, v_2, v_3, v_1)$, $Q_3^1 = (v_i, v_{i+1}, \cdots, v_j, \cdots, v_i)$ $(j > i)$. Because the girth $g = 3$, so $L(Q_3^1) \geq 4$. Let $P_1 = (v_i, v_{i-1}, \cdots, v_7, v_6, v_5, v_4, v_3)$ denote the shortest path from Q_3^1 to C_3^1. Let P_2 denote the shortest path from C_3^1 to Q_3^1. Suppose $L(P_1) = d(v_i, v_3) = d^0(C_3^1, Q_3^1)$. Let $D_1 = C_3^1 \bigcup P_1 \bigcup Q_3^1$.

(i) v_2 is the starting vertex of P_2 and v_e $(v_e \in V(Q_3^1))$ is the end vertex of P_2.

Case 1 $L(P_1) \geq 2$.

We assert $|V(D_1)| \leq n-1$ now. Otherwise, $|V(D_1)| = n$. Now there must be $L(P_2) \geq 2$ and all vertices of P_2 be in $V(P_1)$ but the vertices v_2 and v_e. If there is a vertex $v_m \in V(P_2)$ such that $v_m \neq v_2$, v_e, and $v_m \notin V(P_1)$, then $v_m \in V(C_3^1)$ or $v_m \in V(Q_3^1)$. Suppose $v_m \in V(C_3^1)$, then the length of the shortest path from v_m to v_e is less than that of P_2, which contradicts that P_2 is the shortest path from C_3^1 to Q_3^1. Suppose v_k is the first common vertex of P_1 and P_2 along P_2, there must be $6 \leq k < i$ and $k \equiv 0 \pmod 3$. Otherwise there cause cycle $(v_1, v_2, v_k, v_{k-1}, \cdots, v_5, v_4, v_3, v_1)$, which contradicts $d^*(C_3, Q_3) > 0$ because $\gcd(k, 3) = 1$. There is no arc (v_s, v_h) $(s - h \geq 2)$ in P_2. Otherwise there cause shorter path $P = (v_i, \cdots, v_s, v_h, v_{h-1}, \cdots, v_3)$, $L(P) < L(P_1)$, which contradicts $L(P_1) = d(v_i, v_3) = d^0(C_3^1, Q_3^1)$. So v_5, v_4 are not in P_2 and $L(P_2) \leq L(P_1) - 2$, which contradicts $L(P_1) = d(v_i, v_3) = d^0(C_3^1, Q_3^1)$. Thus, the assertion holds.

Subcase 1.1 $L(Q_3^1) \equiv 1 \pmod 3$.

There exist directed walk $Q_3^1 \bigcup v_i \bigcup P_1$ of length $L(P_1) + L(Q_3^1)$ and directed walk $P_1 \bigcup \frac{L(Q_3^1) - 1}{3} C_3^1$ of length $L(P_1) + L(Q_3^1) - 1$ from v_i to v_3. Note that $|V(D_1)| \leq n-1$ and $L(P_1)+L(Q_3^1)-1 \leq n-4$, so $l(v_i) \leq n-4$ and $l(D) \leq n-4$.

Subcase 1.2 $L(Q_3^1) \equiv 2 \pmod 3$.

Similar to Subcase 1.1, there must be two directed walks of length $L(P_1) + L(Q_3^1)$ and $L(P_1) + L(Q_3^1) + 1$ from v_i to v_3. Note that $|V(D_1)| \leq n-1$ and $L(P_1) + L(Q_3^1) \leq n-3$, so $l(v_i) \leq n-3$ and $l(D) \leq n-3$.

Case 2 $L(P_1) = 1$.

Subcase 2.1 $|V(D_1)| \leq n-1$.

Similar to Subcase 1.1 and Subcase 1.2, there must be $l(D) \leq n-3$.

Subcase 2.2 $|V(D_1)| = n$.

Both P_1, P_2 are arcs now. Along Q_3^1, let a_1 denote the path from v_e to v_i and a_2 denote the path from v_i to v_e.

If $L(Q_3^1) \equiv 1 \pmod 3$, similar to Subcase 1.1, there must be two directed walks of length $L(P_1) + L(Q_3^1)$ and $L(P_1) + L(Q_3^1) - 1$ from v_i to v_3. Note that $L(P_1) + L(Q_3^1) - 1 = n-3$, so $l(v_i) \leq n-3$ and $l(D) \leq n-3$.

If $L(Q_3^1) \equiv 2 \pmod 3$, then $L(a_1) \equiv 2 \pmod 3$, $L(a_2) \equiv 0 \pmod 3$ and $L(a_2) \geq 3$. Otherwise, there cause cycle $C' = (v_i, v_3, v_1, v_2, v_e) \bigcup a_1$ such that $\gcd(L(C'), 3) = 1$, which contradicts $d^*(C_3, Q_3) > 0$.

Let $P_3 = (v_i, v_3, v_1, v_2, v_e)$.

If $L(a_2) = 3$, it is easy to see that $l(D) \leq 3$ because there are two directed walks a_2 and P_3 from v_i to v_e.

If $L(a_2) > 3$, then there exist directed walks $P_1 \bigcup \frac{L(a_2) - 3}{3} C_3^1 \bigcup (v_3, v_1, v_2, v_e)$ of length $L(a_2) + 1$ from v_i to v_e. Note that $L(a_2) \leq n-3$, so $l(v_i) \leq n-3$ and $l(D) \leq n-3$.

In a same way as (i), for the cases: (ii) v_1 is the starting vertex of P_2 and v_e ($v_e \in V(Q_3^1)$) is the end vertex of P_2; (iii) v_3 is the starting vertex of P_2 and v_e ($v_e \in V(Q_3^1)$) is the end vertex of P_2, we can prove that $l(D) \leq n-3$ □

Similar to Lemma 3.1, we have the following Lemma 3.2.

Lemma 3.2 *Let $D \in D_{n,2}$. If $d^*(C_2, Q_2) > 0$, then $l(D) \leq n-3$.*

Corollary 3.3 *Let $D \in D_{n,3}$. If $l(D) = n-2$, then $d^*(C_3, Q_3) = 0$.*

Corollary 3.4 *Let $D \in D_{n,2}$. If $l(D) = n-2$, then $d^*(C_2, Q_2) = 0$.*

Theorem 3.5 *Suppose that $n \equiv i \pmod 3$ where $i = 0, 1, 2$. Let $D \in D_{n,3}$. Then $l(D) = n-2$ if and only if $D \cong D^*_{3(i)}$.*

Proof. We only prove the case that $i = 0$. The other two cases can be proved similarly.

Now we prove the case that $i = 0$. It is clearly that the sufficiency holds by Lemma 2.3. We prove the necessity.

It is clearly that $d^*(C_3, Q_3) = 0$ now by Corollary 3.3. Let $d^0(C_3^1, Q_3^1) = d^*(C_3, Q_3) = 0$. Let $D_1 = C_3^1 \bigcup Q_3^1$.

If $|V(D_1)| \leq n-1$, then $l(D_1) \leq n-3$ by Lemma 2.2 and $l(D) \leq n-3$, which contradicts $l(D) = n-2$. So $|V(D_1)| = n$.

It is easy to check that $D_1 \not\cong D^*_{3(2)}$ because $\gcd(3, L(Q_3^1)) = 1$.

If $D_1 \cong D^*_{3(1)}$, we can suppose $D_1 = D^*_{3(1)}$ for convenience. Then $C_3^1 = (v_{n-2}, v_{n-1}, v_n, v_{n-2})$, $Q_3^1 = (v_1, v_2, \cdots, v_{n-3}, v_{n-2}, v_1)$. There are directed walk of length $n-2$ by going around Q_3^1 once and directed walk of length $n-3$ which is $\frac{n-3}{3}C_3^1$ from v_{n-2} to itself, so $l(D_1) \leq n-3$ and $l(D) \leq n-3$, which contradicts $l(D) = n-2$. So $D_1 \cong D^*_{3(0)}$. For convenience, suppose $D_1 = D^*_{3(0)}$, $C_3^1 = (v_1, v_n, v_{n-1}, v_1)$ and $Q_3^1 = (v_1, v_2, \cdots, v_{n-2}, v_{n-1}, v_1)$.

Assertion 1 There is no arc (v_i, v_j) $(1 \leq i < j < n, j - i \geq 2)$.

Otherwise, suppose that there is arc (v_i, v_j) $(1 \leq i < j < n, j-i \geq 2)$. Let P_1 denote the path from v_i to v_j along Q_3^1, P_2 denote the arc (v_i, v_j), P_3 denote the path from v_1 to v_i along Q_3^1, P_4 denote the path from v_j to v_{n-1} along Q_3^1 and $C = P_3 \bigcup P_2 \bigcup P_4 \bigcup (v_{n-1}, v_1)$. It is easy to check that $L(C) \equiv 0 \pmod 3$ and $L(C) \leq n-3$. Otherwise, if $L(C) \not\equiv 0 \pmod 3$, then $\gcd(L(C), 3) = 1$. Let $D_2 = C \bigcup C_3^1$. Then $l(D_2) \leq n-3$ by Lemma 2.2 and $l(D) \leq n-3$, which contradicts $l(D) = n-2$. So $L(P_1) \equiv 0 \pmod 3$ and $L(P_1) \leq n-3$.

Clearly, there is no case that $v_j = v_{n-1}$ and $v_i = v_1$. Otherwise, there is 2-cycle $C_2 = (v_1, v_{n-1}, v_1)$, which contradicts $g = 3$.

If $v_j \neq v_{n-1}$, then $P_3 \bigcup P_1$ is a directed walk of length $L(P_3) + L(P_1)$ from v_1 to v_j; $\frac{L(P_1)}{3} C_3^1 \bigcup P_3 \bigcup P_2$ is a directed walk of length $L(P_3) + L(P_1) + 1$ from v_1 to v_j. Note that $L(P_3) + L(P_1) \leq n-3$, so $l(v_1) \leq n-3$ and $l(D) \leq n-3$, which contradicts $l(D) = n-2$.

If $v_j = v_{n-1}$, $v_i \neq v_1$, then $P_2 \bigcup \frac{L(P_1)}{3} C_3^1$ is a directed walk of length $L(P_1) + 1$ from v_i to v_{n-1}. Note that $L(P_1) \leq n-3$, so $l(v_i) \leq n-3$ and $l(D) \leq n-3$, which contradicts $l(D) = n-2$.

So the Assertion 1 holds.

Assertion 2 There is no arc (v_j, v_i) $(1 \leq i < j < n, j - i \neq n-2)$.

Otherwise, suppose that there exists arc (v_j, v_i) $(1 \leq i < j < n)$. Note that the girth $g = 3$ in D. Then $j - i \geq 2$. Let P_1 denote the path from v_i to v_j along Q_3^1, P_2 denote the arc (v_j, v_i), P_3 denote the path from v_1 to v_i along Q_3^1, P_4 denote the path from v_j to v_{n-1} along Q_3^1, $C_a = P_1 \bigcup P_2$ and $P = P_4 \bigcup (v_{n-1}, v_1) \bigcup P_3$.

Case 2.1 $L(C_a) \equiv 0 \pmod 3$.

Now $L(P) \equiv 0 \pmod 3$ and $3 \leq L(P) \leq n-3$, $3 \leq L(C_a) \leq n-3$.

If $L(C_a) \geq L(P)$, suppose $L(C_a) - L(P) = 3k$, $k \in N$. $C_a \bigcup \{v_j\} \bigcup P_2$ is directed walk of length $L(C_a) + 1$ and $P_4 \bigcup kC_3^1 \bigcup (v_{n-1}, v_1) \bigcup P_3$ is a directed walk of length $L(C_a)$ from v_j to v_i, so $l(v_j) \leq n-3$ and $l(D) \leq n-3$, which contradicts $l(D) = n-2$.

If $L(C_a) < L(P)$, then $L(P) - L(C_a) \geq 3$. Suppose $L(P) = kL(C_a) + m$, $k \in Z^+, m \in N$. Then $m \equiv 0 \pmod 3$, $0 \leq m \leq L(C_a) - 3$, $L(C_a) - m \equiv 0 \pmod 3$.

If $m = 0$, $kC_a \bigcup \{v_j\} \bigcup P_2$ is a directed walk of length $L(P) + 1$ from v_j to v_i, note that $L(P) \leq n-3$, so $l(v_j) \leq n-3$ and $l(D) \leq n-3$, which contradicts $l(D) = n-2$.

If $m \geq 3$, then $L(P) + L(C_a) - m = (k+1)L(C_a)$. $(k+1)C_a \bigcup \{v_j\} \bigcup P_2$ is a directed walk of length $(k+1)L(C_a)+1$ and $P_4 \bigcup \frac{L(C_a) - m}{3} C_3^1 \bigcup (v_{n-1}, v_1) \bigcup P_3$ is a directed walk of length $L(P) + L(C_a) - m$ from v_j to v_i. Note that $L(P) + L(C_a) = n$ and $3 \leq m$, so $L(P) + L(C_a) - m \leq n-3$, $l(v_j) \leq n-3$ and $l(D) \leq n-3$, which contradicts $l(D) = n-2$.

Case 2.2 $L(C_a) \equiv 1 \pmod 3$.

So there must be just two subcases as follow:

(i) $n - |V(C_3^1 \cup C_a \cup P_3)| \geq 1$;

(ii) $n - |V(C_3^1 \cup C_a \cup P_4)| \geq 1$.

Suppose (i) holds. Then $P_3 \bigcup C_a$ is a directed walk of length $L(P_3) + L(C_a)$ from v_1 to v_i, $\frac{L(C_a)-1}{3} C_3^1 \bigcup \{v_1\} \bigcup P_3$ is a directed walk of length $L(P_3) + L(C_a) - 1$ from v_1 to v_i. Note that $L(P_3) + L(C_a) - 1 \leq n - 4$, so $l(v_1, v_i) \leq n - 4$ and $l(D) \leq n - 4$, which contradicts $l(D) = n - 2$.

Case 2.3 $L(C_a) \equiv 2 \pmod 3$.

If $v_j = v_{n-1}$, then $i \geq 4$. Thus C_a is a directed walk of length $L(C_a)$ from v_{n-1} to itself and $\frac{L(C_a)+1}{3} C_3^1$ is a directed walk of length $L(C_a) + 1$ from v_{n-1} to itself. Because $i \geq 4$, then $L(C_a) \leq n - 4$ and $l(D) \leq l(v_{n-1}) \leq L(C_a) \leq n - 4$, which contradicts $l(D) = n - 2$.

If $v_i = v_1$, then $j \leq n-4$. So there are directed walk of length $L(C_a)$ and directed walk $\frac{L(C_a)+1}{3} C_3^1$ of length $L(C_a) + 1$ from v_1 to itself. Because $j \leq n-4$, then $L(C_a) \leq n-4$ and $l(D) \leq L(C_a) \leq n-4$, which contradicts $l(D) = n - 2$.

If $v_j \neq v_{n-1}, v_i \neq v_1$. There is no case $v_j = v_{n-2}, v_i = v_2$ because $L(C_a) \equiv 2 \pmod 3$. So $L(P_3) \geq 2$ or $L(P_4) \geq 2$. Suppose $L(P_3) \geq 2$, then $C_a \bigcup \{v_j\} \bigcup P_4$ is a directed walk of length $L(C_a) + L(P_4)$ from v_j to v_{n-1} and $P_4 \bigcup \frac{L(C_a)+1}{3} C_3^1$ is a directed walk of length $L(C_a) + L(P_4) + 1$ from v_j to v_{n-1}. Note that $L(C_a) + L(P_4) \leq n - 3$, so $l(D) \leq n - 3$, which contradicts $l(D) = n - 2$.

To sum up, the Assertion 2 holds.

Assertion 3 There is no arc between vertices v_i and v_n.

Otherwise, there are the cases as follows.

Case 3.1 There is arc between v_{n-2} and v_n.

Suppose that there is arc (v_{n-2}, v_n). Then (v_{n-2}, v_n, v_{n-1}) is a directed walk of length 2 and (v_{n-2}, v_{n-1}) is a directed walk of length 1 from v_{n-2} to v_{n-1}, so $l(D) \leq l(v_{n-2}) \leq 1$, which contradicts $l(D) = n - 2$.

If there is arc (v_n, v_{n-2}), then $(v_n, v_{n-2}, v_{n-1}, v_1, v_n)$ is a directed walk of length 4 from v_n to itself. So $l(D) \leq l(v_n) \leq 3$, which contradicts

$l(D) = n-2$.

In a same way, we can prove the following Case 3.2.

Case 3.2 There is arc between v_2 and v_n. Then $l(D) \le 3$.

Case 3.3 There is arc between v_i $(3 \le i \le n-3)$ and v_n.

Along cycle Q_3^1, let P_1 denote the directed path from v_1 to v_i and P_2 denote the directed path from v_i to v_{n-1}. Then $L(P_1) \ge 2$, $L(P_2) \ge 2$.

1° There exists arc (v_i, v_n).

(i) If $L(P_1) \equiv 0 \pmod 3$, then $L(P_2) \equiv 1 \pmod 3$ and $4 \le L(P_2) \le n-5$. Let $W_1 = P_2 \bigcup (v_{n-1}, v_1, v_n)$. Then $L(W_1) \equiv 0 \pmod 3$ and $6 \le L(W_1) \le n-3$. Now $(v_i, v_n) \bigcup \frac{L(W_1)}{3} C_3^1$ is a directed walk of length $L(W_1)+1$ from v_i to v_n, so $l(D) \le l(v_i) \le L(W_1) \le n-3$, which contradicts $l(D) = n-2$.

(ii) If $L(P_1) \equiv 1 \pmod 3$, then $L(P_2) \equiv 0 \pmod 3$, $4 \le L(P_1) \le n-5$ and $3 \le L(P_2) \le n-6$. Let $W_1 = P_2 \bigcup (v_{n-1}, v_1, v_n)$. Then $L(W_1) \equiv 2 \pmod 3$ and $5 \le L(W_1) \le n-4$. $(v_i, v_n) \bigcup \frac{L(W_1)-2}{3} C_3^1$ is a directed walk of length $L(W_1)-1$ from v_i to v_n, so $l(D) \le l(v_i) \le L(W_1) - 1 \le n-5$, which contradicts $l(D) = n-2$.

(iii) $L(P_1) \equiv 2 \pmod 3$, $L(P_1) \ge 2$.

Let $W_2 = P_1 \bigcup (v_i, v_n)$. Then $L(W_2) \equiv 0 \pmod 3$ and $L(W_2) \le n-3$. Now $\frac{L(W_2)}{3} C_3^1 \bigcup (v_1, v_n)$ is a directed walk of length $L(W_2)+1$ from v_1 to v_n, so $l(D) \le l(v_1) \le L(W_2) \le n-3$, which contradicts $l(D) = n-2$.

2° There exists arc (v_n, v_i).

Let $C_b = (v_n, v_i) \bigcup P_2 \bigcup (v_{n-1}, v_1, v_n)$. Then $6 \le L(C_b) \le n-3$, and $L(C_b) \equiv 0 \pmod 3$. Otherwise, let $D_2 = C_b \bigcup (v_n, v_{n-1})$. Then $l(D_2) \le n-5$ by Lemma 2.2, so $l(D) \le l(D_2) \le n-5$, which contradicts $l(D) = n-2$. Thus $L(P_1) \equiv 1 \pmod 3$ and $4 \le L(P_1) \le n-5$. Now $(v_1, v_n) \bigcup \frac{L(P_1)-1}{3} C_3^1 \bigcup (v_n, v_i)$ is a directed walk of length $L(P_1)+1$ from v_1 to v_i, so $l(D) \le l(v_1) \le L(P_1) \le n-5$, which contradicts $l(D) = n-2$.

To sum up, the Assertion 3 holds.

In all, the necessity is proved. □

In a same way, we can prove the following theorem.

Theorem 3.6 *Suppose that* $n \equiv i \pmod 2$ *where* $i = 0, 1$. *Let* $D \in D_{n,2}$. *Then* $l(D) = n - 2$ *if and only if* $D \cong D^*_{2(i)}$.

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